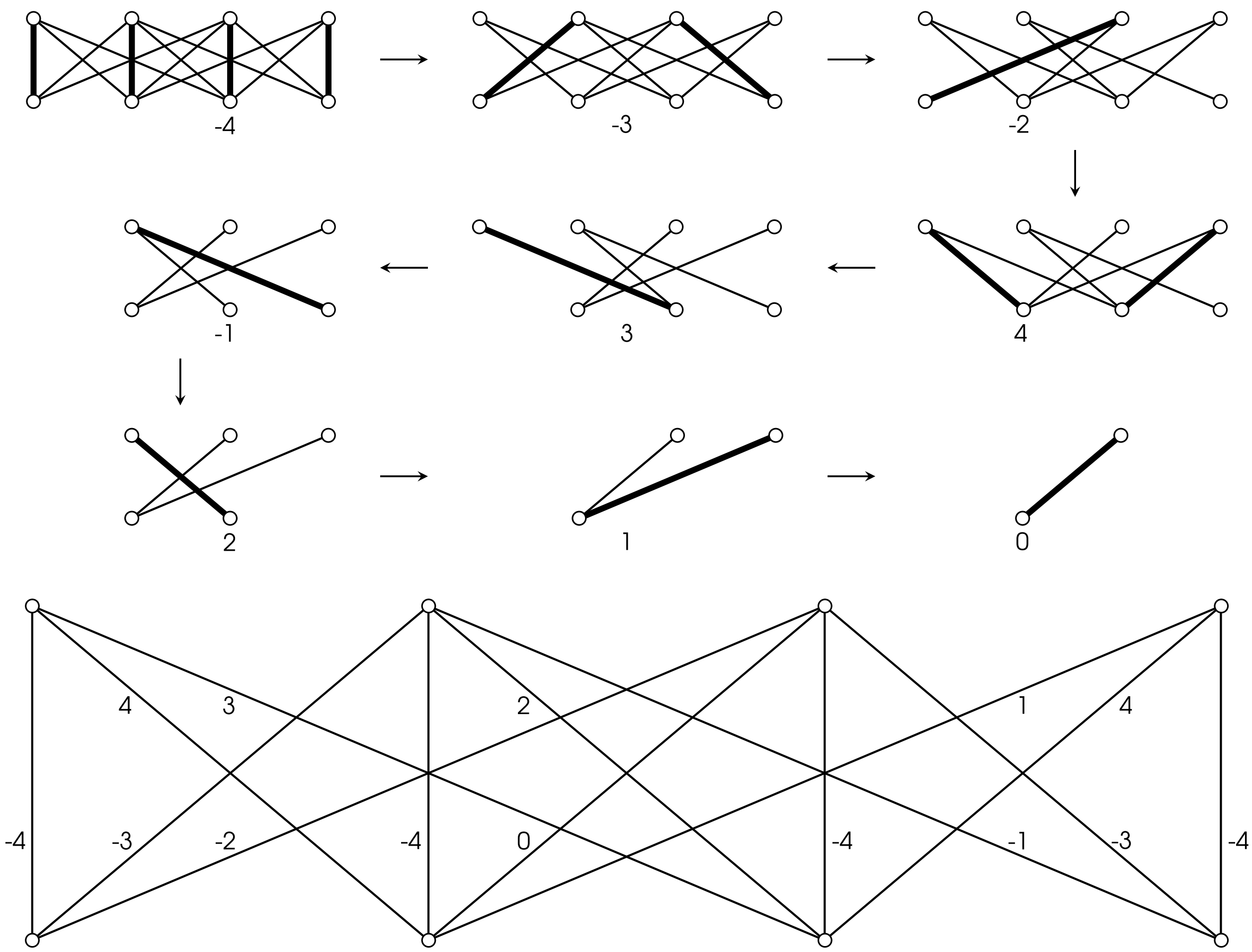
# Graph Coloring Methods

Daniel W. Cranston



Published By Daniel W. Cranston
Richmond, Virginia



ISBN: 979-8-218-46242-0 (paperback)
**1** 2 3 4 5 6 7 8 9 30 29 28 27 26 25 **24**

The cover shows a construction, due to Borodin, Kostochka, and Woodall, of an edge coloring in which each edge $vw$ "defers" to at most $\max\{d(v), d(w)\} - 1$ edges with which $vw$ shares an endpoint. Such a coloring is crucial in their proof of the List Coloring Conjecture for various graph classes. For more details, see Section 5.4.1.

This book was typeset by the author with PDFLaTeX on a MacBook Pro using TeXShop with the fonts XCharter, eulervm, and eucal. The cover uses TeX Gyre Adventor.



Download this book at `https://graphcoloringmethods.com`

To Dad and Beth,
for their encouragement
and support throughout

# Contents

# Preface

> "What is the use of a book," thought Alice, "without pictures or conversations?"
>
> —Lewis Carroll

This book exists for two main reasons: (i) graph coloring is home to some of the most beautiful results, proofs, and techniques in all of mathematics and (ii) I want to help you learn about them in a way that is as easy and fun as possible. Both of these views shape this book.

The book is about ideas, those that are elegant, powerful, and repeated frequently throughout the history of our subject. Each chapter focuses on a single tool, so that you can see lots of examples and develop an intuition for where each technique may be useful in the future.

This books differs from most textbooks in multiple ways. Many of these aim to capture the benefits of an illuminating lecture. We begin most sections with a high-level overview. We also use lots of pictures and worked examples (more than 180 throughout the book). For longer proofs, we break them up into a series of lemmas; usually each lemma and its proof is longer than a page. These convenient "save points" allow you to read the proofs in bite-sized pieces. Reading just the statements of the lemmas provides an outline of the proof. Between the statement of a lemma and its proof, we often include an informal proof sketch. (This sort of intuition is frequently provided in lectures, but is omitted from most papers and many books. The present book is written so that this intuition can be skipped, if you like, and the proofs will still be complete, though perhaps terse.)

To focus attention wholly on techniques, we defer most discussion to a Notes section at the end of each chapter. This contains history and related work, as well as who proved which theorem originally and whose proof we have presented.

This book is about graph coloring, one of the most popular and widely-studied areas of discrete mathematics. The intended reader is a graduate student or early career researcher, although it should be useful for readers who are both less and more experienced. The reader may find it useful to have taken a 1-semester course in graph theory, but this is not strictly necessary. My goal as the author is to help you understand, internalize, and add to (if you like) central results in many areas of graph coloring. This does not mean an exhaustive survey, but rather a curated tour through many areas that are currently flourishing. We focus on topics that have seen significant progress over the past 30 years, emphasizing key techniques that are likely to bear more fruit in the future.

You really should start with Chapter 1, since it introduces many definitions and techniques that are used repeatedly throughout the book. After that, chapters are largely independent. They are organized roughly in order of increasing difficulty, as are the sections within each chapter. In most chapters, a section near the end is marked in the Table of Contents with an $*$, to indicate that its material is more advanced and can be safely skipped on a first reading without missing prerequisites for other sections.

This book is designed for a topics course, although it is also well-suited for self-study. On a first reading, I recommend covering some subset of Sections 1.1–1.4, 2.1–2.5, 3.1, 3.3, 4.1, 4.3, 5.1–5.2, 6.1, 6.3, 7.1–7.3, 8.1, 8.4–8.6, 9.1–9.2, 10.1–10.3, 11.1–11.3, 12.1–12.3. This selection omits most of the hardest proofs, but is still too much material for a single semester. To choose further from among those topics listed, you should simply follow your interests. With all that in mind, feel free to just skim until something catches your eye and start there!

The heart of any thriving research area is not its answers, but its questions. Furthermore, its *tools* and *techniques*, since these give us the best hope for future progress answering these questions. This book is less about what exactly we know (although that is important) and more about how we know it.

*–DWC*

# Chapter 1

# Greedy Coloring

. . . greed—for lack of a better word—is good.
—Gordon Gecko, *Wall Street*

"Greedy, greedy" makes a hungry puppy.
—Aesop's fables (paraphrased)

In this book we study how to partition a set into subsets that satisfy certain constraints. This question arises in contexts as diverse as designing circuits, allocating registers when compiling computer code, solving Sudoku puzzles, and scheduling flight crews. All these problems can be described in the language of graph coloring, where each color represents a subset in the partition, and our goal is to minimize the number of colors.

The general graph coloring problem has no easy answer. More precisely, it is NP-hard[1]. So it is unlikely that we will find an efficient algorithm to optimally solve the coloring problem on an arbitrary input graph. This fundamental hardness result casts a long shadow across the landscape of graph coloring.

In contrast to many problems on spanning trees, connectivity, and matchings (for which we have efficient algorithms that give optimal solutions), for graph coloring we mainly focus on proving upper and lower bounds. We also pay attention to the graph's *structure*, concentrating on planar graphs and other classes of graphs that exhibit some notion of *sparseness*.

A natural idea is to consider elements of the set one-by-one, assigning each to the first part of the partition that is not forbidden by some element already assigned there. This greedy approach can perform quite well, or quite badly, depending on the order in which we consider the elements. In this chapter, we search for good orders, and study what bounds they yield for the minimum number of colors needed to solve the coloring problem.

[1]We give a formal definition in Section A.1.

## 1.1 Degeneracy, Discharging, and Brooks' Theorem

### 1.1.1 Key Ideas and Definitions

graph, vertex set
edge set, edge
endpoints
adjacent
$|G|$, $||G||$
path
endpoints
cycle
connected
acyclic
tree, forest
simple

**Definition 1.1.** A *graph* G consists of a *vertex set* and an *edge set*, where each *edge* is an unordered pair of vertices, called the edge's *endpoints*. (In the problem above, the vertices are the elements of the partition, and the edges are the pairs of vertices forbidden from appearing in the same part.) Two vertices are *adjacent* if they form an edge. For a graph G, we write $|G|$ to denote the size of its vertex set and $\|G\|$ for the size of its edge set. A *path* P in a graph is a sequence of edges $e_1, \ldots, e_\ell$ such that each pair $e_i$ and $e_{i+1}$ have a shared endpoint, and no other edge $e_j$ has that endpoint. (We allow the possibility that $s = 1$ or even $s = 0$, so P is edgeless.) The *endpoints* of a path are the vertices that are endpoints only of $e_1$ and $e_\ell$. A *cycle* is formed from a path with endpoints $v_0$ and $v_\ell$ by adding an edge $v_\ell v_0$. The length of a path or cycle is its number of edges. A graph G is *connected* if, for all $v, w \in V(G)$, some path in G has endpoints $v$ and $w$. A graph is *acyclic* if no subset of its edges form a cycle. A connected acyclic graph is a *tree*, and a disjoint union of trees is a *forest*. Unless stated otherwise, all graphs are undirected and have no loops and no parallel edges. Such a graph is *simple*.

As a first example, the graph G in Figure 1.1 (shown twice, with distinct colorings, which we will get to soon) is simple with $|G| = 8$ and $\|G\| = 12$. It is connected, but far from acyclic; thus, it is neither a tree nor a forest. It contains many paths of lengths 0 to 7 and many cycles of lengths 4 to 8 (but no paths or cycles with other lengths).

coloring
k-coloring
k-colorable, $\chi$
chromatic number
optimal coloring
greedy coloring

A *proper coloring* (or simply *coloring*) of a graph assigns each vertex a color so that adjacent vertices get distinct colors. We denote colors by positive integers. A k-*coloring* is a proper coloring using at most k colors. A graph G is k-*colorable* if it has a k-coloring. The *chromatic number*, $\chi(G)$, of G is the smallest k such that G is k-colorable. A coloring of G using $\chi(G)$ colors is an *optimal coloring*. A simple way to color a graph is to consider its vertices one at a time, in some vertex order $\sigma$, and color each vertex with the smallest color not already used on one of its neighbors. This is a *greedy coloring using* $\sigma$. Figure 1.1 shows two greedy colorings of the same graph using different vertex orders $\sigma$.

In this book we focus on the existence of colorings, rather than on algorithms to produce them, so we typically leave vertex orders $\sigma$ implicit, as in the proof of our first proposition. However, we briefly discuss converting existence proofs into efficient algorithms near the end of Section 1.1.3. (For nearly all of our proofs, we can do this.)

For every graph G, there exists a vertex order $\sigma$ such that coloring G greedily using $\sigma$ produces an optimal coloring. (Given an optimal coloring, form $\sigma$ by starting with all vertices colored 1, followed by those colored 2, etc.) But this observation does not yield an efficient algorithm since each graph G has $|G|!$ possible vertex orders—far too many to try them all. Even for the class of graphs with chromatic number 2, some vertex orders do arbitrarily badly (see Exercise 1). Fortunately, we can often use the structure of G to quickly find an order $\sigma$ that is good, though perhaps not optimal.

**Proposition 1.2.** *If* G *is a forest, then* $\chi(G) \leqslant 2$.

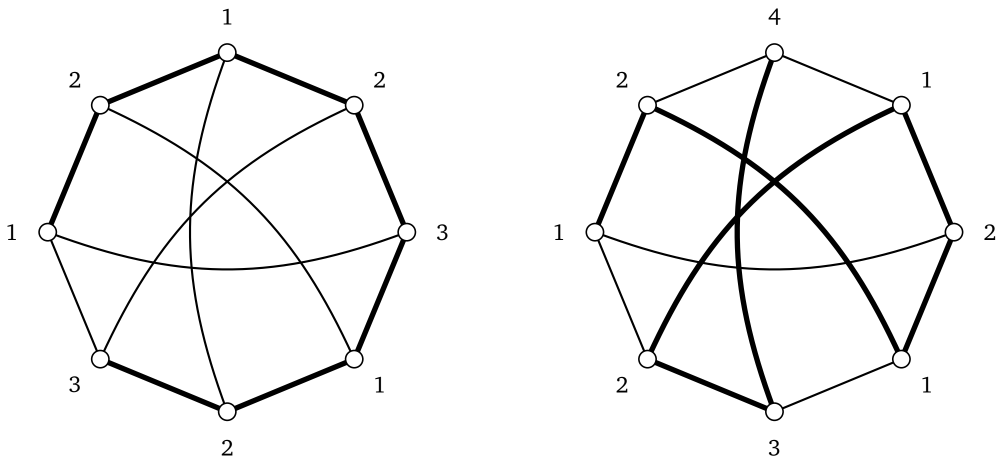



Figure 1.1: Left: Coloring greedily along the bold path uses 3 colors, which is optimal. Right: Coloring greedily along the bold path uses 4 colors. (The order for a greedy coloring need not necessarily follow a path, as shown in Figure 1.9.)

*Proof.* Suppose the proposition is false, and let $G$ be a counterexample with fewest vertices. Since $G$ is a forest, it contains a vertex $v$ of degree at most 1[2]. Let[3] $G' := G - v$. Since $G'$ is also a forest, and $G'$ has fewer vertices than $G$, a smallest counterexample, the theorem is true for $G'$. That is, $G'$ has a 2-coloring $\varphi'$. Since $\varphi'$ uses at most one color on a neighbor of $v$, we can greedily extend the coloring $\varphi'$ to $G$. So $G$ is not a counterexample after all. □

**Remark 1.3.** The proof of Proposition 1.2 is simple, but it illustrates a template that we will see often. The proof breaks into three steps: (i) show that $G$ has a vertex $v$ of degree at most 1, (ii) show that $G - v$ has a 2-coloring, and (iii) show that every 2-coloring of $G - v$ extends to a 2-coloring of $G$. Step (i) is *unavoidability*. Step (iii) is *reducibility*, because proving the theorem for $G$ reduces to the (easier) problem of proving the theorem for $G - v$. We call $v$ a *reducible configuration*, or simply *reducible*, for the problem of 2-coloring. Step (ii) seems to come for free. Subgraph $G - v$ cannot be a counterexample, since we chose $G$ to be a smallest counterexample, and $G - v$ is smaller.

unavoidability
reducibility
reducible configuration

Step (ii) implicitly uses that $G - v$ is again a forest, so it satisfies the hypothesis of the theorem. In fact, this proof is by induction on $|G|$. (To construct $\sigma$ explicitly, we unpack the, recursively constructed, vertex order $\sigma'$ for $G'$, and append $v$.) Since the theorem is trivially true for an isolated vertex, we omit the details of the base case, by choosing $G$ to be a smallest counterexample, also called a *minimal counterexample*. When we color $G - v$ by the induction

minimal counterexample

[2]Suppose, to the contrary, that each vertex has degree at least 2. Starting from an arbitrary vertex, we walk arbitrarily, never following the same edge twice in immediate succession. Since $G$ is finite, we eventually return to a vertex previously visited. Thus, $G$ contains a cycle, a contradiction.

[3]Throughout the book, we write ":=" for assignment. So "Let $G' := G - v$" can be read as "Let $G'$ be defined as $G - v$". In contrast "$2 + 2 = 4$" should be read as "$2 + 2$ *equals* 4".

color $G - v$ by minimality

hypothesis, for short we write that we *color* $G - v$ *by minimality*.

The proof of Proposition 1.2 has only a single reducible configuration, a vertex of degree at most 1. But in general, our proofs may have many. When proving a coloring theorem for a graph class $\mathcal{G}$, we show that each configuration is reducible (independent of the others), and that every graph $G$ in $\mathcal{G}$ contains at least one of these reducible configurations, so $G$ cannot be a minimal counterexample to our theorem. To prove more results with this unavoidability/reducibility template, we use the following definitions, many of which we illustrate in Figure 1.2.

subgraph of G induced by $W$

$G[W]$

hereditary

**Definition 1.4.** For a graph $G$, we write $V(G)$ and $E(G)$ for its vertex and edge sets. A *subgraph* of $G$ is formed from $G$ by possibly deleting some of its edges and possibly deleting some of its vertices. For $W \subseteq V(G)$, the *subgraph of* $G$ *induced by* $W$ has vertex set $W$ and edge set $\{vw : v, w \in W \text{ and } vw \in E(G)\}$. We denote this subgraph by $G[W]$. A graph class $\mathcal{G}$ is *hereditary* if $G \in \mathcal{G}$ implies $H \in \mathcal{G}$ for every induced subgraph $H$ of $G$.

The *degree* of a vertex is the number of edges containing it. The *length* of a face is the number of edges on a walk along its boundary. We write $d(v)$ and $\ell(f)$ for the degree of vertex $v$ and the length of face $f$, and $\overline{d}(G)$ for the average degree of $G$. The maximum and minimum degrees of $G$ are $\Delta(G)$ and $\delta(G)$. We often write $\Delta$ for $\Delta(G)$.

adjacent, $N(v)$

neighborhood

$k/k^+/k^-$-vertex

k-neighbor

planar

plane graph

face

k-face, girth

Vertices $v$ and $w$ are *adjacent* if $vw \in E(G)$. In this case $v$ and $w$ are *neighbors* and they are the *endpoints* of edge $vw$. The *neighborhood* of $v$, denoted $N(v)$, is the set of all of its neighbors. A $k$-*vertex* is a vertex of degree $k$. A $k^+$-*vertex* or a $k^-$-*vertex* is one, respectively, of degree at least $k$ or at most $k$. A $k$-*neighbor* is an adjacent $k$-vertex, and $k^+$-*neighbors* and $k^-$-*neighbors* are defined similarly. A graph is *planar* if it can be embedded in the plane (with edges shown as contiguous curves between their endpoints) so that no edges intersect, except possibly at their endpoints. Such an embedding is a *plane graph*. A *face* of a plane graph $G$ is a maximal connected region of the plane that contains no vertex or edge of $G$. For plane graphs we define $k$-*face*, $k^+$-*face*, and $k^-$-*face* analogously. The *girth* of a graph is the length of its shortest cycle. If the graph is acyclic, then the girth is infinite. When we forbid loops and parallel edges, as we usually do, our girth is always at least 3.

mad(G)

degeneracy

d-degenerate

col(G)

witnesses

The *maximum average degree of* $G$, denoted $\mathrm{mad}(G)$, is the maximum, over all subgraphs $H$ of $G$, of $\overline{d}(H)$. Formally, $\mathrm{mad}(G) := \max_{H \subseteq G} 2\|H\|/|H|$. Note that this maximum is attained by an induced subgraph, since including more edges in a subgraph increases its average degree. The *degeneracy* of a graph $G$ is the maximum over all subgraphs $H$ of $\delta(H)$. If $G$ has degeneracy at most $d$, then $G$ is $d$-*degenerate*. Finally, the *coloring number* of $G$, denoted $\mathrm{col}(G)$, is 1 plus its degeneracy. A vertex order $\sigma$ *witnesses*, or shows, that $\mathrm{col}(G) \leqslant k+1$ if each vertex in $\sigma$ has at most $k$ neighbors earlier in $\sigma$. When $\mathrm{col}(G)$ is bounded by some constant, we often work with a vertex order witnessing this bound.

Figure 1.2 shows a plane graph with every face of length 3, except for the outer face, which has length 8. Here $N(v_1) = \{v_5, v_{11}\}$ and $N(v_{12}) = \{v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$. Finally, the degeneracy is 3. That it is at most 3 is witnessed by the order $\{v_1, v_2, \ldots, v_{12}\}$. That it is at least 3 is shown by the subgraph induced by $\{v_5, v_6, \ldots, v_{12}\}$, which has minimum degree 3. The graph $G$ on the left in Figure 1.3 has $\Delta(G) = 3$ and $\delta(G) = 2$ and girth 8. Each 3-vertex has

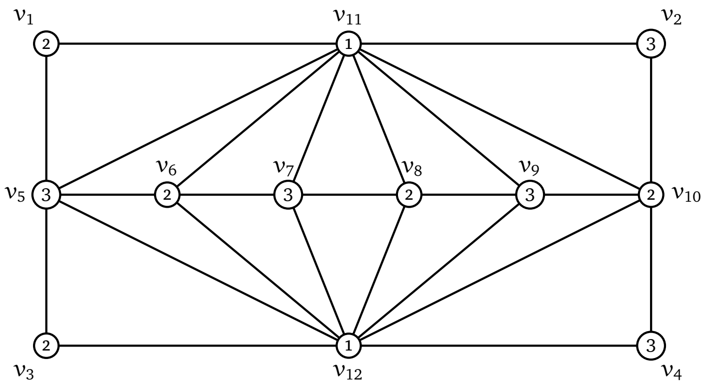



Figure 1.2: The order $v_1, \dots, v_{12}$ witnesses that G is 3-degenerate. So, coloring greedily using $v_{12}, \dots, v_1$ is guaranteed to use at most 4 colors. In fact, it uses only 3 colors.

three 2-neighbors and lies on three 8-faces. We can easily check that $\overline{d}(G) = 12/5$. In fact, $\text{mad}(G) = \overline{d}(G)$. Finally, $\chi(G) = 2$ and $\text{col}(G) = 3$, since G has degeneracy 2.

We study $\text{col}(G)$ because $\chi(G) \leqslant \text{col}(G)$, as we see in our next proposition (and $\text{col}(G)$ is also easy to compute efficiently, as we prove in Lemma 1.23). Specifically, $\text{col}(G)$ is the best bound on $\chi(G)$ that we can prove by coloring greedily, when we remember which vertices are already colored, but forget which colors we used where. A graph is 1-degenerate precisely when it is a forest. All planar graphs are 5-degenerate and triangle-free planar graphs are 3-degenerate, as we see in Proposition 1.7. For each integer $k$, the class of $k$-degenerate graphs is hereditary. Likewise, for each $\varepsilon > 0$, the class of graphs with $\text{mad}(G) < \varepsilon$ is hereditary. Hereditary classes are convenient to study, since they facilitate proofs by minimal counterexample, as illustrated by the proofs of Propositions 1.2 and 1.5.

**Proposition 1.5.** *Every graph* $G$ *satisfies* $\chi(G) \leqslant \text{col}(G) \leqslant \Delta + 1$.

*Proof.* The second inequality follows from the definition of $\text{col}(G)$; now we prove the first. Suppose the proposition is false, and let G be a minimal counterexample. Choose a vertex $v$ of minimum degree. By definition $d(v) \leqslant \text{col}(G) - 1$. By minimality, $\chi(G - v) \leqslant \text{col}(G - v)$ and $\text{col}(G - v) \leqslant \text{col}(G)$, so $G - v$ has a coloring $\varphi$ using at most $\text{col}(G)$ colors. Since $\varphi$ uses at most $\text{col}(G) - 1$ colors on neighbors of $v$, we can extend $\varphi$ by coloring $v$ greedily. So G is not a counterexample. Thus, the proposition is true. $\square$

Unfortunately, the difference $\text{col}(G) - \chi(G)$ can be arbitrarily large. For example, when G is the complete bipartite graph $K_{n,n}$ we get $\chi(G) = 2$, but $\text{col}(G) = n + 1$. This phenomenon is unsurprising, since we can compute $\text{col}(G)$ efficiently (as we do in Lemma 1.23), but it is NP-hard to even approximate $\chi(G)$ within a constant factor.

We often prove coloring results for planar graphs (possibly with girth at least some constant $g$), as in Corollary 1.7. In these proofs, the bound below on maximum average degree helps us

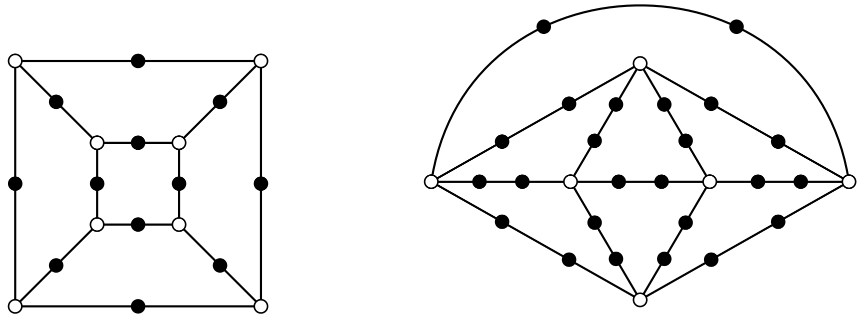


Figure 1.3: When $g$ is a multiple of 3, 4, or 5, the bound on $\|G\|$ in Lemma 1.6 can hold with equality.[4] We form $G$ from a platonic solid by dividing each edge the same number of times. Vertices of the platonic solid are white and vertices newly created are black.

show that each graph contains some reducible configuration; this is the "unavoidability" step that we discussed in Remark 1.3.

**Lemma 1.6.** *Let* $G$ *be a simple planar graph with girth at least* $g$*. Now* $\mathrm{mad}(G) < \frac{2g}{g-2}$*. If also* $|G| \geqslant g$*, then* $\|G\| \leqslant (|G|-2)\frac{g}{g-2}$*. In particular, every simple planar* $G$ *with at least three vertices has* $\|G\| \leqslant 3|G| - 6$ *and* $\mathrm{mad}(G) < 6$*. (Figure 1.3 shows two examples.)*

*Proof.* For each statement, we assume $G$ is connected; otherwise we add edges to form a connected graph $G'$, and proving the statement for $G'$ also proves it for $G$. The final statement follows from the first two, when $g = 3$. So we must prove these first two.

For the first statement, note that planar graphs with girth at least $g$ form a hereditary class. So it suffices to show that $\frac{2\|G\|}{|G|} < \frac{2g}{g-2}$. If $|G| < g$, then $G$ has no cycle; since $G$ is connected, $G$ is a tree. Thus, $\frac{2\|G\|}{|G|} = \frac{2(|G|-1)}{|G|} < 2 < \frac{2g}{g-2}$, as desired. If $|G| \geqslant g$, then the bound follows from the second statement, which we prove below.

Now we prove the second statement; assume that $|G| \geqslant g$. Recall Euler's formula: If $G$ is a connected planar graph with $|G|$ vertices, $\|G\|$ edges, and $|F|$ faces, then $|G| - \|G\| + |F| = 2$. By summing the lengths of walks along each face boundary, we count each edge twice, so $2\|G\| \geqslant g|F|$ (since each face boundary contains a cycle, which has length at least $g$). Now substituting for $|F|$ and solving for $\|G\|$ shows that $\|G\| \leqslant (|G|-2)\frac{g}{g-2}$. □

We also get the following easy corollary.

**Corollary 1.7.** *If mad*$(G) < k$*, then col*$(G) \leqslant \lceil k \rceil$*. In particular, col*$(G) \leqslant 6$ *(resp.* 4, 3*) when* $G$ *is planar with girth at least* 3 *(resp.* 4, 6*).*

*Proof.* By Pigeonhole, if $\mathrm{mad}(G) < k$, then $\delta(H) \leqslant \lceil k \rceil - 1$, for every subgraph $H$, so $\mathrm{col}(G) \leqslant \lceil k \rceil$. Taking $g \in \{3, 4, 6\}$ in Lemma 1.6 gives the stated values of $\mathrm{col}(G)$. □

[4] In fact, the bound on $\|G\|$ can hold with equality whenever $g \geqslant 3$. See Exercise 6.

### 1.1.2 Coloring Graphs on Surfaces: Heawood's Bound

The 4 Color Theorem asserts that every graph embeddable in the plane has a 4-coloring. This bound is sharp, since $K_4$ is planar and needs four colors. A variation on this problem fixes a surface $\mathcal{S}$, and asks for the maximum chromatic number, $\chi(\mathcal{S})$, of graphs that can be embedded in $\mathcal{S}$. For the plane $\mathcal{P}$, showing that $\chi(\mathcal{P}) \geqslant 4$ is trivial, while proving that $\chi(\mathcal{P}) \leqslant 4$ is a monumental task. For almost every other surface $\mathcal{S}$, the opposite is true: determining a sharp upper bound on $\chi(\mathcal{S})$ is easy, but giving a matching lower bound is harder. The goal of this section is to prove Theorem 1.11, which gives this upper bound on $\chi(\mathcal{S})$. To do so, we need a few definitions, which we will only use in the present section.

**Definition 1.8.** To add a *handle* to a surface $\mathcal{S}$, we cut from $\mathcal{S}$ two unit disks, and add a cylinder, identifying the boundary of each of its ends with the boundary of one of the disks. To add a *crosscap* to $\mathcal{S}$, we cut from $\mathcal{S}$ a single disk and add a disk with each pair of antipodal points on its boundary identified. Adding $h$ handles to the sphere gives $S_h$, and adding $k$ crosscaps gives $N_k$. (Theorem 1.9 implies that adding a combination of handles and crosscaps is equivalent to either adding only handles or only crosscaps.) A 2-*cell embedding* of a graph $G$ in a surface $\mathcal{S}$ is a drawing of $G$ in $\mathcal{S}$, with no edges crossing, so that each face can be continuously contracted to a single point. If $G$ has a 2-cell embedding in a surface $\mathcal{S}$, then $G$ is *embeddable* in $\mathcal{S}$.

handle

crosscap

$S_h$, $N_k$

2-cell embedding

embeddable

To prove Theorem 1.11, we need a well-known classification of surfaces, given in Theorem 1.9. A proof can be found in Thomassen [375] or Mohar and Thomassen [304, Section 3.1], but we omit it, since it is mainly topological, and thus strays from our study of coloring. We also use a generalization to arbitrary surfaces of the edge bound for planar graphs in Lemma 1.6. We leave the proof to Exercise 8.

**Theorem 1.9 (Classification of Surfaces).** *Every surface can be formed from the sphere by either repeatedly adding handles or repeatedly adding crosscaps. More precisely, every surface $\mathcal{S}$ is homeomorphic to exactly one $S_h$ (with $h \geqslant 0$) or one $N_k$ (with $k \geqslant 1$).*

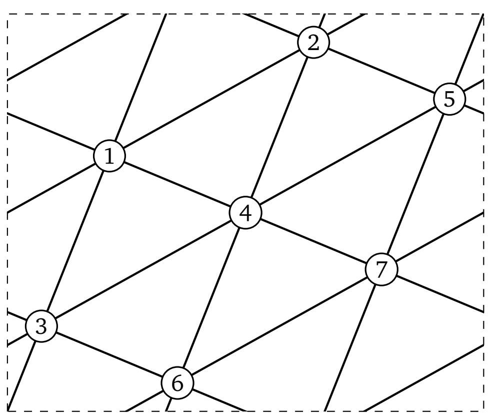



Figure 1.4: A 2-cell embedding of $K_7$ on the torus, $S_1$.

To unify results for the orientable surfaces $S_h$ and the nonorientable surfaces $N_k$, we define

Euler genus

the *Euler genus*, $\mathrm{eg}(\mathcal{S})$, of a surface $\mathcal{S}$. The Euler genus of $S_h$ is $2h$, twice the number of handles, and the Euler genus of $N_k$ is simply $k$, the number of crosscaps.

**Lemma 1.10.** *If* $G$ *has a 2-cell embedding in surface* $\mathcal{S}$*, then* $\|G\| \leqslant 3|G| + 3\mathrm{eg}(\mathcal{S}) - 6$.

A good intuition for Lemma 1.10 is that each handle allows 6 extra edges. Let $H$ be a maximal plane subgraph of $G$ and let $G'$ be a plane triangulation with $H$ as a subgraph and $|G'| = |H|$. Each time that we add a handle from the interior of one triangular face of $G'$ to another, the handle can accommodate 6 more edges. So $\|G\| \leqslant \|G'\| + 6h \leqslant 3|G'| - 6 + 6h = 3|G| + 3\mathrm{eg}(\mathcal{S}) - 6$.

**Theorem 1.11.** *For every surface* $\mathcal{S}$ *with* $\mathrm{eg}(\mathcal{S}) > 0$*, every graph* $G$ *embeddable in* $\mathcal{S}$ *satisfies*

$$\mathrm{col}(G) \leqslant \left\lfloor \frac{7 + \sqrt{1 + 24\mathrm{eg}(\mathcal{S})}}{2} \right\rfloor. \tag{1.1}$$

*Proof.* Fix a surface $\mathcal{S}$. Choose a positive integer $k$ and a graph $G$ that is embeddable in $\mathcal{S}$ such that $\mathrm{col}(G) = k$; among such graphs, choose $G$ to minimize $\|G\|$. We will show that $k$ is at most the right side of inequality (1.1). For every proper subgraph $H$, the minimality of $G$ implies that $\mathrm{col}(H) < k$. So $\mathrm{col}(G) = 1 + \delta(G) \leqslant 1 + 2\|G\|/|G| \leqslant 1 + (2(3|G| + 3\mathrm{eg}(\mathcal{S}) - 6))/|G| = 7 + (6\mathrm{eg}(\mathcal{S}) - 12)/|G|$; the second inequality uses Lemma 1.10. Clearly, $\mathrm{col}(G) \leqslant |G|$, so $\mathrm{col}(G) \leqslant \min\{|G|, 7 + (6\mathrm{eg}(\mathcal{S}) - 12)/|G|\}$.

When $\mathrm{eg}(\mathcal{S}) > 2$, the second argument in the minimum above decreases as $|G|$ increases; so this minimum is maximized when its arguments are equal. Solving the resulting quadratic in $|G|$ gives (1.1), since $\mathrm{col}(G)$ is an integer. When $\mathrm{eg}(\mathcal{S}) = 2$, the minimum is 7 whenever $|G| \geqslant 7$; this agrees with Theorem 1.11. When $\mathrm{eg}(\mathcal{S}) = 1$, the second argument is always less than 7, so the floor of the minimum never exceeds 6, which again agrees with (1.1). □

Ringel and Youngs [343] showed that inequality (1.1) is sharp, for every surface $\mathcal{S}$ except the plane and $N_2$, which is the Klein bottle. To do so, they embedded in $\mathcal{S}$ a complete graph $K_t$ with $t$ equal to the right side of (1.1). Figure 1.4 shows $K_7$ embedded in the torus, $S_1$.

### 1.1.3 The Discharging Method and Efficient Coloring Algorithms

Most proofs in this book are more subtle than simply computing $\mathrm{col}(G)$. One reason is that by choosing colors for more than one vertex at a time, we can often prove stronger bounds on $\chi(G)$. Another reason is that for some types of coloring, such as injective coloring, low degree vertices are not always reducible.[5]

[5]Still another reason is that otherwise the book would be far shorter.

**Definition 1.12.** An *injective coloring*, $\varphi$, of a graph $G$ assigns colors to its vertices so that $v$ and $w$ get distinct colors whenever they have a common neighbor (but $\varphi$ need not be proper)[6]. The *injective chromatic number*, $\chi^i(G)$, is the smallest $k$ such that $G$ has an injective coloring with at most $k$ colors. The *neighboring graph*, $G^{(2)}$, has $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{vw \text{ such that } v \neq w \text{ and } v \text{ and } w \text{ have a common neighbor in } G\}$. So $\chi^i(G) = \chi(G^{(2)})$.

injective coloring

injective chromatic number

neighboring graph

Our next proof illustrates an important idea: multiple reducible configurations.

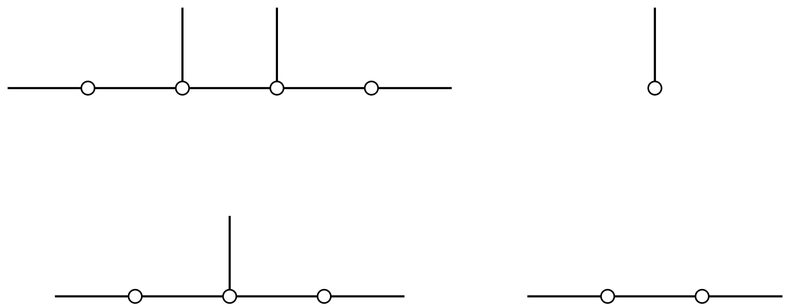


Figure 1.5: Theorem 1.13: configurations (i)–(iv) in clockwise order from top right.

**Theorem 1.13.** *If* $\mathrm{mad}(G) < 36/13$ *and* $\Delta \leqslant 3$*, then* $\mathrm{col}(G^{(2)}) \leqslant 5$*; thus,* $\chi^i(G) \leqslant 5$.

*Proof.* The second statement follows from the first, since $\mathrm{col}(G^{(2)}) \geqslant \chi(G^{(2)}) = \chi^i(G)$. Note that graphs satisfying the hypotheses form a hereditary class. So assume the first statement is false and let $G$ be a minimal counterexample. We show that $G$ contains none of the following four configurations, shown in Figure 1.5, since each is reducible: (i) a $1^-$-vertex, (ii) adjacent 2-vertices, (iii) a 3-vertex adjacent to at least two 2-vertices, and (iv) adjacent 3-vertices, each adjacent to a 2-vertex. (The label of each configuration matches its number of vertices.)

In each case, we denote the configuration by $H$. Let $\sigma$ be a vertex order for $G - H$ showing that $\mathrm{col}((G-H)^{(2)}) \leqslant 5$; such an order $\sigma$ exists since $G$ is a minimal counterexample, and $G - H$ is smaller than $G$. In each of cases (i)–(iii), we simply append to $\sigma$ the vertices of $H$, in any order. This approach succeeds because every pair of vertices adjacent in $G^{(2)}$ that is not adjacent in $(G-H)^{(2)}$ contains a vertex in $H$. In Case (iv), let $v_1$ and $v_2$ denote adjacent 3-vertices, and let $w_1$ and $w_2$ denote their 2-neighbors[7]. Now we append to $\sigma$ the order $v_1$, $v_2$, $w_1$, $w_2$; we can check that each vertex has at most 4 neighbors in $G^{(2)}$ earlier in $\sigma$. This finishes the proof of reducibility.

To prove unavoidability, we use a counting argument. We show that if $\Delta(G) \leqslant 3$ and $G$ has none of configurations (i)–(iv) then $\overline{d}(G) \geqslant 36/13$, which contradicts the hypothesis. A *charge* is simply a number. We assign to each vertex a charge, equal to its degree. We redistribute charge using the following two *discharging rules*.

charge

discharging rules

[6]Injective coloring is less central to graph coloring than many topics we will study. However, it provides a setting for some easy examples: Theorems 1.13 and 1.41.

[7]We allow the possibility that the 3-vertices have the same 2-neighbor.

(R1) Each 2-vertex gets $3/13$ from each 3-neighbor.

(R2) Each 2-vertex gets $1/13$ from each 3-vertex at distance 2, via each length 2 path.

Consider the final charges after applying (R1) and (R2) everywhere possible, simultaneously. Note that $\delta(G) \geqslant 2$, by the absence of (i). Since (iii) and (iv) are forbidden, each 3-vertex loses at most $\max\{3/13, 3(1/13)\} = 3/13$, so ends with at least $36/13$. Since (ii), (iii), and (iv) are forbidden, each 2-vertex $v$ gains exactly $2(3/13) + 4(1/13) = 10/13$, from its two 3-neighbors and (up to) four 3-vertices at distance two. So $v$ ends with $36/13$. Each vertex has final charge at least $36/13$, so the average is at least $36/13$. Since the total charge was preserved by discharging, the average initial charge, which is the average degree, is also at least $36/13$. This contradicts the hypothesis $\text{mad}(G) < 36/13$, which finishes the proof. □

**Example 1.14.** Now we show that Theorem 1.13 is sharp. The Heawood graph, $H$, is 3-regular and $|H| = 14$ (see Figure 1.6). Further, $H$ is vertex-transitive and bipartite, and has girth 6. Form $H_0$ from $H$ by deleting a single vertex. Now $\Delta(H_0) = 3$, $\overline{d}(H_0) = 36/13$, and $\chi^i(H_0) = 6$, since the six vertices in the smaller part need distinct colors. In fact, it is easy to check that $\text{mad}(H_0) = 36/13$; see Exercise 2. Thus Theorem 1.13 is sharp. ♢

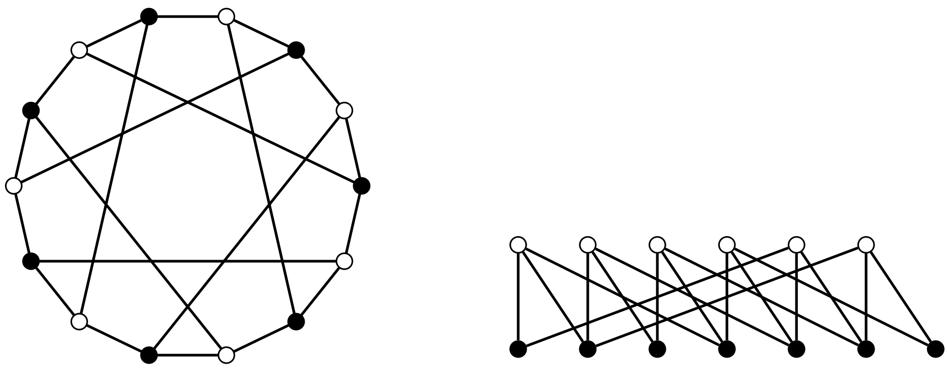


Figure 1.6: The Heawood graph $H$, and a bipartite drawing of $H - v$.

**Remark 1.15.** The proof of Theorem 1.13 illustrates a new key idea. For the first time we have multiple reducible configurations, here (i)–(iv), rather than just a single one. A *configuration* is an induced subgraph $H$, along with specified values $d_v$, such that $d_G(v) \leqslant d_v$, for each $v \in V(H)$. Our counting argument, used to prove unavoidability, is an example of the *discharging method*. Whenever we have a hypothesis $\text{mad}(G) < \varepsilon$, we assume that $G$ contains no reducible configuration, and we follow the same general 3-step framework. (1) Assign each vertex an initial charge equal to its degree. (2) Redistribute charge, maintaining its sum, by a set of discharging rules. (3) Prove that each vertex finishes with charge at least $\varepsilon$, which contradicts the hypothesis $\text{mad}(G) < \varepsilon$. This final step uses the fact that each reducible configuration is forbidden. (If a vertex $v$ lacks the high degree neighbors needed to give it sufficient charge, then

configuration

discharging method

we try to prove that $v$ appears in some reducible configuration, which yields a contradiction.) Once we settle on our reducible configurations and initial charges, picking discharging rules is often easy, as we discuss below in Example 1.17.

When $G$ is a plane graph, we can also assign charge to its faces, which allows more flexibility in our choice of initial charges. Lemma 1.16 gives three options for initial charge functions that yield negative sums. When we use one of these initial charge functions, we change step (3) in the previous paragraph, to require that each vertex finishes with charge at least 0. Otherwise, the blueprint is the same. When a vertex or face $x$ ends with the desired charge, we say that $x$ *ends happy*. So to reach a contradiction, we must show that every vertex and face ends happy.

ends happy

**Lemma 1.16.** *Let $G$ be a plane graph. Assign charges to all vertices and faces using one of the following initial charge functions (i) ch$(v) := d(v)-6$ and ch$(f) := 2\ell(f)-6$, (ii) ch$(v) := d(v)-4$ and ch$(f) := \ell(f)-4$, or (iii) ch$(v) := 2d(v)-6$ and ch$(f) := \ell(f)-6$. In each case, the sum of the initial charges is negative. For easy reference, we refer to the charges in (i), (ii), and (iii) as* vertex charging, balanced charging, *and* face charging.[8]

vertex charging
balanced charging
face charging

*Proof.* The proofs for each of parts (i), (ii), and (iii) are nearly identical. In each case, the crucial step is rewriting the sum of the initial charges using Euler's formula. We provide the details for (i) and leave those for (ii) and (iii) to Exercise 4. Let $V$ and $F$ denote the sets of $G$'s vertices and faces.

$$\begin{aligned}
\sum_{v\in V} \mathrm{ch}(v) + \sum_{f\in F} \mathrm{ch}(f) &= \sum_{v\in V}(d(v)-6) + \sum_{f\in F}(2\ell(f)-6) \\
&= (2\|G\| - 6|V|) + (4\|G\| - 6|F|) \\
&= -6(|F| - \|G\| + |V|) \\
&= -12.
\end{aligned}$$
□

Assigning initial charges is generally easy. (If $G$ is planar, then we use face charging, balanced charging, or vertex charging. Otherwise, we typically use ch$(v) := d(v)$.) But how do we choose discharging rules? There is no easy answer, but an example will be useful.

**Example 1.17.** How did we choose the discharging rules we used to prove Theorem 1.13? Since 2-vertices start with less charge than 3-vertices, the former should take charge from the latter. A natural choice is to have 2-vertices take charge from their 3-neighbors. If each 2-vertex takes $1/3$ from each 3-neighbor, then the reducibility of configurations (i), (ii), and (iii) implies the result when $\mathrm{mad}(G) < 8/3$.

To take advantage of (iv), we need 2-vertices to also take charge from 3-vertices at distance 2. So suppose we try the rules (R1) Each 2-vertex takes $a$ from each 3-neighbor and (R2) Each 2-vertex takes $b$ from each 3-vertex at distance 2. Now only the values $a$ and $b$ remain undetermined. As in the proof of Theorem 1.13, each 3-vertex loses at most $\max\{a, 3b\}$. Each

[8]If $G$ is a triangulation, then each face has charge 0 under (i). Similarly, if $G$ is 3-regular, then each vertex has charge 0 under (iii). This is the source of the names vertex charging and face charging.

2-vertex gains $a$ from each of two neighbors, and $b$ from each of four vertices at distance 2, for a total of $2+2a+4b$. (As we write out the details, we notice that a 2-vertex $v$ could have two paths of length 2 that lead to the same 3-vertex $w$. To ensure that $v$ receives enough, we modify (R2) so that $w$ sends $b$ along each of these paths.) Thus, we should maximize the minimum of the quantities $3-a$, $3-3b$, and $2+2a+4b$. We set these three quantities equal, and solve for $a$ and $b$. This yields the optimum $36/13$, when $a = 3/13$ and $b = 1/13$. $\diamondsuit$

To generalize the previous example, suppose we fix $\varepsilon > 0$ and are proving an upper bound on some coloring parameter for every graph $G$ with $\text{mad}(G) < \varepsilon$. Say we fix a set of reducible configurations and a set of discharging rules, but with the amount sent in each rule undetermined. We can often optimize our threshold on $\text{mad}(G)$ by solving a linear program to determine the amount to send in each rule. (Typically, this LP is simple enough that we can solve it by hand.) So our focus when presenting such proofs is primarily on how to prove reducibility. However, unavoidability plays a crucial supporting role and, more often than not, relies on the discharging method. We discuss this technique further in the Notes.

$G^2$ **Definition 1.18.** The *square*, $G^2$, of a graph $G$ is formed from $G$ by adding an edge $vw$ whenever
$N^2(v)$ $v$ and $w$ are distance 2 in $G$. Let $N^2(v)$ denote the neighbors of $v$ in $G^2$.

Clearly, $\chi(G^2) \geqslant \Delta(G)+1$, since in $G^2$ each vertex $v$ and its neighbors in $G$ form a clique. We can easily check that $\Delta(G^2) \leqslant \Delta(G)^2$, so Proposition 1.5 implies that $\chi(G^2) \leqslant \text{col}(G^2) \leqslant \Delta(G^2)+1 \leqslant \Delta(G)^2+1$. This bound can be sharp, such as when $G$ is the 5-cycle or the Petersen graph. But often we can improve the bound significantly. Wegner conjectured that if $G$ is planar and $\Delta \geqslant 8$, then $\chi(G^2) \leqslant 1 + \left\lfloor \frac{3}{2}\Delta \right\rfloor$, and he constructed graphs to show this is best possible; see Exercise 2.5. As a warmup, we prove an upper bound on $\text{col}(G^2)$ for all graphs with $\text{col}(G) \leqslant k+1$. For $k \geqslant 5$, this includes planar graphs.

**Lemma 1.19.** *If* $\text{col}(G) \leqslant k+1$ *and* $\Delta(G) \geqslant k-1$, *then* $\text{col}(G^2) \leqslant (2k-1)\Delta(G) - k^2 + k + 1$.

*Proof.* The class of graphs with $\Delta \geqslant k-1$ is *not* hereditary. However, if $\Delta \leqslant k-2$, then Proposition 1.5 gives $\text{col}(G^2) \leqslant \Delta(G^2)+1 \leqslant (k-2)^2+1 = k^2-4k+5$. Thus, for all graphs with $\text{col}(G) \leqslant k+1$ (which *is* a hereditary class), we prove the more general bound $\text{col}(G^2) \leqslant \max\{k^2-4k+5, (2k-1)\Delta(G)-k^2+k+1\}$. This bound implies the stated result, since when $\Delta \geqslant k-1$ (and $\Delta \neq 0$), this maximum is attained by the second argument.

It suffices to prove the bound when $\Delta \geqslant k-1$, since otherwise it holds trivially, as shown above. Let $\sigma$ be a vertex order witnessing that $\text{col}(G) \leqslant k+1$. We show that $\sigma$ also witnesses that $\text{col}(G^2) \leqslant (2k-1)\Delta(G) - k^2 + k + 1$. Consider a vertex $v$ in $\sigma$. Each neighbor of $v$ in $G^2$ either (i) is a neighbor of $v$ in $G$ or (ii) shares a common neighbor $w$ with $v$ in $G$. Each neighbor $w$ of $v$ that precedes $v$ in $\sigma$ serves as a common neighbor for $v$ and at most $\Delta(G)-1$ other vertices. Thus, $w$ is responsible for at most $\Delta$ neighbors of $v$ in $G^2$ (including $w$) that precede $v$ in $\sigma$. Each neighbor $w$ that follows $v$ in $\sigma$ may also serve as a common neighbor for $v$ and up to $\Delta(G)-1$ other vertices, but at most $k-1$ of these precede $v$ in $\sigma$. So, when $\Delta(G) \geqslant k-1$, the number of neighbors of $v$ in $G^2$ that precede $v$ in $\sigma$ is at most $k\Delta(G) + (\Delta(G)-k)(k-1) = (2k-1)\Delta(G) - k^2 + k$. $\square$

By Corollary 1.7, every planar graph G has $\mathrm{col}(G) \leqslant 6$. So, if $\Delta(G) \geqslant 4$, then Lemma 1.19 implies $\mathrm{col}(G^2) \leqslant 9\Delta(G) - 19$. To strengthen this bound when $\Delta$ is large, we first prove a structural lemma, which encapsulates the unavoidability step. The reducibility step comes within the proof of Theorem 1.22.

**Lemma 1.20.** *Every planar graph contains a $5^-$-vertex with at most two $12^+$-neighbors.*

The idea of the proof is simple. We assume G is planar and each $5^-$-vertex $v$ has three "big" neighbors ($v$ exists, by Lemma 1.6). We use vertex charging, and let each $5^-$-vertex take its needed charge equally from each of its big neighbors. Finally, we set the degree threshold for big vertices to be as small as possible, so that each big vertex ends happy.

*Proof.* Suppose the lemma is false and let G be a counterexample. Adding edges cannot give any vertex fewer $12^+$-neighbors, so assume that G is a triangulation. We use vertex charging: each vertex $v$ has initial charge $d(v)-6$. By Lemma 1.16(i), the sum of these charges is negative. (Since G is a triangulation, each face gets charge 0, so we can safely ignore the face charges.) We use the following discharging rule; see Figure 1.7.

(R) Each $5^-$-vertex $v$ takes $\frac{6-d(v)}{3}$ from each $12^+$-neighbor.

Because G is a counterexample, each $5^-$-vertex $v$ gets charge from three or more neighbors, so $v$ ends happy, as defined in Remark 1.15. Each vertex $v$ with $6 \leqslant d(v) \leqslant 11$ loses no charge, so again $v$ ends happy.

Consider a $12^+$-vertex $v$. Since G is a triangulation, the neighbors of $v$ induce a cycle C, possibly with chords. Let H denote the subgraph of C induced by $5^-$-vertices (excluding any chords of C that may exist in G). Since each $5^-$-vertex $w$ has at least three $12^+$-neighbors, $d_H(w) \leqslant d_G(w) - 3 \leqslant 2$ for each vertex $w$ in H. If H is a cycle, then each of its vertices is a 5-vertex, so $v$ is happy, since $d(v) - 6 - d(v)/3 > 0$ when $d(v) \geqslant 12$. Otherwise H is a proper subgraph of a cycle, so H is a disjoint union of paths, which we handle below.

For each path P of H containing j vertices, we compute the average sent from $v$ to vertices of P and the next vertex of C, which receives no charge. When $j = 1$, this average is at most $(1 + 0)/2 = 1/2$. When $j = 2$, it is at most $(2(2/3) + 0)/3 < 1/2$. And when $j \geqslant 3$, the average is smaller, since each internal vertex of P is a 5-vertex, which takes only $1/3$ from $v$. So $v$ is happy, since $d(v) - 6 - d(v)/2 \geqslant 0$ when $d(v) \geqslant 12$. □

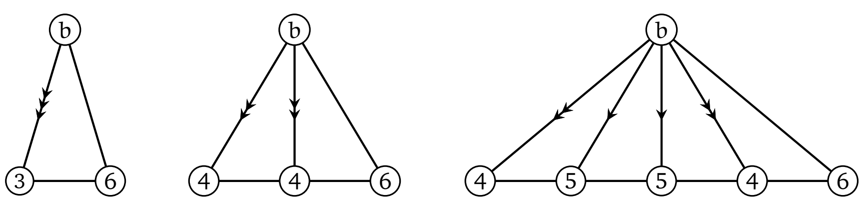



Figure 1.7: Lemma 1.20: Three examples of big vertices and paths in H. The arrow ↠↠ denotes that the tail gives the head 3/3. Likewise, ↠ and → denote 2/3 and 1/3.

Earlier we discussed how to optimize a hypothesis bounding $\text{mad}(G)$, given a set of reducible configurations. Lemma 1.20 presents a complementary problem. What is the smallest value of $k$ for which the lemma is true, with "$k^+$-neighbors" in place of "$12^+$-neighbors"? Note that we only used $k \geqslant 12$ in the final line of the proof. In fact, the value $k = 12$ comes directly from solving the inequality $k - 6 - k/2 \geqslant 0$. The lemma also remains true when $k = 11$, although the discharging argument for that proof is more complicated, as we discuss briefly in the Notes. The version with $k = 11$ is best possible, as shown by the following example.

**Example 1.21.** Form $H$ from an icosahedron by adding a new vertex, $v$, inside each face, $f$, and making $v$ adjacent to all vertices on the boundary of $f$ (see Figure 1.8). The only $5^-$-vertices in $H$ are these new 3-vertices. Each neighbor of a 3-vertex is now a 10-vertex. So each 3-vertex has three 10-neighbors. Thus, Lemma 1.20 becomes false if we replace 12 by 10. The same construction also works if, rather than starting from an icosahedron, we begin with any plane triangulation with minimum degree 5. ♢

When $G$ is planar, we can use Lemma 1.20 to strengthen our bound on $\text{col}(G^2)$.

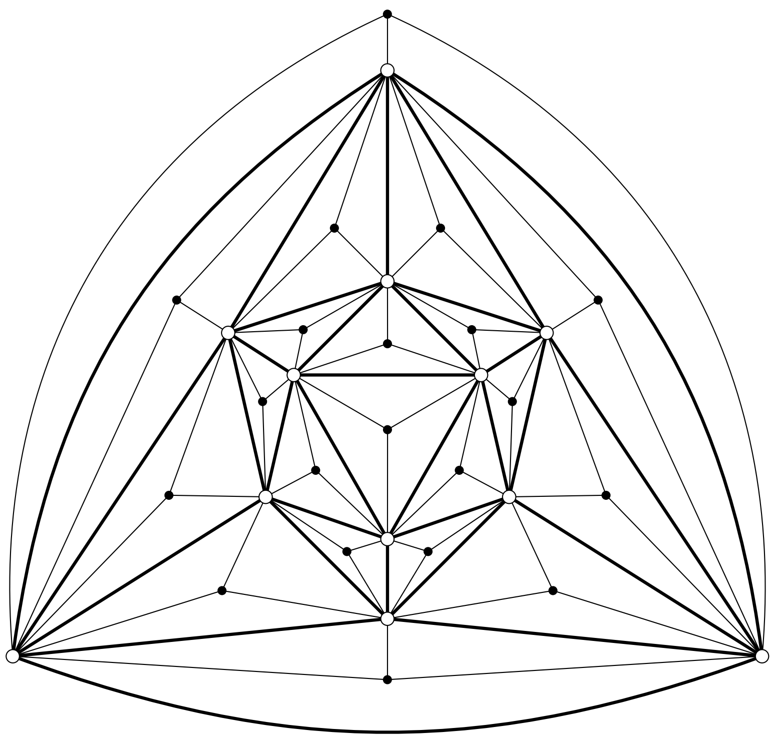


Figure 1.8: The icosahedron with bold edges and white vertices, with a new vertex (black) added inside each face, adjacent to all vertices of that face, as in Example 1.21. This witnesses that Lemma 1.20 is nearly sharp.

**Theorem 1.22.** *Every planar graph* $G$ *satisfies col*$(G^2) \leqslant 2\Delta(G) + 72$.

Let $v$ be a vertex guaranteed by Lemma 1.20. We might naturally try to bound $\text{col}(G^2)$ by getting a vertex order $\sigma$ witnessing the bound for $\text{col}((G - v)^2)$, and appending $v$ to $\sigma$. But this approach fails, since vertices with $v$ as their common neighbor in $G$ may be non-adjacent in $(G - v)^2$. To sidestep this obstacle, rather than deleting $v$, we form $H$ by *contracting an edge* $e$ incident to $v$. (Here we use that the class of planar graphs is closed under edge contraction.) Note that all vertex pairs that are adjacent in $G^2$, but do not include $v$, are also adjacent in $H^2$. However, we must choose edge $e$ carefully, to ensure that $\Delta(H) \leqslant \Delta(G)$.

*Proof.* Assume the theorem is false, and let $G$ be a minimal counterexample. When $\Delta(G) \leqslant 3$, Proposition 1.5 gives $\text{col}(G^2) \leqslant \Delta(G^2) + 1 \leqslant 10$. Since $G$ is planar, $\text{col}(G) \leqslant 6$. So when $4 \leqslant \Delta(G) \leqslant 13$, Lemma 1.19 implies that $\text{col}(G^2) \leqslant 9\Delta(G) - 19 \leqslant 2\Delta(G) + 72$. Thus, we assume that $\Delta(G) \geqslant 14$.

Let $v$ be a vertex guaranteed by Lemma 1.20, and note that $N^2(v) \leqslant 2\Delta(G) + 3(11)$. If $d(v) \geqslant 3$, then form $H$ from $G$ by contracting edge $vw$, where $w$ is some $11^-$-neighbor of $v$. Otherwise, form $H$ by contracting any edge $vw$ incident to $v$. (In each case $v$ is "merged into" $w$, so we still call the new vertex $w$.) Now $\Delta(H) \leqslant \max\{\Delta(G), 5 + 11 - 2\} = \Delta(G)$. So by minimality there exists a vertex order $\sigma$ witnessing the bound for $H^2$. To get a vertex order for $G^2$, we simply append $v$ to $\sigma$. □

This book is not primarily about algorithms. But since our proofs are typically constructive, they do immediately yield coloring algorithms. We explore this idea in the next lemma and the discussion that follows.

**Lemma 1.23.** *If* $d(v) = \delta(G)$, *then col*$(G) = \max\{1 + d(v), \textit{col}(G - v)\}$. *Further, we can color* $G$ *with col*$(G)$ *colors in time* $O(\Delta \cdot |G|)$.

*Proof.* In Definition 1.4, we let $\text{col}(G) := 1 + \max_{H \subseteq G} \delta(H)$, but it is easy to check that we can restrict this maximum to be taken over all *induced* subgraphs $H$. Choose $v$ such that $d(v) = \delta(G)$ and let $t := \max\{1+d(v), \text{col}(G-v)\}$. By definition $\text{col}(G) \geqslant \text{col}(H)$ for every induced subgraph $H$ of $G$. Also, $\text{col}(G) \geqslant 1 + \delta(G)$, so $\text{col}(G) \geqslant t$. Conversely, $\text{col}(G) \leqslant t$, since if $H \subseteq G$ and $v \in V(H)$, then $\delta(H) \leqslant d_H(v) \leqslant d_G(v)$; and if $v \notin V(H)$, then $1 + \delta(H) \leqslant \text{col}(G - v)$. This proves the first statement.

Given a vertex order $\sigma$ showing that $\text{col}(G) \leqslant k$, we can $k$-color $G$ by coloring greedily, using the reverse of $\sigma$. So coloring $G$ efficiently reduces to efficiently constructing $\sigma$.

To convert our structural statement into an algorithm to construct $\sigma$, we unpack the recursion. So $\text{col}(G) = 1 + \max_{1 \leqslant i \leqslant |G|} \delta(H_i)$, where $H_1 := G$ and each other $H_i$ is formed from $H_{i-1}$ by deleting a vertex of minimum degree. This statement does imply a polynomial time algorithm, but to improve the running time we need a more general statement. In fact, $\text{col}(G) = 1 + \max_{1 \leqslant i \leqslant |G|} \delta(H_i)$, whenever each $H_i$ is formed from $H_{i-1}$ by deleting any vertex with degree at most $\max_{j=1}^{i-1}\delta(H_j)$. The proof is essentially the same; we just note that deleting such a vertex will never increase the maximum over the whole sequence.

Now we analyze the running time. We assume a data structure that stores, for each vertex $v$, both the degree of $v$ and a doubly linked list of its neighbors. We begin with a preprocessing phase, in which we determine the minimum degree $\delta$ of $G$, and also form a list of all vertices of degree $\delta$; call this the *candidate list*. Now we begin a loop in which we delete a vertex $v$ in the candidate list, and update the degrees and neighbor lists of each of its neighbors. If any neighbor $w$ of $v$ now has degree at most $\delta$, then we add $w$ to the candidate list. This finishes an iteration of the loop. Since $d(v) \leqslant \Delta$, each iteration of the loop takes time $O(\Delta)$. We repeat the loop until either (i) all vertices are deleted or (ii) the candidate list is empty.

When (i) holds, we are done. So instead assume (ii) holds. Now we repeat the preprocessing phase, increasing $\delta$ to the current minimum degree. Since $\delta$ can increase at most $\Delta$ times, we run the preprocessing phase at most $\Delta$ times. Since each preprocessing phase runs in time $O(|G|)$, together they run in time $O(\Delta|G|)$. We iterate the loop $O(|G|)$ times, and each iteration runs in time $O(\Delta)$, for a total running time of $O(\Delta|G|)$. □

Most coloring theorems proved by discharging translate into coloring algorithms using the approach above. First, we repeatedly delete reducible configurations, until we reach the empty graph. Next, we reassemble the graph, adding back one configuration at a time. At each point, we can extend our coloring to the new configuration $H$, precisely because $H$ is reducible. The proof of unavoidability guarantees that the candidate list will never be empty. As a result, the preprocessing phase runs only once, at the start. This approach works easily when each reducible configuration $H$ has bounded size, and each vertex of $H$ has bounded degree. In Section 1.3, we introduce reducible configurations of unbounded size, which play a key role in many of our proofs. Translating such proofs into algorithms is still typically simple, although the running times are often longer.

As a final remark on algorithms, we note that it is straightforward to compute $\mathrm{mad}(G)$ in polynomial time. One method is to phrase the question as a network flow problem. This problem can be solved efficiently by the Max-flow/Min-cut algorithm. In Exercise 7 we consider an alternate formulation.

### 1.1.4 Brooks' Theorem

By Proposition 1.5, every graph $G$ satisfies $\chi(G) \leqslant \Delta + 1$. Brooks refined this result by showing that if $G$ is connected, then equality holds only when $G$ is an odd cycle or a complete graph $K_{\Delta+1}$. We can prove Brooks' Theorem in many ways. Here we mainly use greedy coloring. But at one crucial point we specifically color one vertex with a color used on a certain other vertex. And at two points we are able to finish because the final vertex to be colored has two neighbors using the same color. These extra wrinkles are inevitable, since $\mathrm{col}(G) = \Delta + 1$ for every $\Delta$-regular graph $G$.

**Theorem 1.24 (Brooks' Theorem).** *Let* $G$ *be a connected graph. If* $G$ *is neither an odd cycle nor a complete graph, then* $\chi(G) \leqslant \Delta$*. Otherwise,* $\chi(G) = \Delta + 1$.

*Proof.* The theorem holds if $G$ is a path, a cycle (either odd or even), or a complete graph, so assume it is none of these. Thus, $\Delta \geqslant 3$. We often implicitly use the following observation. Coloring greedily, with some order $\sigma$, uses no color larger than $\Delta$ on each vertex that has a neighbor later in $\sigma$. Similarly, if a vertex has two neighbors with a common color, then coloring it greedily (even if all its neighbors are colored) uses no color larger than $\Delta$.

**Case 1: Some vertex $v$ has $d(v) < \Delta$.** Let $\sigma$ be an order of the vertices by non-increasing distance from $v$. Now coloring $G$ greedily by $\sigma$ uses at most $\Delta$ colors.

*Below we assume that* $G$ *is regular. Since* $G$ *is* $\Delta$*-regular, but not* $K_{\Delta+1}$*,* $G$ *contains a vertex* $v_2$ *with neighbors* $v_1$ *and* $v_3$ *that are non-adjacent. Let* $v_1 \cdots v_r$ *be a maximal path starting with* $v_1v_2v_3$*. That is,* $v_r$ *has all its neighbors among* $v_1, \ldots, v_{r-1}$.

**Case 2: $r = |G|$.** Since $d(v_2) = \Delta$, there exists $j > 3$ such that $v_2v_j \in E(G)$. Color greedily in the order $v_1, v_3, v_4, \ldots, v_{j-1}, v_{|G|}, v_{|G|-1}, \ldots, v_j, v_2$.

**Case 3: $r < |G|$.** Let $v_i$ denote the neighbor of $v_r$ that comes earliest among $v_1, \ldots, v_r$. Note that $v_i, \ldots, v_r$ induce a cycle, with one or more chords. Let $H := G[\{v_i, \ldots, v_r\}]$; see Figure 1.9. Recall that $V(G) \setminus V(H) \neq \emptyset$, since $r < |G|$. Since $G$ is connected, some vertex in $H$ has a neighbor outside $H$. But all neighbors of $v_r$ are in $H$. Fix $v_j$ such that $v_{j-1}$ has a neighbor $w$ outside $H$ but $v_j$ has all neighbors in $H$. Let $\sigma$ be a vertex order of $V(G) \setminus V(H)$ by non-increasing distance in $G$ from $H$. First, color $G - V(H)$ greedily using $\sigma$. Now color $v_j$

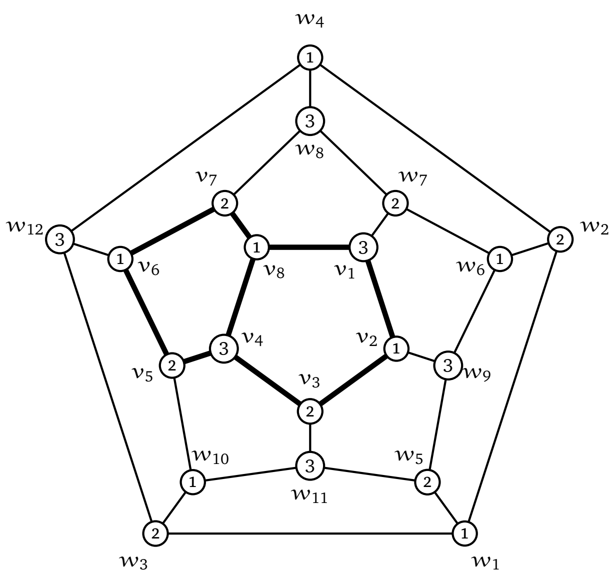



Figure 1.9: A 3-coloring of the icosahedron as in Case 3 of the proof of Brooks' Theorem. Subgraph $H$ is shown in bold, with $j := 4$ and $w := w_{11}$. The vertex order $\sigma$ of $G - V(H)$ is denoted by $w_1, \ldots, w_{12}$.

with the same color used on $w$. (This is possible since $N_G(v_j) \subseteq V(H)$, so no neighbor of $v_j$ is colored.) Finally, finish by coloring greedily using the order $v_{j+1}, v_{j+2}, \ldots, v_r, v_1, \ldots, v_{j-1}$. □

## 1.2 Choice, Paint, and Alon–Tarsi Numbers

When we prove a coloring result by induction, we often delete some configuration H, color $G-H$ by minimality, and find colors for $V(H)$. We usually do not know which colors are available for vertices of H; instead we have only a lower bound on the *number* of colors available. This approach naturally leads to list-coloring, the main topic of this section. We also study two further generalizations.

### 1.2.1 Definitions and Basic Inequalities

choice number
paint number
Alon–Tarsi number
$\chi_\ell(G), \chi_p(G)$
$AT(G)$

Here we introduce three variations on chromatic number (all defined below). In order of increasing generality, these are *choice number* (also called *list-chromatic number*), denoted $\chi_\ell(G)$, *paint number*, denoted $\chi_p(G)$, and *Alon–Tarsi number*, denoted $AT(G)$. We compare these parameters with those of the previous section, chromatic number and coloring number. The focus of this section is the following theorem, which summarizes the relationships among these five parameters.

**Theorem 1.25.** *Every graph* G *satisfies*

$$\chi(G) \leqslant \chi_\ell(G) \leqslant \chi_p(G) \leqslant AT(G) \leqslant col(G).$$

*Further, if we let* $G := K_{n,n}$ *and* $n \to \infty$*, then each of the differences* $\chi_\ell(G)-\chi(G)$*,* $\chi_p(G)-\chi_\ell(G)$*,* $AT(G)-\chi_p(G)$*, and* $col(G)-AT(G)$ *is unbounded.*

Below we define these new terms and sketch a proof of Theorem 1.25.

list assignment
L-coloring
list-coloring
f-choosable
k-assignment
degree-choosable
k-choosable
choice number

**Definition 1.26.** For a graph G, a *list assignment* L assigns to each vertex $v$ a list $L(v)$ of allowable colors. An L-*coloring* of G, also called a *list-coloring*, is a proper coloring $\varphi$ such that $\varphi(v) \in L(v)$ for each vertex $v$. For a positive integer-valued function f, an f-*assignment* is a list assignment L such that $|L(v)| = f(v)$ for each vertex $v$. If G has an L-coloring for every f-assignment L, then G is f-*choosable* (or f-*list-colorable*). If $|L(v)| = k$ for some constant k and all $v \in V(G)$, then L is a k-*assignment*. If $|L(v)| = d(v)$ for each $v$, then L is a *degree-assignment*. If G has an L-coloring for every degree-assignment L, then G is *degree-choosable*. If G has an L-coloring for every k-assignment L, then G is k-*choosable* (or k-*list-colorable*). The *choice number*, $\chi_\ell(G)$, (or *list-chromatic number*) is the smallest k such that G is k-choosable.

Clearly $\chi(G) \leqslant \chi_\ell(G)$ for every graph G, since the lists for all vertices might be identical. However the difference $\chi_\ell(G) - \chi(G)$ may be arbitrarily large, even for bipartite graphs.

**Proposition 1.27.** *Let* k *be a positive integer, and let* $n := \binom{2k-1}{k}$*. Now* $\chi_\ell(K_{n,n}) > k$*. Thus, the difference* $\chi_\ell(K_{n,n}) - \chi(K_{n,n})$ *is unbounded as* $n \to \infty$*.*

*Proof.* Let $G := K_{n,n}$. See Figure 1.10. Let $L$ assign to the vertices of each part of $G$ the distinct $k$-element subsets of[9] $[2k-1]$. Suppose that $G$ has an $L$-coloring $\varphi$. Now $\varphi$ uses at least $k$ colors on each part, since if $\varphi$ avoids some set of size $k$ on a part, then some vertex in that part must be uncolored. Since $\varphi$ uses at least $k$ colors on each part, and $2(k) > 2k-1 = |\cup_{v\in V} L(v)|$, some color is used on both parts. So $\varphi$ is not proper, which is a contradiction. $\square$

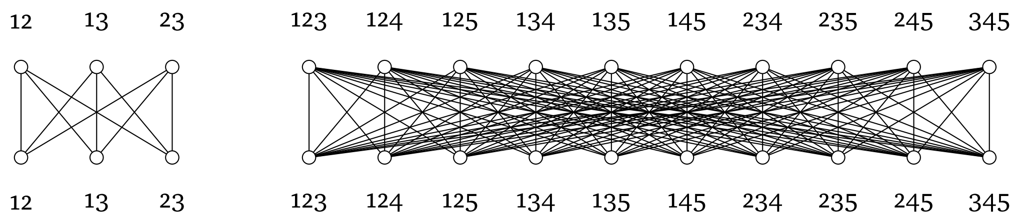



Figure 1.10: Left: A list assignment showing that $K_{3,3}$ is not 2-choosable. Right: A list assignment showing that $K_{10,10}$ is not 3-choosable.

To translate Proposition 1.27 into a lower bound on $\chi_\ell(K_{n,n})$ in terms of $n$, we use Stirling's approximation, which shows that $n = \binom{2k-1}{k} \leqslant \frac{4^k}{2\sqrt{\pi k}}$. This implies that $k \geqslant (1+o(1))\frac{1}{2}\log n$. Erdős, Rubin, and Taylor [152] improved this bound to $\chi_\ell(K_{n,n}) \geqslant (1+o(1))\log n$, which is sharp. The following short probabilistic argument proves a matching upper bound.

Let $U$ and $W$ be the parts of $K_{n,n}$. Given a $(1+\lceil\log n\rceil)$-assignment $L$, we designate each color in $\cup_{v\in V(G)} L(v)$ to be used on either $U$ or $W$, each with probability $\frac{1}{2}$. It is straightforward to check that with positive probability each vertex $v$ has some color in $L(v)$ designated for use on its part. Thus, $G$ has an $L$-coloring. (In the proof of Lemma 1.32 we reformulate this proof, and provide a few more details.)

**Definition 1.28.** Fix $f : V(G) \to \mathbb{Z}^+$. The $f$-*painting game* is played by two players, *Lister* and *Painter*. Initially, all vertices are unpainted. On each round Lister lists some of the unpainted vertices, and Painter paints some independent subset of these. If Lister lists any vertex $v$ on $f(v)$ rounds without Painter painting $v$, then Lister wins the game. Otherwise Painter wins. If Painter can always win the $f$-painting game, then $G$ is $f$-*paintable*. The $k$-*painting game* is the case when $f(v) = k$ for some constant $k$, and $k$-*paintable* is defined analogously. The smallest integer $k$ such that $G$ is $k$-paintable is the *paint number* of $G$, which we denote by $\chi_p(G)$.

$f$-painting game
Lister, Painter
$f$-paintable
$k$-paintable
paint number

**Proposition 1.29.** *Every graph* $G$ *satisfies* $\chi_\ell(G) \leqslant \chi_p(G)$.

*Proof.* Fix a graph $G$ and integer $k < \chi_\ell(G)$, and let $L$ be a $k$-assignment such that $G$ is not $L$-colorable. By symmetry among the colors, we can assume that $\cup_{v\in V(G)} L(v) = [s]$ for some integer $s$. Now Lister has a winning strategy in the $k$-painting game, by listing on each round $i$ every vertex with color $i$ in its list that is not yet painted. Lister wins precisely because $G$ has no $L$-coloring. This proves the proposition. $\square$

[9]Here, and throughout the book, for each positive integer $a$ we let $[a] := \{1, \ldots, a\}$.

So $\chi_p(G) \geqslant \chi_\ell(G)$. But can $\chi_p(G) - \chi_\ell(G)$ be arbitrarily large? For many years it was unknown whether this difference ever exceeds 1. We saw above that $\chi_\ell(K_{n,n}) = (1 + o(1))\log n$. But more refined arguments show that $\chi_\ell(K_{n,n}) = \log n - (\frac{1}{2} + o(1))\log\log n$. With this in mind, Duraj, Gutowski, and Kozik [126] showed that $\chi_p(K_{n,n}) \geqslant \log n + O(1)$. Together with the refined bound on $\chi_\ell(K_{n,n})$, their result implies that $\chi_p(K_{n,n}) - \chi_\ell(K_{n,n}) \geqslant (\frac{1}{2} + o(1))\log\log n$.

circulation

Alon–Tarsi orientation

*EE, OE*

f-Alon–Tarsi

f-AT, k-AT

$AT(G)$

**Definition 1.30.** For an orientation $D$ of a graph $G$, a subdigraph $H$ of $D$ is a *circulation* if in $H$ the indegree equals the outdegree at each vertex, that is, $d^-_H(v) = d^+_H(v)$ for all $v$. An *Alon–Tarsi orientation* $D$ is one in which the number of circulations $H$ with $\|H\|$ even differs from the number with $\|H\|$ odd; see Figure 1.11. We denote these numbers of circulations by EE and OE (for "even" and "odd"; the second E is for "eulerian" graph, which is a synonym for circulation). An Alon–Tarsi orientation $D$ is $f$-*Alon–Tarsi*, or $f$-*AT* for short, if $f(v) > d^+_H(v)$ for each vertex $v$. A graph $G$ is $k$-*AT* if it is $f$-AT, when $f(v) = k$ for all $v$. The *Alon–Tarsi number*, *AT*$(G)$, is the smallest $k$ such that $G$ is $k$-AT.

Alon and Tarsi [20] used algebraic methods to show that if $G$ is $f$-AT, then $G$ is $f$-choosable. This implies that $\chi_\ell(G) \leqslant AT(G)$ for every $G$. Schauz later gave a constructive proof, showing that if $G$ is $f$-AT, then $G$ is $f$-paintable. In particular, $\chi_p(G) \leqslant AT(G)$ for every $G$. In Chapter 8 we extensively study Alon–Tarsi orientations. There we give a short proof that $\chi_\ell(G) \leqslant AT(G)$, and a longer proof that $\chi_p(G) \leqslant AT(G)$. For the present, we only prove the easy bound $AT(G) \leqslant col(G)$, and that the differences $AT(K_{n,n}) - \chi_p(K_{n,n})$ and $col(K_{n,n}) - AT(K_{n,n})$ are both unbounded.

**Proposition 1.31.** *Every graph $G$ satisfies AT$(G) \leqslant$ col$(G)$.*

*Proof.* Let $G$ have degeneracy $k$, and choose order $\sigma$ where each vertex has at most $k$ neighbors earlier in $\sigma$. Form orientation $D$ from $G$ by directing each edge towards its endpoint that comes earlier in $\sigma$. So $D$ is Alon–Tarsi, since it is acyclic, and thus its only circulation is the edgeless digraph. Further, $d^+(v) \leqslant k$ for every vertex $v$. So $D$ witnesses $AT(G) \leqslant k + 1 = col(G)$. □

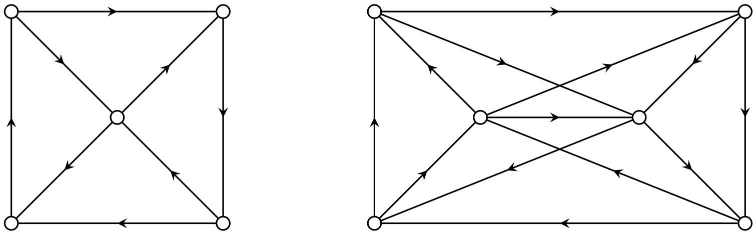


Figure 1.11: Left: Each vertex has outdegree at most 2 and EE $= 3$ while OE $= 4$. So the underlying graph is 3-AT. Right: The left center vertex, call it $w$, has outdegree 3 and all others have outdegree 2. Now EE $= 30$ and OE $= 28$. So the underlying graph is $f$-AT, where $f(w) = 4$ and $f(v) = 3$ for each other vertex $v$.

**Proposition 1.32.** *As $n \to \infty$, the differences (i) AT$(K_{n,n}) - \chi_p(K_{n,n})$ and (ii) col$(K_{n,n}) -$ AT$(K_{n,n})$ are both unbounded.*

*Proof.* (i) By Pigeonhole, every orientation of $K_{n,n}$ has a vertex with outdegree at least $\lceil \|G\|/|G| \rceil = \lceil \frac{n}{2} \rceil$. So, to prove that $\text{AT}(K_{n,n}) - \chi_p(K_{n,n})$ is unbounded, it suffices to show that $\chi_p(K_{n,n}) \leqslant \log n + O(1)$. The proof below is similar to that for list-coloring, which we presented above (after Proposition 1.27). Our key new idea here is to derandomize that probabilistic argument.

Let $A$ and $B$ denote the parts of $K_{n,n}$, and let $k := 2 + \lceil \log n \rceil$; we show that $\chi_p(K_{n,n}) \leqslant k$. For each vertex $v$, let the *danger of* $v$ equal 0 if $v$ is already painted, and equal $2^{t-k}$ if $v$ has been listed $t$ times but never painted. On each round, Painter paints all of the vertices listed in either $A$ or $B$. Consider the sum of the dangers of vertices in $A$ and of vertices in $B$, and how these sums increase when Lister lists the vertices for round $i$. If the sum of dangers for $A$ increases more than the sum for $B$, then Painter paints all the vertices listed in $A$; otherwise, he paints the vertices in $B$. Note that the potential increase of dangers in $A$, if Painter does not paint its listed vertices, is exactly equal to the decrease if Painter does paint them. So, after each round, the sum of dangers in the whole graph has not increased. The initial sum of dangers is $|G|\, 2^{-k} \leqslant 2n(2^{-2-\log n}) = \frac{1}{2}$. If Lister ever wins, then the sum of dangers is at least 1. Thus, Painter always wins.

danger of $v$

(ii) Now we consider $\text{col}(K_{n,n}) - \text{AT}(K_{n,n})$. Given a subgraph $H$ of a graph $G$ and an orientation $D$ of $G$, by Pigeonhole some vertex of $H$ has outdegree at least $\|H\|/|H|$. Thus $\text{AT}(G) \geqslant 1 + \left\lceil \frac{1}{2}\text{mad}(G) \right\rceil$. For $K_{n,n}$, this bound holds with equality. Denote by $U$ and $W$ the parts of $K_{n,n}$, with $x_1, \ldots, x_n \in U$ and $y_1, \ldots, y_n \in W$. Let $S := \{x_1, \ldots, x_{\lceil \frac{n}{2} \rceil}, y_1, \ldots, y_{\lceil \frac{n}{2} \rceil}\}$, and let $T := V(K_{n,n}) \setminus S$. When an edge $e$ has both endpoints in $S$, or both endpoints in $T$, direct $e$ toward $U$. Otherwise, direct $e$ toward $W$. The resulting orientation $D$ has $d^+_D(v) \leqslant \left\lceil \frac{n}{2} \right\rceil$ for all $v$. Since $K_{n,n}$ is bipartite, every directed cycle is even. Every circulation $H$ of $D$ is an edge-disjoint union of directed cycles, so $\|H\|$ even. Thus, $\text{AT}(G) = 1 + \left\lceil \frac{n}{2} \right\rceil$, because $D$ has $OE = 0$ and $EE \geqslant 1$. In contrast, $\text{col}(K_{n,n}) = n + 1$, since in every vertex order the final vertex has $n$ earlier neighbors. Hence, $\text{col}(K_{n,n}) - \text{AT}(K_{n,n}) = \left\lfloor \frac{n}{2} \right\rfloor$. □

For the rest of this chapter, we mainly focus on $\chi_\ell(G)$. But often our arguments also prove analogous bounds for $\chi_p(G)$ and $\text{AT}(G)$. We elaborate on this point in Section 1.2.3. The kernel method, which we study in Chapter 5, works wonderfully for $\chi_p(G)$, but not for $\text{AT}(G)$. In Chapter 8, we prove the Alon–Tarsi Theorem, and apply it to the problem of edge-choosability. In contrast, the recoloring technique used in Chapter 3 (Kempe swaps) already fails for choosability.

### 1.2.2 2-Choosable Graphs and Degree-Choosable Graphs

For each $k$ at least 3, it is NP-hard to decide if an input graph $G$ is $k$-colorable. Not surprisingly, the same is true for $k$-choosable. In contrast, a graph is 2-colorable precisely when it has no odd cycle. The problem of 2-choosability is a bit harder, but not much.

trunk
$\theta$-graph
**Definition 1.33.** The *trunk* of a graph is what results when we repeatedly delete its leaves. More formally, the trunk is the maximum subgraph $H$ with $\delta(H) \geqslant 2$. A $\theta$-*graph*, $\theta_{a,b,c}$, is formed by subdividing three parallel edges, so that the resulting three edge-disjoint paths joining the 3-vertices have lengths $a$, $b$, and $c$. When vertices $v$ and $w$ are adjacent, we write $v \leftrightarrow w$; otherwise $v \not\leftrightarrow w$.
$\leftrightarrow$, $\not\leftrightarrow$

**Lemma 1.34.** *A connected graph $G$ is 2-choosable if its trunk is $\theta_{2,2,2p}$, $C_{2p}$, or the empty graph.*

*Proof.* Let $G$ be a graph and $L$ be a 2-assignment for $G$. If $v$ is a $1^-$-vertex of $G$, then we can extend any $L$-coloring of $G - v$ to an $L$-coloring of $G$. So it suffices to prove the lemma when $\delta(G) \geqslant 2$ (or when $G$ is the empty graph, for which the lemma is trivial). Our proof primarily uses greedy coloring, but we consider a few possible vertex orders, depending on $L$. We also may specify colors non-greedily for one or two vertices.

Suppose $G$ is an even cycle.[10] If all lists are identical, then $G$ is $L$-colorable, since $\chi(G) = 2$. So assume not, and let $v$ and $w$ be adjacent vertices with distinct lists. Color $w$ from $L(w) \setminus L(v)$, and proceed around the cycle away from $v$, coloring greedily and ending with $v$. Thus, $G$ is $L$-colorable. This finishes the case that $G$ is an even cycle.

$v_1, v_2$
$x_1, x_2, w_i$
$\varphi$
Suppose instead that $G$ is $\theta_{2,2,2p}$, as in Figure 1.12. Let $v_1$ and $v_2$ denote the 3-vertices, let $x_1$ and $x_2$ denote the 2-vertices adjacent to both $v_1$ and $v_2$, and let $w_1, \ldots, w_{2p-1}$ denote the vertices of the longer path, with $w_1 \leftrightarrow v_1$. We color (almost) greedily, but the vertex order we use depends on $L$. Once we have colored a vertex $z$, we denote its color by $\varphi(z)$.

**Case 1: All path vertices $w_i$ have identical lists.** If $v_1$ and $v_2$ have a common available color $\alpha$, then we use $\alpha$ on both of them and color greedily in the order $x_1, x_2, w_1, \ldots, w_{2p-1}$. The resulting coloring is proper, since the path vertices alternate colors, so $\varphi(w_{2p-1}) = \varphi(w_1) \neq \varphi(v_1) = \varphi(v_2)$. Assume instead that $v_1$ and $v_2$ have no common color. Now Pigeonhole implies that one 3-vertex, say $v_1$ by symmetry, has a color $\alpha \in L(v_1)$ that is available for at most one of its neighbors, say either $w_1$ or $x_1$. If $\alpha \in L(x_1)$, then we use $\alpha$ on $v_1$, and color greedily in the order $x_1$, $v_2$, $x_2$, $w_{2p-1}, \ldots, w_1$. Otherwise, $\alpha$ is available for no neighbor of $v_1$ other than $w_1$. So now we use $\alpha$ on $v_1$, and color greedily in the order $w_1, \ldots, w_{2p-1}$, $v_2$, $x_1$, $x_2$.

**Case 2: Vertices $w_i$ and $w_{i+1}$ have distinct lists, for some $i \in [2p-1]$.** Let $\sigma_1$ be the vertex order $w_i, \ldots, w_1, v_1, x_1, x_2, v_2$, $w_{2p-1}$, $\ldots, w_{i+1}$, and let $\sigma_2$ be the reverse of $\sigma_1$. We color by either $\sigma_1$ or $\sigma_2$, with each vertex colored greedily, except for $w_i, x_1, x_2$ in $\sigma_1$ and $w_{i+1}, x_1, x_2$ in $\sigma_2$. Suppose we color $w_i$ with $\alpha \in L(w_i) \setminus L(w_{i+1})$, and try to color greedily by $\sigma_1$, but we fail. We must fail at $v_2$, since every other vertex (except for $w_{i+1}$) has only a single neighbor colored before it. So, after coloring $w_i$ through $v_1$ by $\sigma_1$, we cannot extend the coloring to $\{x_1, x_2, v_2\}$. There must exist colors $\beta$ and $\gamma$ such that $L(v_2) = \{\beta, \gamma\}$, $L(x_1) = \{\beta, \varphi(v_1)\}$ and $L(x_2) = \{\gamma, \varphi(v_1)\}$. Note that $L(v_2) \cap L(x_1) \cap L(x_2) = \emptyset$. Now we abandon $\sigma_1$ and instead color by $\sigma_2$. Specifically, we color $w_{i+1}$ from $L(w_{i+1}) \setminus L(w_i)$ and

[10]The 2-choosability of an even cycle, $C_{2p+2}$, is actually implied by the 2-choosabilty of a $\theta$-graph, $\theta_{2,2,2p}$, since $C_{2p+2} \subseteq \theta_{2,2,2p}$. So we prove the first statement explicitly mainly as a gentle introduction to the ideas used to prove the second.

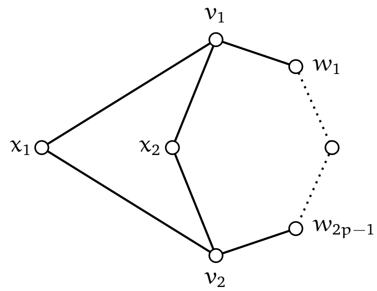



Figure 1.12: The θ-graph $\theta_{2,2,2p}$.

color greedily by $\sigma_2$, except that we choose colors for $x_1$ and $x_2$ so that we can still color $v_1$. Since $L(v_2) \cap L(x_1) \cap L(x_2) = \emptyset$, we must succeed. □

The converse of Lemma 1.34 is also true. A connected graph is 2-choosable *only if* its trunk is $\theta_{2,2,2p}$, $C_{2p}$, or the empty graph. Proving this requires showing that every other graph has a subgraph that is an even subdivision of one of six specific graphs and, for each of these graphs $H$, constructing a 2-assignment $L$ such that $H$ is not $L$-colorable. Constructing such 2-assignments fits better into the context of Chapter 2, so we defer this lower bound to Exercises 2.6 and 2.7.

The proof of Brooks' Theorem in Section 1.1.4 does not immediately extend to list-coloring, since it twice uses that a pair of vertices at distance two are colored with a common color, which may be impossible if their lists are disjoint (however, with one more idea it can be extended; see Exercise 12). But arguments similar to those used in that proof do yield the following partial list-coloring analogue of Brooks' Theorem.

**Theorem 1.35.** *Let $G$ be a connected graph and $L$ be an assignment such that $|L(v)| \geqslant d(v)$ for all $v$. Now $G$ has an $L$-coloring if either of the following holds.*

*(a)* $|L(w)| > d(w)$ *for some vertex $w$; or*

*(b)* $G$ *is 2-connected and not all lists are identical.*

Next we prove a full list-coloring analogue of Brooks' Theorem. In fact, the same proof works for Alon–Tarsi orientations (so also for paintability), but not for coloring number.

**Definition 1.36.** A *cut-set* in a connected graph $G$ is $S \subseteq V(G)$ such that $G - S$ is disconnected. A graph is $k$-*connected* if each cut-set has size at least $k$. A *block* of a graph $G$ is a maximal 2-connected subgraph. The blocks of $G$ partition its edge set (but not vertex set). A *Gallai tree* is a connected graph in which each block is a complete graph or an odd cycle. In the example Gallai tree in Figure 1.13 the 15 blocks are, in order approximately from left to right: $C_7$, $K_2$, $K_3$, $K_2$, $K_2$, $K_4$, $K_3$, $C_5$, $K_5$, $K_2$, $K_3$, $K_3$, $K_2$, $K_2$, $K_2$. Recall that $L$ is a *degree-assignment* if $|L(v)| = d(v)$ for all $v$. A $v, w$-*path* is a path with vertices $v$ and $w$ as its endpoints.

cut-set
$k$-connected
block
Gallai tree
degree-assignment
$v, w$-path

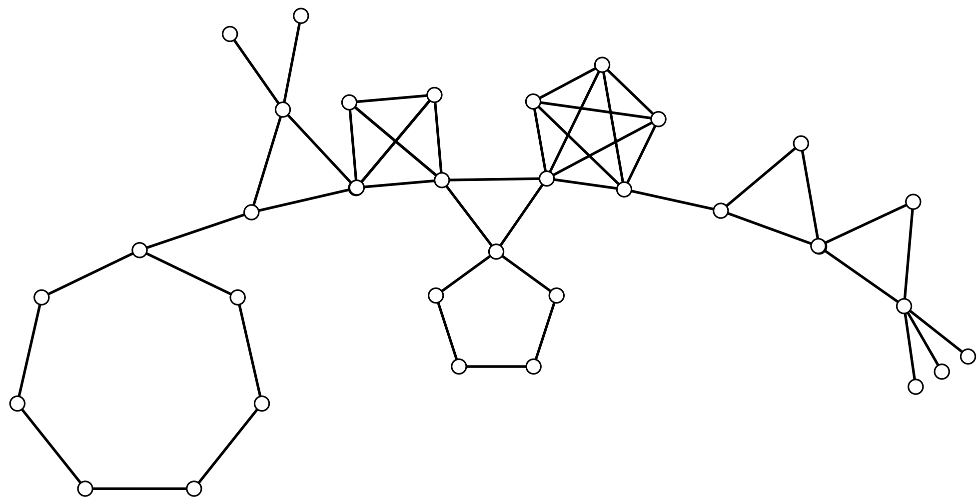


Figure 1.13: A Gallai tree with 15 blocks.

**Theorem 1.37.** *If* G *is a connected graph, then* G *is degree-choosable if and only if* G *is not a Gallai tree. So* $\chi_\ell(G) \leqslant \Delta$ *when* G *is connected and not a clique or an odd cycle.*

To prove Theorem 1.37, the key is showing unavoidability, which we do in Lemma 1.38. To prove reducibility (of a configuration $H$), we $L$-color $G - H$ by Theorem 1.35(a). This differs from our examples so far, which have colored $G - H$ recursively.

**Lemma 1.38 (Rubin's Block Lemma).** *If* G *is a 2-connected graph that is not a complete graph or an odd cycle, then* G *contains an induced even cycle with at most one chord.*

*Proof.* Since $G$ is not complete, $G$ has a minimal cut-set $S$. Because $S$ is minimal, each vertex of $S$ has a neighbor in every component of $G - S$. Since $G$ is 2-connected, $|S| \geqslant 2$. Pick $v, w \in S$ and form a cycle $C$ from the union of shortest $v, w$-paths through each of two components of $G - S$, as on the left in Figure 1.14. Since these paths are shortest, $C$ has at most one chord, $vw$. If $C$ is even, then the lemma is true; so assume $C$ is odd. Now one of the $v, w$-paths $P$ in $C$ is odd; so if $vw$ is present, then $P + vw$ is a chordless even cycle, and again the lemma is true. So we assume $vw$ is absent. Thus, $C$ is an induced odd cycle of length at least 5.

Since $G$ is not an induced odd cycle, there exists $x \in V(G) \setminus V(C)$. First suppose that each such $x$ has at most one neighbor on $C$, and fix such an $x$. Let $R$ be a shortest path containing $x$ and having both endpoints on $C$. Now $C \cup R$ induces a subgraph consisting of two 3-vertices joined by three vertex-disjoint paths. Since two of these paths have the same parity, their vertices induce an even cycle with at most one chord, as desired.

So instead assume that some $x \in V(G) \setminus V(C)$ has at least two neighbors on $C$. Denote these neighbors by $v_1, \ldots, v_k$, in order along $C$, as on the right in Figure 1.14. These $v_i$ partition the edges of $C$ into paths $P_i$, each with endpoints $v_i$ and $v_{i+1}$. If any $P_i$ is even, then $x \cup V(P_i)$

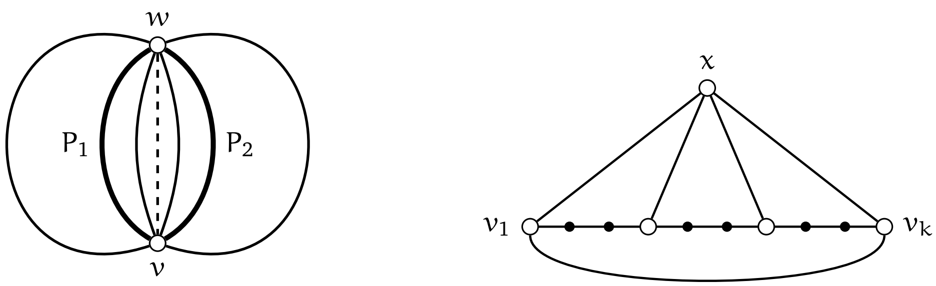



Figure 1.14: Left: Paths $P_1$ and $P_2$ induce a cycle $C$. Right: Vertex $x$ is not on $C$, but has neighbors on $C$.

induces a chordless even cycle; so assume each $P_i$ is odd. If $k \geqslant 4$, then $x \cup V(P_1) \cup V(P_2)$ induces an even cycle with one chord; so assume $k \leqslant 3$. Since each $P_i$ is odd and $C$ is odd, we must have $k = 3$. Since $C$ has length at least 5, some $P_i$ has length at least 3; by symmetry, say it is $P_3$. Now again, $x \cup V(P_1) \cup V(P_2)$ induces an even cycle with one chord. $\square$

*Proof of Theorem 1.37.* We can easily check that Gallai trees are not degree-choosable, by induction on the number of blocks. We leave the details as Exercise 2.3.

Let $L$ be a degree-assignment for $G$. Since $G$ is not a Gallai tree, $G$ has a block $B$ that is neither a complete graph nor an odd cycle. By Lemma 1.38, $B$ has an induced even cycle with at most one chord; call this cycle $C$. Let $vw$ be an edge of $C$. By Theorem 1.35(a), we can $L$-color $G - vw$, and this $L$-coloring induces an $L$-coloring $\varphi'$ of $G - V(C)$. If $C$ is a chordless even cycle, then we can extend $\varphi'$ to an $L$-coloring of $G$, since $C$ is 2-choosable, by Lemma 1.34. So assume instead that $C$ is a cycle with a chord. Let $v_1, \ldots, v_p$ denote its vertices; by symmetry we assume the chord is incident with $v_1$. Since $d(v_1) = 3 > d(v_p)$, we can color $v_1$ with some color not in $L(v_p)$. Now we can color greedily around the cycle, in order of increasing index, ending with $v_p$. $\square$

### 1.2.3 Reducibility for Paint Number and Alon–Tarsi Number

**Definition 1.39.** Fix a function $f : V(G) \to \mathbb{Z}^+$. Recall that a graph $G$ is $f$-*paintable* if Painter can always win the $f$-painting game on $G$, as in Definition 1.28. Similarly, a graph $G$ is $f$-*AT* if $G$ has an Alon–Tarsi orientation $D$ such that $f(v) > d^+_D(v)$ for all $v$, as in Definition 1.30. A graph $H$ is $f$-*paint-reducible* if $H$ is $f'$-paintable, where $f'(v) := f(v) - (d_G(v) - d_H(v))$. (Intuitively, this captures the idea that we can paint $G - H$ "first" and paint $H$ "second". But this intuition neglects the subtlety that by the time $G - H$ is painted some vertices of $H$ might have no colors remaining.) Similarly, a graph $H$ is $f$-*AT-reducible* if $H$ is $f'$-AT, where $f'(v) := f(v) - (d_G(v) - d_H(v))$. We define $f$-*list-reducible* analogously. When the context is clear, we often say that $H$ is $f$-reducible, or simply that $H$ is reducible.

f-paintable
f-AT
f-paint-reducible
f-AT-reducible
f-list-reducible

Our discussion of reducibility thus far has focused on $f$-list-reducibility. The point of Definition 1.39 is that we can take the same approach for paintability and Alon–Tarsi orientations. We formalize this intuition in the next lemma.

**Lemma 1.40.** *Suppose we are proving a theorem of the form: (i) Every graph $G$ in hereditary class $\mathcal{G}$ is $f$-paintable or (ii) Every graph $G$ in $\mathcal{G}$ is $f$-AT. A hypothetical minimal counterexample to (i) contains no induced subgraph that is $f$-paint-reducible; a hypothetical minimal counterexample to (ii) contains no induced subgraph that is $f$-AT-reducible.*

The analogous statement for list-coloring is trivial. If $G$ contains an $f'$-reducible induced subgraph $H$, then by minimality $G - H$ has an $f$-list-coloring $\varphi'$. Each $v \in V(H)$ loses to $\varphi'$ at most $d_G(v) - d_H(v)$ colors used on its neighbors. So $v$ has at least $f(v) - (d_G(v) - d_H(v))$ colors remaining. Thus, we can extend $\varphi'$ to an $L$-coloring $\varphi$, precisely because $H$ is $f$-list-reducible. Our approach in proving Lemma 1.40 is similar, but the details are more numerous and more technical, since we cannot simply paint $H$ "after" we paint $G - H$.

*Proof.* (i) Suppose $H$ is $f$-paint-reducible. By minimality, $G - H$ is $f$-paintable; that is, Painter has a winning strategy $A$ for the $f$-painting game on the graph $G - H$. By hypothesis, $H$ is $f'$-paintable, with $f'$ as in Definition 1.39. So Painter also has a winning strategy, $B$, for the $f'$-painting game on the graph $H$.

Now we use $A$ and $B$ to show that Painter has a winning strategy for the $f$-painting game on $G$, so $G$ is not a counterexample. On each round $i$, Lister lists a set of unpainted vertices, $\mathcal{S}_i$. First painter plays strategy $A$ on $G - H$, as if Lister listed $\mathcal{S}_i - H$. Suppose $A$ dictates that Painter paints the subset $\mathcal{T}_i$. Next Painter plays strategy $B$ on $H$, as if Lister listed $\mathcal{S}_i - N[\mathcal{T}_i]$. Say $B$ dictates $\mathcal{W}_i$. Altogether, Painter paints $\mathcal{T}_i \cup \mathcal{W}_i$. In the painting game that Painter simulates on $H$, each vertex $v$ is listed at least $f(v) - (d_G(v) - d_H(v))$ times (or is painted before that). Since Painter wins both the $f$-painting game on $G - H$ and the simulated $f'$-painting game on $H$, Painter wins the $f$-painting game on $G$.

(ii) By minimality, $G - H$ is $f$-AT, so let $D_1$ be an orientation of $G - H$ witnessing this. By hypothesis, $H$ is $f'$-AT, so let $D_2$ be an orientation of $H$ witnessing this. Now orient $G$ by directing edges of $G - H$ and of $H$ as in $D_1$ and $D_2$, and directing all remaining edges away from $V(H)$. Note that $d_D^+(v) < f(v)$ for all $v$. We show that $D$ is Alon–Tarsi, that is, that the number of circulations $J$ of $D$ with $\|J\|$ even differs from the number with $\|J\|$ odd. Let EE and OE denote these numbers of circulations. Analogously, define $EE_1$ and $OE_1$, for $D_1$, and $EE_2$ and $OE_2$, for $D_2$.

Note that no circulation $J$ of $D$ contains edges joining $V(H)$ to $V(G - H)$; if it did, then in $V(G - H)$ the sum of in-degrees would exceed the sum of out-degrees, a contradiction. So each circulation of $D$ is the disjoint union of a circulation of $D_1$ and a circulation of $D_2$. In fact, $EE = EE_1EE_2 + OE_1OE_2$ and $OE = EE_1OE_2 + EE_2OE_1$. We must show that $EE - OE \neq 0$. Since $D_1$ and $D_2$ are both AT, we know that $EE_1 - OE_1 \neq 0$ and $EE_2 - OE_2 \neq 0$. Thus, we have $EE - OE = (EE_1EE_2 + OE_1OE_2) - (EE_1OE_2 + EE_2OE_1) = (EE_1 - OE_1)(EE_2 - OE_2) \neq 0$. $\square$

## 1.3 Bigger Reducible Configurations: 3 Easy Applications

Here we present 3 applications that use degree-choosable graphs as reducible configurations. It is straightforward to modify the proofs to get analogous results for paint number and also for Alon–Tarsi number. (But that needlessly complicates the exposition, so we omit those details.)

### 1.3.1 More Injective Coloring

Recall the notion of injective coloring, from Definition 1.12. Since $\chi^i(G) = \chi(G^{(2)})$, we write $\chi^i_\ell(G)$ to denote $\chi_\ell(G^{(2)})$. We can also define $\chi^i_p$, $AT^i$, and $col^i$ analogously.

**Theorem 1.41.** *If a graph* $G$ *has* $\Delta \leqslant 3$ *and mad*$(G) < \frac{5}{2}$, *then* $\chi^i_\ell(G) \leqslant 4$.

*Proof.* Suppose the theorem is false. Let $G$ be a counterexample minimizing $|G|$, and let $L$ be a 4-assignment such that $G$ has no injective $L$-coloring. Now $G$ has no $1^-$-vertex $v$; otherwise $G - v$ has an injective $L$-coloring $\varphi$ by minimality, and we can extend $\varphi$ to $v$. Similarly, $G$ has no adjacent 2-vertices, $v$ and $w$. If so, then $G - \{v, w\}$ has an injective $L$-coloring $\varphi$, again by minimality. Since $v$ and $w$ each have at most 3 neighbors in $G^{(2)}$, we can extend $\varphi$ to $v$ and $w$.

Let $H$ be the subgraph of $G$ induced by edges incident to 2-vertices. (When we write $k$-vertex or $k$-neighbor, $k$ denotes the degree in $G$, not in $H$.) Since adjacent 2-vertices are forbidden in $G$, every edge in $H$ has as its endpoints one 2-vertex and one 3-vertex. Note that every vertex $v$ with $d_H(v) = 1$ is a 3-vertex. If a component $H_1$ of $H$ has no cycles, then it has more 3-vertices then 2-vertices, precisely because a tree has more vertices than edges. Similarly, if $H_1$ has at most one cycle, then it has at least as many 3-vertices as 2-vertices, so

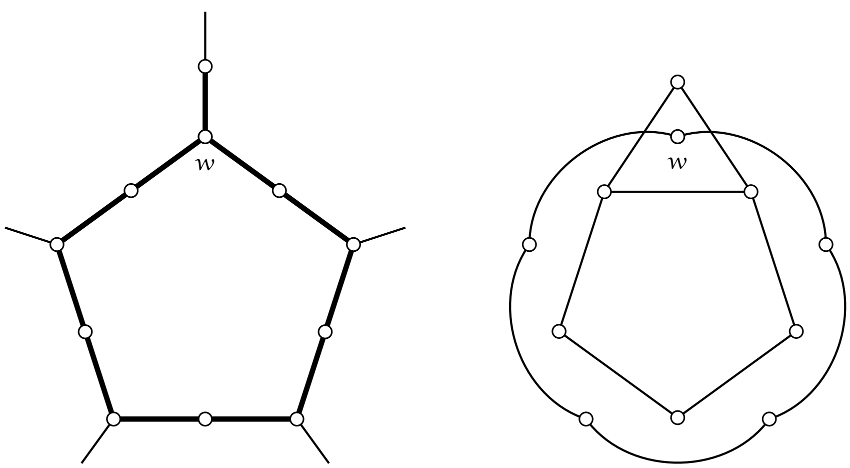



Figure 1.15: Left: $J$ is shown in bold, as a subgraph of $G$. Right: $G^{(2)}[J]$ has two components. The component containing $w$ is a cycle, and the other component is a cycle with a chord.

$\frac{1}{|H_1|}\sum_{v\in H_1} d_G(v) \geqslant \frac{5}{2}$. Since $\overline{d}(G) \leqslant \mathrm{mad}(G) < \frac{5}{2}$, some component $H_1$ has more 2-vertices
$w$ than 3-vertices. So $H_1$ has a cycle $C$ containing a 3-vertex $w$ with $d_{H_1}(w) = 3$, as on the left
$J$ in Figure 1.15. Let $J := V(C) \cup N(w)$. Since $G$ is a minimal counterexample, $G - J$ has an
$\varphi$ injective L-coloring $\varphi$. We must extend $\varphi$ to an injective L-coloring of all of $G$. For each $v \in J$, form $L'(v)$ from $L(v)$ by deleting all colors forbidden on $v$ by $\varphi$.

Since $C$ is even, $G^{(2)}[J]$ has two components. Let $J_1$ denote the component containing $w$, and $J_2$ the other component. Each vertex $v$ of $J_1$ has one neighbor outside $J_1$, so $\varphi$ forbids at most 2 colors on $v$. Further, $w$ has a 2-neighbor outside $J_1$, so $\varphi$ forbids only one color on $w$. Thus $|L'(v)| \geqslant 2 = d_{J_1}(v)$ for all $v \in V(J_1)$. Also, $|L'(w)| \geqslant 3 > d_{J_1}(w)$. So $J_1$ has an $L'$-coloring $\varphi_1$, by Theorem 1.35(a). Now consider $J_2$. Since each 3-vertex of $J$ other than $w$ has exactly two 2-neighbors in $J$, each vertex $v$ of $J_2$, except for the two neighbors of $w$, has $d_{J_2}(v) = 2$ and at most two colors forbidden by $\varphi$. The remaining two vertices of $J_2$ have $d_{J_2}(v) = 3$ and at most one color forbidden by $\varphi$. Thus, $d_{J_2}(v) \leqslant |L'(v)|$ for each $v \in V(J_2)$. So $J_2$ has an $L'$-coloring $\varphi_2$, by Theorem 1.35(b). Together, $\varphi$, $\varphi_1$, and $\varphi_2$ give an injective L-coloring of $G$, a contradiction. □

### 1.3.2 3-Choosability of Planar Graphs

10-sun **Definition 1.42.** In a planar graph, a 10-*sun* is a 10-face such that each incident vertex $v$ has $d(v) = 3$ and $v$ is also incident to a 3-face; see the left of Figure 1.16. It is easy to check that 10-suns are reducible for 3-choosability, as we do in the proof of Theorem 1.44. Our interest in 10-suns stems from Lemma 1.43.

**Lemma 1.43.** *If* $G$ *is a planar graph with* $\delta(G) \geqslant 3$ *and with no cycles of lengths 4 to 9, then* $G$ *contains a* 10-*sun.*

Before proving this lemma, we show how to apply it.

**Theorem 1.44.** *If* $G$ *is a planar graph with no cycles of lengths* 4 *to* 9*, then* $G$ *is* 3*-choosable.*

*Proof.* Assume the theorem is false. Let $G$ be a minimal counterexample, and let $L$ be a 3-assignment such that $G$ has no L-coloring. If $G$ has a $2^-$-vertex $v$, then $G - v$ has an L-coloring $\varphi$ by minimality. To extend $\varphi$ to $G$, color $v$ greedily. So assume $\delta(G) \geqslant 3$. By Lemma 1.43, $G$ contains a 10-sun $f$. By minimality, $G - V(f)$ has an L-coloring $\varphi$. Each vertex $v$ of $f$ has exactly one neighbor colored by $\varphi$, so $v$ has a list of two allowable colors. Thus, we can extend $\varphi$ to $G$ since (by Lemma 1.34) the 10-cycle is 2-choosable. □

*Proof of Lemma 1.43.* Assume the lemma is false and $G$ is a counterexample. To get a contradiction, we use balanced charging (as in Lemma 1.16), and the following three discharging rules.

(R1) Each vertex gives $1/3$ to each incident 3-face.

(R2) Each $10^+$-face gives $2/3$ to each incident 3-vertex that is incident to a 3-face.

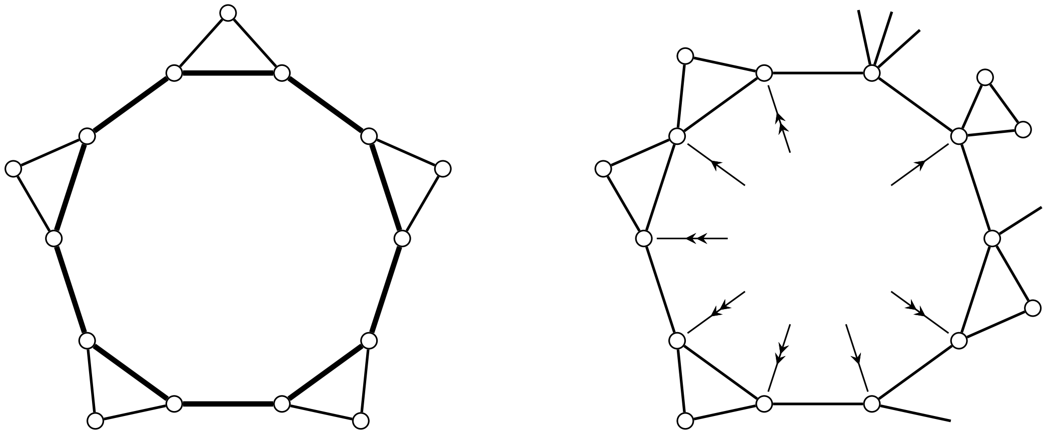


Figure 1.16: Left: A 10-sun, drawn in bold, has five other adjacent vertices on its five adjacent 3-faces. Right: A 10-face $f$ gives charge to incident vertices via (R2) and (R3). The arrows $\longrightarrow$ and $\twoheadrightarrow$ denote, respectively, that $f$ gives the vertices $1/3$ and $2/3$.

(R3) Each $10^+$-face $f$ gives $1/3$ to each (a) incident 3-vertex that is not incident to a 3-face, (b) incident 4-vertex that is incident to two 3-faces, and (c) incident 4-vertex that is incident to one 3-face that shares no edge with $f$.

Now we show that each vertex and face ends happy, which gives a contradiction. Recall that $\mathrm{ch}(v) := d(v) - 4$ and $\mathrm{ch}(f) := \ell(f) - 4$ for all $v$ and $f$. Note, since $G$ has no 4-cycle, that each vertex $v$ is incident with at most $d(v)/2$ 3-faces. $\mathrm{ch}(v), \mathrm{ch}(f)$

**Case 1: $\mathbf{d(v) = 3}$.** If $v$ is incident to a 3-face, then $v$ gives $1/3$ by (R1) and receives $2(2/3)$ by (R2), so $v$ ends happy since $-1 - 1/3 + 2(2/3) = 0$. If $v$ is not incident to a 3-face, then $v$ receives $3(1/3)$ by (R3a), so $v$ ends happy since $-1 + 3(1/3) = 0$.

**Case 2: $\mathbf{d(v) = 4}$.** If $v$ is incident to two 3-faces, then $v$ ends happy by (R1) and (R3b), since $-2(1/3) + 2(1/3) = 0$. If $v$ is incident to one 3-face, then $v$ ends happy by (R1) and (R3c), since $-1/3 + 1/3 = 0$. If $v$ is incident to no 3-faces, then $v$ clearly ends happy.

**Case 3: $\mathbf{d(v) \geqslant 5}$.** Now $v$ has at most $d(v)/2$ incident 3-faces, since $G$ has no 4-cycle. So $v$ ends happy, since $d(v) - 4 - (d(v)/2)/3 = (5d(v) - 24)/6 > 0$.

**Case 4: $\boldsymbol{\ell(f) = 3}$ or $\boldsymbol{\ell(f) \geqslant 11}$.** If $\ell(f) = 3$, then $f$ ends happy by (R1), since $-1 + 3(1/3) = 0$. By (R2) and (R3), each face $f$ gives at most $2/3$ to each incident vertex. If $\ell(f) \geqslant 12$, then $f$ ends happy, since $\ell(f) - 4 - 2\ell(f)/3 = (\ell(f) - 12)/3 \geqslant 0$. If $\ell(f) = 11$, then by parity $f$ cannot give $2/3$ to each incident vertex; some incident vertex receives at most $1/3$ from $f$. So $f$ ends happy, since $11 - 4 - 10(2/3) - 1/3 = 0$.

**Case 5: $\boldsymbol{\ell(f) = 10}$.** By assumption, $f$ is not a 10-sun. If $f$ gives $2/3$ to at most 8 incident vertices, then $f$ ends happy, since $10 - 4 - 8(2/3) - 2(1/3) = 0$. If $f$ gives $2/3$ to at least 9 incident vertices, then $f$ is similar to a 10-sun, with at least one additional edge incident at some vertex $v$. But now $v$ gets no charge from $f$; so $f$ ends happy, since $10 - 4 - 9(2/3) = 0$. $\square$

Given the decision in the proof of Lemma 1.43 to use balanced charging, our choice of discharging rules is fairly straightforward. Each 3-face $f$ has initial charge $3 - 4 = -1$, so $f$ needs to receive total charge at least $1$. It is natural to take charge for $f$ from its incident vertices, and the simplest option is to take equally from all 3 incident vertices. This motivates (R1). If a 3-vertex $v$ is incident to a 3-face $f$, then after applying (R1), $v$ has charge $-4/3$. The remaining two faces incident to $v$, say $f_1$ and $f_2$, are both $10^+$-faces, so each such $f_i$ clearly has extra charge. Again, it is simplest to have $v$ take its needed charge equally from $f_1$ and $f_2$; this motivates (R2). Even after settling on (R1) and (R2), we still have a few types of $4^-$-vertices that need more charge. This motivates (R3). It is a bit more ad hoc, but not difficult to discover.

### 1.3.3 Planar Graphs with $\Delta \geqslant 9$ are $(\Delta + 1)$-Edge-choosable

In Section 3.1 we show that $\chi'(G) \leqslant \Delta + 1$ for every graph $G$. Vizing conjectured the stronger bound $\chi'_\ell(G) \leqslant \Delta + 1$. Our next theorem proves his conjecture for planar graphs with $\Delta \geqslant 9$.

**Theorem 1.45.** *If $G$ is planar with $\Delta \geqslant 9$, then $\chi'_\ell(G) \leqslant \Delta + 1$.*

Each edge $vw$ with $d(v)+d(w) \leqslant \Delta+2$ is clearly reducible for Theorem 1.45 (as we show in Claim 1). Our other reducible configurations are even cycles with degrees alternating between 3 and $\Delta$. These are reducible precisely because even cycles are 2-choosable. The rest of the proof is some clever counting and discharging (which we motivate further below).

*Proof.* Planar graphs with $\Delta \geqslant 9$ do not form a hereditary class. So instead of Theorem 1.45, we prove a more general result, which holds for a larger hereditary class: Every planar graph
$k$ $G$ satisfies $\chi'_\ell(G) \leqslant k$, where we let $k := \max\{10, \Delta + 1\}$. Suppose this theorem is false. Let $G$ be a counterexample with fewest edges, and let $L$ be an edge-$k$-assignment such that $G$ has no $L$-coloring. We prove two structural claims.

**Claim 1.** *Every edge $vw$ has $d(v) + d(w) \geqslant k + 2$. In particular, $G$ has no $2^-$-vertices, and if $G$ has 3-vertices, then $\Delta \geqslant 9$.*

Proof. Suppose $vw$ has $d(v) + d(w) \leqslant k + 1$. By minimality $G - vw$ has an $L$-coloring $\varphi$. The number of colors $\varphi$ forbids on $vw$ is at most $d(v) + d(w) - 2 \leqslant k - 1$. Since $|L(vw)| = k$, we can extend $\varphi$ to $vw$, a contradiction. The first statement implies the second and third. $\diamondsuit$

$H$ Let $H$ denote the subgraph induced by edges with an endpoint of degree 3. Claim 1 implies
$n_3$ that each edge $vw$ of $H$ has $d(v) + d(w) = k + 2$. Let $n_3$ denote the numbers of 3-vertices in
$n_\Delta$ $G$ and let $n_\Delta$ denote the number of $\Delta$-vertices with a 3-neighbor.

**Claim 2.** *Subgraph $H$ is acyclic; furthermore, if $H$ is non-empty, then $2n_3 < n_\Delta$.*

Proof. When $H$ is empty, the lemma holds trivially, so assume it is not. Suppose that $H$ contains some cycle $C$. Since $H$ is bipartite, the degrees along $C$ alternate $3, \Delta, \ldots, 3, \Delta$. By minimality, $G - E(C)$ has an $L$-coloring $\varphi$. For each $e \in E(C)$, form $L'(e)$ from $L(e)$ by deleting all colors forbidden on $e$ by $\varphi$. The number of edges incident to $e$ and not in $C$ is $3+\Delta-2(2)=\Delta-1$. Thus, we always have $|L'(e)| \geqslant k-(\Delta-1) \geqslant 2$. Since even cycles are 2-choosable (so also edge-2-choosable), we can color $E(C)$ from $L'$. Together with $\varphi$, this gives an edge-$L$-coloring of $G$, a contradiction. Thus $H$ is acyclic, so it has fewer edges than vertices. Now $\|H\| = 3n_3$ and $\|H\| < |H| = n_3 + n_\Delta$. So $3n_3 < n_3 + n_\Delta$, which implies the desired inequality. $\diamondsuit$

We show that every planar graph $G$ violates either Claim 1 or 2. Fix a plane embedding of $G$ and let $F$ denote its set of faces; we use balanced charging. We also use a "bank", which starts with charge 0. To reach a contradiction, we show that each vertex and face ends happy, and the bank does too. We use two discharging rules, applied in succession.

(R1) Every $\Delta$-vertex with a 3-neighbor sends $1/2$ to a central bank, and every 3-vertex takes 1 from the bank.

(R2) Every $5^+$-vertex splits its charge after (R1) equally among all incident 3-faces.

Since $2n_3 \leqslant n_\Delta$, by Claim 2, the bank ends happy. Each 3-vertex $v$ gets 1 by (R1), so $v$ ends happy. Each 4-vertex starts and ends with 0. Each $5^+$-vertex ends happy, by (R2). Similarly, each $4^+$-face starts and ends happy. So we only need to check that each 3-face ends happy. The following claim is helpful.

**Claim 3.** *By (R2), each 3-face gets at least $1/5$ from each incident 5-vertex; at least $1/3$ from each 6-vertex; at least $3/7$ from each 7-vertex; and at least $1/2$ from each $8^+$-vertex.*

Proof. By Claim 1, every $\Delta$-vertex with a 3-neighbor is a $9^+$-vertex. So for each $p \in \{5,6,7,8\}$, the charge $f$ receives from a $p$-vertex is at least $(p-4)/p$, as stated. For $p \geqslant 9$, the charge is at least $(p-4-1/2)/p$. This expression equals $1/2$ when $p=9$, and it increases with $p$. $\diamondsuit$

Consider a 3-face $f$, and let $p$ be the smallest degree of any vertex incident to $f$. Since $f$ has initial charge $-1$, it must receive at least 1; Figure 1.17 shows two examples. By Claim 1, each edge $vw$ satisfies $d(v)+d(w) \geqslant k+2 \geqslant 12$. If $p \geqslant 6$, then $f$ gets at least $3(1/3)$. If $p=5$, then $f$ gets at least $1/5+2(3/7)$. If $p \in \{3,4\}$, then $f$ has two incident $8^+$-vertices, so gets at least $2(1/2)$. Thus we are done, since all faces, all vertices, and the bank end happy. $\square$

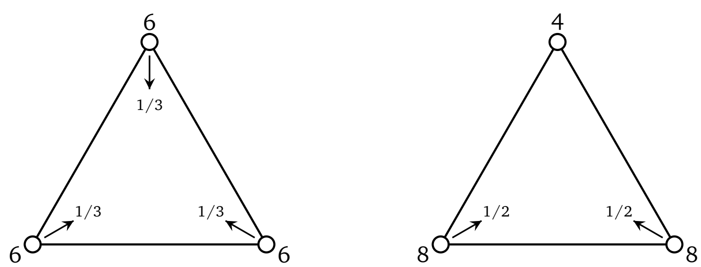



Figure 1.17: Two examples of triangles that finish with charge 0.

As is often the case, we can start writing the proof of the previous result before we know what we will prove. More precisely, we know the whole statement of Theorem 1.45 except for the lower bound on $\Delta$, which we aim to minimize as we work out the details of the proof. Below we provide more intuition about this process.

Similar to the proof of Lemma 1.43, once we choose balanced charging, we need to find charge 1 for each 3-face and for each 3-vertex (Claim 1, which is typical for such proofs, ensures that $\delta(G) \geqslant 3$). A first attempt might be to have each 3-vertex take charge $1/3$ from each neighbor (which, by Claim 1 must be a $\Delta$-vertex). However, with this approach, a $\Delta$-vertex $v$ might lose as much as $\Delta/3$ to 3-neighbors, which would compromise $v$'s ability to give charge to 3-faces. However, here Claim 2 comes to our rescue; essentially, it says that each $\Delta$-vertex should need to sponsor (that is, supply the needed charge for) on average only half of one 3-vertex. (R1) is how we make this intuition formal. And (R2) is an obvious first try to get the needed charge to 3-faces.

Having settled on (R1) and (R2), but not yet having formulated Claim 3 or the analysis afterward, we can be confident that we will get a complete proof, at least when $\Delta$ is big enough. Here's why. As $d(v)$ grows, the charge that $v$ sends, by (R2), to each incident 3-face tends to 1. By Claim 1, each such $f$ must have at least two $(\Delta/2)^+$-vertices. So when $\Delta$ is big enough, we will win. (More formally, each 3-face receives total charge no less than arbitrarily close to 2.) Claim 3 and the analysis that follows simply calculates how big is big enough.

## 1.4 A Harder Application: Squares of Planar Graphs with Girth at Least 6

In this section we prove that if $G$ is planar with girth at least 6 and $\Delta \geqslant 295$, then $\chi_\ell(G^2) \leqslant \Delta+2$. This bound is best possible: For every $D \geqslant 3$, there exists a planar graph $G_D$ with $\Delta = D$ and girth 6, such that $\chi(G_D^2) = D+2$, as we will see in Section 2.1. The hypothesis bounding $\Delta$ can be weakened significantly, as we discuss in the Notes. But we present the theorem as stated, since it admits a simpler proof. As usual, we prove a more general result.

**Theorem 1.46.** *If $G$ is planar with girth at least 6, then $\chi_\ell(G^2) \leqslant \max\{295, \Delta(G)\} + 2$.*

The proof of this result uses discharging. Because it is longer than any we have seen yet, we dedicate Section 1.4.1 to reducibility, and Section 1.4.2 to unavoidability. In both sections, we use the following definitions.

$k$, $L$
minimal counterexample
weak neighbors
$N^2(v)$, $s$-thread

**Definition 1.47.** Assume Theorem 1.46 is false. Let $G$ be a counterexample with the fewest edges, let $k := \max\{295, \Delta(G)\}$, and let $L$ be a $(k+2)$-assignment such that $G^2$ has no $L$-coloring. Such a pair $(G, L)$ is a *minimal counterexample*. Vertices $v$ and $w$ are *weak neighbors* if they have a common 2-neighbor, but are not adjacent. Recall that $N^2(v)$ denotes the neighbors of $v$ in $G^2$. An $s$-*thread* in $G$ is a path with $s$ internal vertices, each of which has degree 2 in $G$.

### 1.4.1 Reducible Configurations

The following simple lemma is crucial. It implies that, for every edge in $G$, either an endpoint has high degree or an endpoint has a neighbor with high degree. This suggests that low degree vertices will have high degree vertices nearby, to give them enough charge to end happy.

**Lemma 1.48.** *Let $(G, L)$ be a minimal counterexample, and choose $vw \in E(G)$ such that $|N^2(v)| \leqslant |N^2(w)|$. Now either $|N^2(v)| \geqslant k+2$ or $|N^2(w)| \geqslant k+3$.*

*Proof.* Suppose the contrary. By minimality, $(G - vw)^2$ has an $L$-coloring. Uncolor $v$ and $w$; now greedily color $w$, followed by $v$. This gives an $L$-coloring of $G^2$, a contradiction. □

**Lemma 1.49.** *Let $(G, L)$ be a minimal counterexample. Now (i) $\delta(G) \geqslant 2$; (ii) $G$ has no $3^+$-thread, and every 2-thread has a $k$-vertex at each end; and (iii) the subgraph induced by 2-threads is acyclic.*

*Proof.* (i) If $G$ has a $1^-$-vertex $v$, then let $H := G - v$. By minimality, $H^2$ has an $L$-coloring $\varphi$, and we extend $\varphi$ to $G^2$ by coloring $v$ greedily. (ii) Suppose $G$ has either (a) a $3^+$-thread, beginning with 2-vertices $v$ and $w$ or (b) a 2-thread with 2-vertices $v$ and $w$, and $w$ adjacent to a $(\Delta-1)^-$-vertex. Let $H := G - \{v, w\}$. By minimality, we $L$-color $H^2$. Now we greedily color $v$ followed by $w$. This gives an $L$-coloring of $G^2$, a contradiction. (iii) Suppose to the contrary that the subgraph induced by 2-threads has a cycle $C$. Form $H$ from $G$ by deleting all 2-vertices on $C$. By minimality $H^2$ has an $L$-coloring $\varphi$. The uncolored 2-vertices on $C$ induce in $G^2$ an even cycle. Each 2-vertex has $\Delta$ colored neighbors in $G^2$, so at least two available colors. Thus, we can extend $\varphi$ to an $L$-coloring of $G^2$, precisely because even cycles are 2-choosable. □

We need one more important reducible configuration. To motivate it, we say a bit about the initial charges and discharging rules. Initially, each vertex $v$ has charge $2d(v) - 6$ and each face $f$ has charge $\ell(f) - 6$. Intuitively, high degree vertices split their charge equally among their neighbors. Since $G$ has girth at least 6, each face is happy. Also, each $3^+$-vertex is happy. So we just need to get more charge to 2-vertices. By Lemma 1.49, $G$ has fewer 2-threads than $\Delta$-vertices, so 2-vertices on 2-threads can receive charge from $\Delta$-vertices without much trouble. But $G$ can also have many 2-vertices on 1-threads. Each such 2-vertex $v$ needs charge 2, so we want $v$ to take 1 from each neighbor. Now consider a 3-vertex $v$ with a $\Delta$-neighbor and two 2-neighbors (such as $v_3$ in Figure 1.18). Each 2-neighbor takes 1 from $v$. As $\Delta$ grows, the amount that $v$ gets from its $\Delta$-neighbor approaches 2, but is always less. To end happy, $v$ needs a little extra charge from somewhere.

If $v$ is on a $7^+$-face $f$, then $v$ gets this charge from $f$. Often $v$ can also get some charge from an incident 6-face, if $f$ got charge from incident vertices. But some 6-faces have no incident vertices that can give them charge. A 6-face cannot give $v$ charge if the degrees of its incident vertices, in order, are $\Delta$, 3, 2, 3, 2, 3, and each incident 3-vertex has a $\Delta$-neighbor and two weak neighbors of small degree, as is true of vertex $v_3$ in Figure 1.18. Our final reducible configuration shows that $G$ has no 3-vertex incident to three such 6-faces. Thus, some incident face $f$ will

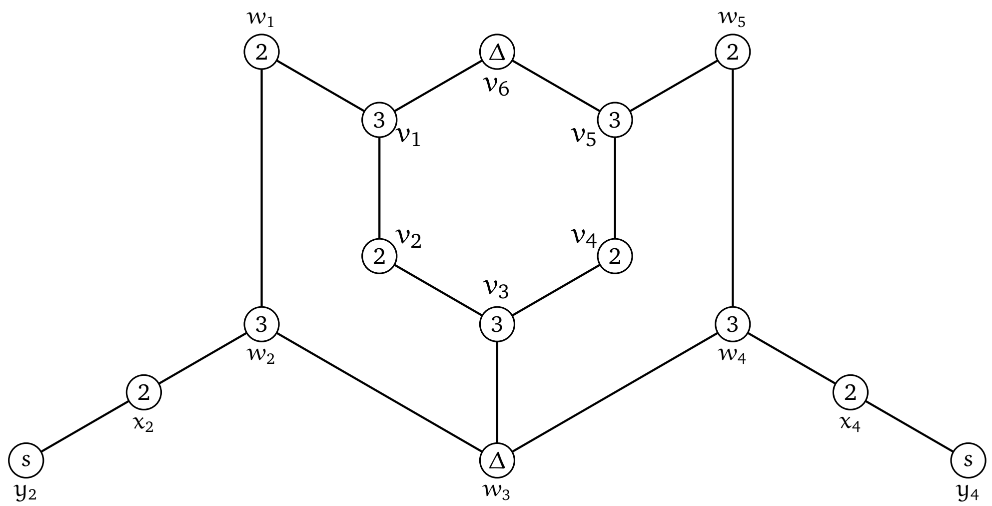



Figure 1.18: A reducible configuration for Theorem 1.46: here a 3-vertex $v_3$ lies on three 6-faces, each of which has vertices with degrees 3, 2, 3, 2, 3, $\Delta$, and none of these vertices has too many vertices at distance two (except for possibly the $\Delta$-vertices). More precisely, the label in each vertex denotes its degree, and s stands for *small*, which means degree at most 59.

give charge to $v$ whenever $v$ is a 3-vertex with a $\Delta$-neighbor and two weak 3-neighbors. The particular amount of charge, $\varepsilon$, that $v$ gets from $f$ is less important, as long as it is constant. For each $\varepsilon > 0$, we can increase $\Delta$ until $\frac{2\Delta-6}{\Delta} \geqslant 2-\varepsilon$, which simplifies to $\Delta \geqslant \frac{6}{\varepsilon}$. Of course, as $\varepsilon$ increases our hypothesis $\Delta \geqslant \frac{6}{\varepsilon}$ becomes weaker, which yields a stronger result. In the proof we will let $\varepsilon := 1/10$, which requires $\Delta \geqslant 60$. So, in fact, the hypothesis $\Delta \geqslant 295$ arises elsewhere (specifically, in Case 3).

**Lemma 1.50.** *The configuration in Figure 1.18 is reducible for Theorem 1.46. That is, it cannot appear as an induced subgraph of $G$ in a minimal counterexample $(G, L)$.*

*Proof.* Suppose instead that $G$ contains this configuration. Form $H$ from $G$ by deleting every 2-vertex and 3-vertex in Figure 1.18. By minimality, $H^2$ has an $L$-coloring $\varphi$. To extend $\varphi$ to $G^2$, greedily color $v_3$. Now the remaining uncolored 3-vertices ($v_1$, $w_2$, $w_4$, $v_5$) induce a 4-cycle in $G^2$. Each 3-vertex has at least two allowable colors; thus, we can color the 3-vertices, since the 4-cycle is 2-choosable. We now greedily color the remaining 2-vertices, since each has at most $59+3$ neighbors in $G^2$. □

### 1.4.2 Discharging

big, small
special

**Definition 1.51.** A vertex $v$ is *big* if $d(v) \geqslant 60$ and is *small* if $59 \geqslant d(v) \geqslant 3$. A 3-vertex is *special* if it has at least two 2-neighbors.

**Lemma 1.52.** *Let* $(G, L)$ *be a minimal counterexample. When we use face charging and the following seven discharging rules, shown in Figure 1.19, every vertex and face (and the bank) ends happy. Thus, no minimal counterexample exists, so Theorem 1.46 is true.*

*(R1) Each* $2$*-vertex on a* $1$*-thread takes* $1$ *from each neighbor. Each* $2$*-vertex on a* $2$*-thread takes* $2$ *from the bank, and each* $\Delta$*-vertex at the end of a* $2$*-thread sends* $4$ *to the bank.*

*(R2) Each special vertex takes* $19/10$ *from each big neighbor, and each non-special small vertex takes* $3/2$ *from each big neighbor.*

*(R3) Each small vertex (possibly special) takes* $9/10$ *from each big weak neighbor.*

*(R4) Each special vertex takes* $1/10$ *from each small weak neighbor.*

*(R5) Each* $3$*-vertex with a* $2$*-neighbor and two small neighbors takes* $1/20$ *from each small neighbor.*

*(R6) Each special vertex takes* $1/5$ *from each incident* $7^+$*-face.*

*(R7) Let* $f$ *be a* $6$*-face with incident big vertex* $v$*. Now* $f$ *takes* $1/5$ *from* $v$ *unless both neighbors of* $v$ *along* $f$ *are special, and* $f$ *splits this* $1/5$ *equally among incident special vertices.*

Before analyzing final charges, we motivate some of these rules. Most of our work goes into getting enough charge to 3-vertices, particularly special vertices. By Lemma 1.49, G has no $1^-$-vertex and no $3^+$-thread. Further, every 2-thread has $\Delta$-vertices at its ends, and the subgraph induced by the 2-threads is acyclic. Thus, G has fewer 2-threads than $\Delta$-vertices ending those 2-threads. So (R1) ensures that all 2-vertices end happy and the bank ends happy, while each $\Delta$-vertex loses at most $\Delta + 4$. Big vertices can afford to send in each direction $\frac{2(60)-6}{60} = \frac{19}{10}$. Rules (R2)–(R4) send charge from big vertices to small vertices (including special vertices), and from small vertices to special vertices. To get more charge to special vertices, we add (R5)–(R7).

*Proof.* First we show that the faces and bank end happy. By Lemma 1.49, G has fewer 2-threads than $\Delta$-vertices at the ends of these 2-threads. So the bank ends happy. Each $7^+$-face $f$ has at most $\frac{2}{3}\ell(f)$ incident special vertices, since each special vertex has a 2-neighbor on each incident face. Thus, each $7^+$-face $f$ ends happy, since $\ell(f) - 6 - \frac{1}{5}(\frac{2}{3}\ell(f)) = \frac{13}{15}\ell(f) - 6 > 0$. Each 6-face starts with 0 and sends only what it receives, so ends happy.

**Case 1: $v$ is a big vertex.** We show that the total $v$ sends to its neighbors, weak neighbors, and incident faces is at most $\frac{19}{10}d(v)$. Consider a neighbor $w$ of $v$. If $w$ is special, then $w$ takes $\frac{19}{10}$ by (R2). If $w$ is small and non-special, then $w$ takes $\frac{3}{2}$, by (R2), and the two faces incident to $vw$ each take at most $\frac{1}{5}$, by (R7), for a total of $\frac{19}{10}$. Finally, if $w$ is a 2-vertex, then $w$ takes at most 1, by (R1), and the other neighbor of $w$ takes at most $\frac{9}{10}$ by (R3), again a total of $\frac{19}{10}$. If $v$ sends no charge to the bank, then $v$ ends happy since $2d(v) - 6 \geqslant \frac{19}{10}d(v)$ when $d(v) \geqslant 60$. If $v$ does send charge to the bank, then let $vwxy$ be a 2-thread starting at $v$. By

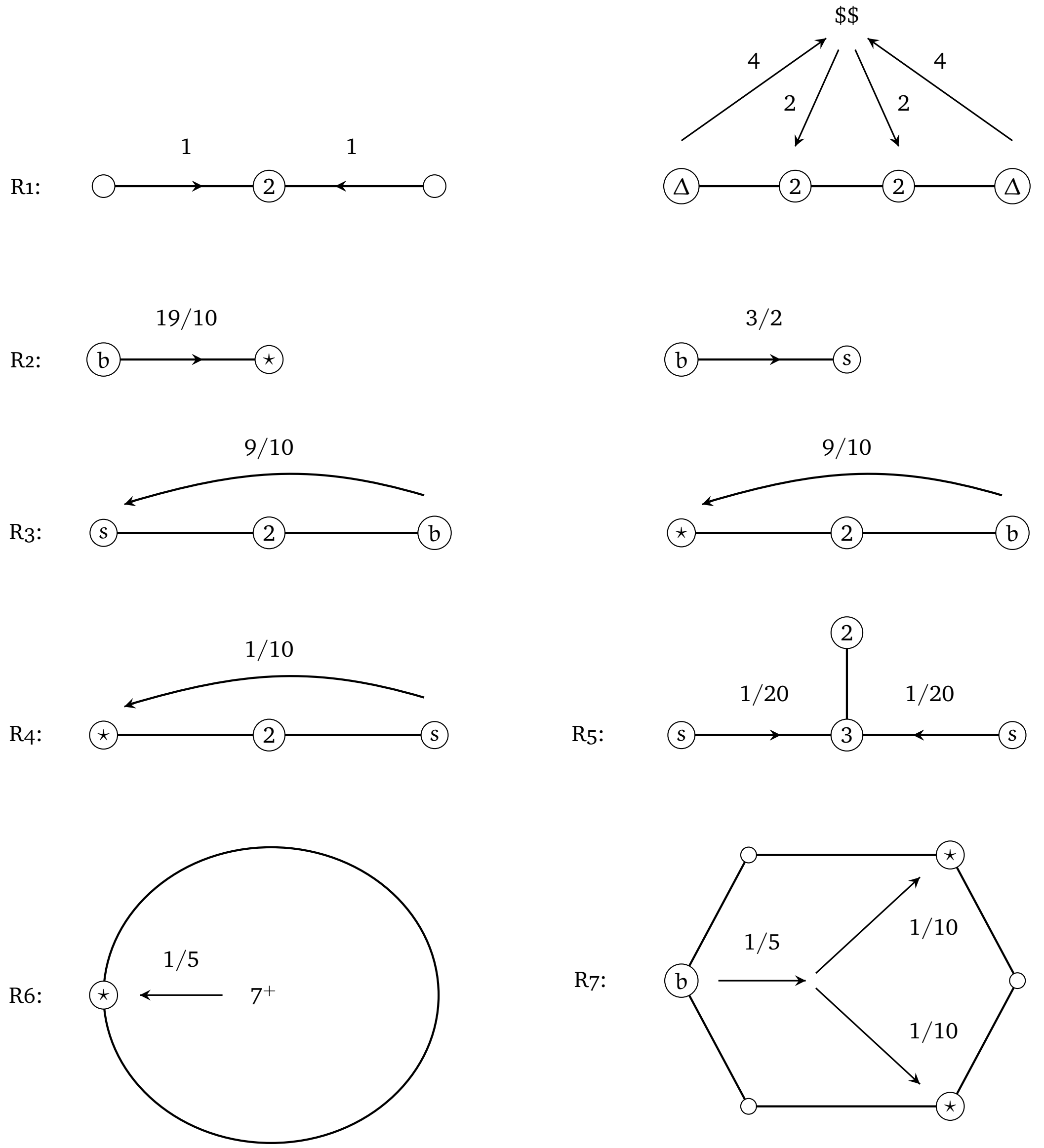



Figure 1.19: Examples of (R1)–(R7) in Lemma 1.52.

applying Lemma 1.48 to edge $wx$, we see that $\max\{|N^2(w)|, |N^2(x)|\} = \Delta + 2 \geqslant k + 2 \geqslant 297$; so $\Delta \geqslant 295$. Thus, $v$ ends happy, since $2\Delta - 6 - 4 \geqslant \frac{19}{10}\Delta$.

**Case 2: $v$ is a small $6^+$-vertex.** By (R1), (R4), and (R5) $v$ sends at most $1 + \frac{1}{10}$ in each direction. If $v$ has a $3^+$-neighbor $w$, then $v$ sends $w$ at most $\frac{1}{20}$, by (R5), so $v$ ends happy, since $2d(v) - 6 - (d(v) - 1)(1 + \frac{1}{10}) - \frac{1}{20} = \frac{9}{10}d(v) - \frac{99}{20} = \frac{9}{10}(d(v) - \frac{11}{2}) > 0$. Otherwise, $v$ has only 2-neighbors. Applying Lemma 1.48 to each edge incident to $v$ shows that every weak neighbor $w$ of $v$ is big, so $w$ sends $\frac{9}{10}$ to $v$. Thus $v$ ends happy, since $2d(v) - 6 - (1 - \frac{9}{10})d(v) = \frac{19}{10}(d(v) - \frac{60}{19}) > 0$.

**Case 3: $v$ is a 4-vertex or 5-vertex.** Suppose that $v$ has a big neighbor $w$. By (R2), $v$ takes $\frac{3}{2}$ from $w$. Now $v$ sends at most $1+\frac{1}{10} = \frac{11}{10}$ in the direction of each neighbor other than $w$, by (R1), (R4), and (R5). Thus, $v$ ends happy, since $2d(v)-6-\frac{11}{10}(d(v)-1)+\frac{3}{2} = \frac{9}{10}d(v)-\frac{34}{10} > 0$. So suppose instead that $v$ has no big neighbor. Now $|N^2(v)| \leqslant 59d(v) \leqslant 295 \leqslant k$, so Lemma 1.48 implies that each weak neighbor of $v$ is big. Thus, the net charge that $v$ sends in each direction is at most $\max\{1 - \frac{9}{10}, \frac{1}{20}\} = \frac{1}{10}$, either by (R1) and (R3) or by (R5). So $v$ ends happy, since $2d(v) - 6 - d(v)(\frac{1}{10}) > 0$.

**Case 4: $v$ is a 3-vertex.** Suppose $v$ has no 2-neighbors. If $v$ does not send charge by (R5), then $v$ does not send any charge, so $v$ ends happy. Now suppose $v$ does send charge by (R5), say to a 3-neighbor $w$. Since $|N^2(w)| < k$, Lemma 1.48 implies that $|N^2(v)| > k \geqslant \Delta$, which means that $v$ has a big neighbor. So $v$ ends happy, because $3(2) - 6 - 2(\frac{1}{20}) + \frac{3}{2} > 0$.

Suppose $v$ has exactly one 2-neighbor. If $v$ has no big neighbor, then $|N^2(v)| < k$, so Lemma 1.48 implies that every neighbor of $v$ has a big neighbor. Thus, $v$ has a big weak neighbor, $w$. Now $v$ gets $\frac{9}{10}$ from $w$, by (R3), and $2(\frac{1}{20})$ from its $3^+$-neighbors, by (R5), so $v$ ends happy, since $2(3) - 6 - 1 + \frac{9}{10} + 2(\frac{1}{20}) = 0$. If $v$ has a big neighbor $w$, then $w$ sends $\frac{3}{2}$ to $v$ by (R2). Thus, $v$ ends happy, since $2(3) - 6 - (1 + \frac{1}{10}) - \frac{1}{20} + \frac{3}{2} > 0$.

Suppose $v$ is special. We first consider the case that $v$ has three 2-neighbors, since it is easier. By Lemma 1.48, each weak neighbor of $v$ is big, which implies that $v$ gets charge from each incident face $f$, by (R6) or (R7). If $\ell(f) \geqslant 7$, then $f$ sends $\frac{1}{5}$ to $v$ by (R6). Otherwise, suppose $\ell(f) = 6$. Now $f$ has at least two incident big vertices (neighbors of the 2-neighbors of $v$ along $f$), and each incident big vertex $w$ has a non-special neighbor along $f$ (its 2-neighbor in common with $v$), so each $w$ sends $\frac{1}{5}$ to $f$. Further, $f$ has exactly one incident special vertex, $v$, so $v$ gets at least $\frac{2}{5}$ from $f$. Thus, $v$ is happy, since $2(3) - 6 - 3(1) + 3(\frac{9}{10}) + 3\min\{\frac{1}{5}, \frac{2}{5}\} > 0$.

Finally, suppose that $v$ has exactly two 2-neighbors. If $v$ has a small neighbor, then both weak neighbors of $v$ must be big, by Lemma 1.48. So each big weak neighbor sends $\frac{9}{10}$ to $v$, by (R3). As above, the face incident with $v$ and its two weak neighbors sends at least $\frac{1}{5}$ to $v$. So $v$ ends happy, since $2(3)-6-2(1)+2(\frac{9}{10})+\frac{1}{5} = 0$. Thus, we assume that $v$ has a big neighbor.

If either weak neighbor, $w$, of $v$ is non-special (either big or small), then $w$ sends $v$ at least $\frac{1}{10}$, by (R3) or (R4). Now $v$ ends happy, since $2(3) - 6 - 2(1) + \frac{19}{10} + \frac{1}{10} = 0$. (When special vertices are weak neighbors, the charge they send each other by (R4) cancels.) So assume that $v$ has a big neighbor and two special weak neighbors. Suppose that $v$ has a 2-neighbor $w$ and the other neighbor of $w$, call it $x$, is a special vertex without a big neighbor. Now edge $wx$ violates Lemma 1.48, a contradiction. Thus each special weak neighbor of $v$ has a big neighbor.

If $v$ is on a $7^+$-face $f$, then $f$ sends $v$ at least $\frac{1}{5}$, so $v$ ends happy, since $2(3)-6-2(1)+\frac{19}{10}+\frac{1}{5} > 0$. Thus, we conclude that $v$ is on three 6-faces. By Lemma 1.50, these three incident 6-faces are not as shown in Figure 1.18, with $v$ in the role of $v_3$. So at least one face $f$ incident to $v$ receives $\frac{1}{5}$ from an incident big vertex and sends at least $\frac{1}{10}$ to $v$, by (R7). Again, $v$ ends happy, since $2(3) - 6 - 2(1) + \frac{19}{10} + \frac{1}{10} = 0$.

**Case 5: $v$ is a 2-vertex.** Now $v$ ends with 0, by (R1). □

## Notes

Proposition 1.5 and Lemma 1.23 are due to Szekeres and Wilf [369]. Lemmas 1.6 and 1.16 are folklore. Injective coloring was introduced by Doyon, Hahn, and Raspaud [121]. Theorem 1.13 and the sharpness example $H-v$ are due to Cranston, Kim, and Yu [93]; so is [94] Theorem 1.41.

Theorem 1.11 gives an easy upper bound on the maximum chromatic number of a graph embeddable in an arbitrary surface S. This bound was proved by Heawood [212], who believed (incorrectly) that he had also shown equality. Proving a matching lower bound reduces to the problem of determining, for each $n$, the minimum Euler genus of a surface into which $K_n$ embeds. This problem splits into 12 cases, depending on the value of $n$ mod 12. Throughout the 1960s, different groups contributed to the effort, and in 1968 Ringel and Youngs [343] announced a complete proof. Dirac [118] strengthened Heawood's upper bound by showing that if G satisfies inequality (1.1) with equality, then G contains as a subgraph a complete graph of order $\text{col}(G)$. More precisely, Dirac proved this for all but a few exceptional surfaces. These remaining cases were handled by Albertson and Hutchinson [9]. This strengthening is clearly false for the plane (as Dirac observed), and it is also false for the Klein bottle.

The Discharging Method is one of the most widely used techniques in graph coloring. For a thorough treatment of this topic, we recommend "An Introduction to the Discharging Method via Graph Coloring", by the author and West [105], as well as the extended version "A Guide to the Discharging Method" [104]. As we mentioned in Section 1.1.3, discharging is used to prove unavoidability. But to get our coloring theorems, we must also prove that each configuration in our unavoidable set is reducible for the coloring problem of interest.

In the present book we group these results by the method used for reducibility, rather than that used for unavoidability. As a consequence, this book has no chapter dedicated solely to discharging. However, to the reader seeking one, we offer the following suggested reading list which may comprise a "virtual chapter". Introductory: Sections 1.1.1, 1.1.3, 1.3, Theorem 3.15, Lemma 4.8, Lemma 4.17, Lemma 5.23, Theorem 5.25, Theorem 9.15, Proof of Theorem 12.2 (near the end of Section 12.1.1), Proof of Theorem 12.18 (near the end of Section 12.2), and Section 12.3.2. Advanced: Section 1.4, Lemma 4.37, Section 12.5.1, and Section 12.5.5.

Balogh, Kochol, Pluhar, and Yu [32] proved a stronger version of Lemma 1.20: every planar graph has a $5^-$-vertex with at most two $11^+$-neighbors. This version is sharp, as shown by Example 1.21. The main difference in their proof is that if two 4-vertices are adjacent, then each receives less charge from each high degree vertex that is a neighbor of both, and more charge from each high degree neighbor that is not adjacent to the other 4-vertex. This result was refined further by Harant and Jendrol [202], who strengthened the upper bounds on the degrees of the neighbors of the $5^-$-vertex other than the two neighbors with largest degrees.

Wegner [411] conjectured that every planar graph G with $\Delta \geqslant 8$ satisfies $\chi(G^2) \leqslant 1 + \left\lfloor \frac{3}{2}\Delta \right\rfloor$. (This bound is best possible, as witnessed by the construction in Exercise 2.5.) Lemma 1.19 was implicit in work of Jonas [232]. Theorem 1.22 is due to van den Heuvel and McGuinness [397]; for every planar G, they showed that $\text{col}(G^2) \leqslant 2\Delta + 25$. Their main extra work, beyond what we presented, was proving a more technical asymmetric version of Lemma 1.20, and proving stronger lemmas when $\Delta$ is small. (This more technical asymmetric lemma was later subsumed by the work of Harant and Jendrol mentioned in the previous paragraph.)

Agnarsson and Halldórsson [3] and Borodin, Broersma, Glebov, and van den Heuvel [53, 54] both showed that $\mathrm{col}(G^2) \leqslant \lceil \frac{9}{5}\Delta \rceil + 1$ for $\Delta$ sufficiently large. Exercise 10 demonstrates that this bound is sharp. Molloy and Salavatipour [308] used a complicated discharging argument to show that every planar G satisfies $\chi(G^2) \leqslant \lceil \frac{5}{3}\Delta \rceil + 78$; this is the best progress towards Wegner's Conjecture (at least when $\Delta$ is not too big).

Asymptotically, Wegner's conjecture was confirmed by Havet, van den Heuvel, McDiarmid, and Reed [205]. They showed that if G is planar then $\chi_\ell(G^2) = \frac{3}{2}\Delta(1 + o(1))$. Suppose G is a planar graph. Wegner also conjectured that $\chi(G^2) \leqslant 7$ when $\Delta = 3$ and $\chi(G^2) \leqslant \Delta + 5$ when $4 \leqslant \Delta \leqslant 7$. Nearly 40 years after Wegner posed this problem, two groups confirmed the case $\Delta = 3$: Thomassen [384] and Hartke, Jahanbekam, and Thomas [204]. The first paper used a detailed structural analysis. In contrast, the second paper used a relatively straightforward discharging argument (to prove unavoidability) with extensive computer case-checking (to prove reducibility).

Brooks [74] proved his eponymous theorem in 1941, and it has frequently been reproved and strengthened. We mainly follow Zając [426], which in turn slightly simplifies Lovász [288]. For numerous alternate proofs, see [99].

List-coloring was introduced by Vizing [402] (on one side of the Iron Curtain) and independently by Erdős, Rubin, and Taylor [152] (on the other side). The former paper proved Theorem 1.35, and also Brooks' Theorem for list-coloring. The latter characterized both degree-choosable graphs, also done by Borodin [49], and 2-choosable graphs; it also proved Proposition 1.27 and determined the asymptotics of $\chi_\ell(K_{n,n})$. This paper was highly influential, due to its wealth of ideas and open questions. The authors conjectured that (i) every planar graph is 5-choosable and (ii) some planar graph is not 4-choosable. They were right on both accounts. Thomassen [376] proved (i), which we state as Theorem 11.1; Voigt [403] confirmed (ii), which is Theorem 2.3. Explaining the origin of the problem, Erdős, Rubin, and Taylor wrote:

> It got started when we tried to solve Jeff Dinitz's problem... Given a $m \times m$ array of $m$-sets, is it always possible to choose one from each set, keeping the chosen elements distinct in every row, and distinct in every column? To the best of our knowledge Jeff Dinitz's problem remains unsolved for $m \geqslant 4$.

A moment's reflection shows that Dinitz was asking whether $\chi'_\ell(K_{m,m}) = m$. In Theorem 5.11, we will see more generally that $\chi'_\ell(G) = \Delta$ for *every* bipartite graph!

Rubin's Block Lemma (Lemma 1.38) appeared in [152], where it was was attributed to Rubin. The proof given was a tedious case analysis, so a few years later Entringer [149] gave a shorter proof. In 2010 Hladký, Král, and Schauz [219] gave one that is even shorter; this is what we presented. In fact, Rubin's Block Lemma appeared as early as 1963 in work of Gallai [169, Theorem 1.9].

Alon and Tarsi [20] proved that $\chi_\ell(G) \leqslant AT(G)$; we revisit the Alon–Tarsi Theorem in Chapter 8. Paintability was introduced by Schauz [354] and by Zhu [435]. Schauz modified the proof of Theorem 11.1 to show that planar graphs are 5-paintable. He later strengthened the Alon–Tarsi Theorem [355] to show that always $\chi_p(G) \leqslant \mathrm{AT}(G)$. Zhu proved that $\chi_p(G) \leqslant \chi(G) \lg |G| + 1$ by generalizing the proof in Section 1.2.1 that $\chi_p(K_{n,n}) \leqslant \lg n + 2$.

Grötzsch [185] proved that triangle-free planar graphs are 3-colorable, which is Theorem 4.5. This result inspired many questions on 3-colorability of planar graphs without cycles of certain lengths. Steinberg (see [7]) conjectured that a planar graph is 3-colorable when it has neither 4-cycles nor 5-cycles. The question seemed hard, so Erdős generalized it: What is the smallest integer $k$, if it exists, such that a planar graph is 3-colorable if it has no cycles of lengths 4 to $k$? Abbot and Zhou [1] proved that $k \leqslant 11$. Theorem 1.44 shows that $k \leqslant 9$. The result is due to Borodin [52] and also to Sanders and Zhao [351]. The current best bound is $k \leqslant 7$, by Borodin, Glebov, Raspaud, and Salavatipour [59]. Cohen-Addad, Hebdige, Král', Li, and Salgado [89] disproved Steinberg's Conjecture; see Theorem 2.5. Thus, $k \in \{6, 7\}$.

Vizing [398, 399] proved that $\chi'(G) \leqslant \Delta + 1$ for every graph $G$; we will see three proofs in Section 3.1. He conjectured that also $\chi'_\ell(G) \leqslant \Delta+1$. Theorem 1.45, due to Borodin [50], verifies this for planar graphs with $\Delta \geqslant 9$. Our proof follows Cohen and Havet [88]. Bonamy [43] strengthened this result to include planar graphs with $\Delta = 8$. Borodin [50] also proved that $\chi'_\ell(G) = \Delta$ for planar graphs with $\Delta \geqslant 14$, which we leave as Exercise 19. Borodin, Kostochka, and Woodall [69] extended the result to planar graphs with $\Delta \geqslant 12$; this is Theorem 5.24.

Wang and Lih [409] conjectured that for each $g \geqslant 5$ there exists $D_g$ such that $\chi(G^2) = \Delta(G) + 1$ when $G$ has girth at least $g$ and $\Delta(G) \geqslant D_g$. Borodin et al. [56] proved this for $g \geqslant 7$ and disproved it for $g \in \{5, 6\}$; these results were later extended to list-coloring [62], for which it suffices to let $D_7 := 30$.

For $g = 6$, Dvořák, Král', Nejedlý, and Škrekovski [132] showed that a single extra color suffices, when $\Delta(G)$ is large enough. They proved Theorem 1.46 with 8821 in place of 295 (they stated the result only for coloring, but their proof also works for list coloring). Our presentation more closely follows Borodin and Ivanova [61], who improved 8821 to 36.

All of these (upper bound) results in the previous two paragraphs were subsumed by work of Bonamy, Lévêque, and Pinlou [47], who proved that $\chi_\ell(G^2) \leqslant \Delta(G) + 2$ when $\Delta(G) \geqslant 17$ and $\mathrm{mad}(G) < 3$ (this includes all planar graphs of girth at least 6, by Lemma 1.6). For planar graphs with girth at least 5, Bonamy, Cranston, and Postle [45] showed that $\chi_\ell(G^2) \leqslant \Delta(G)+2$ when $\Delta(G)$ is sufficiently large. These ideas were extended by Choi, Cranston, and Pierron [83] to prove the same result for planar graphs with no 4-cycle.

## Exercises

Most exercises have a hint provided in a Hints section near the end of the book.[11]

1.1. For each positive integer $k$, construct a tree $T$ and vertex order $\sigma$ such that coloring $T$ greedily by $\sigma$ uses $k$ colors.

[11] The absence of a link to the Hints section is intentional; it supports the aim of encouraging the reader to try to solve the problem *before* reading the hint.

1.2. Recall from Example 1.14 that the Heawood graph $H$ is the incidence graph of the Fano plane, so it is vertex-transitive. Show that $\mathrm{mad}(H - v) = 36/13$.

1.3. (a) A graph is *chordal* if it has no induced chordless $4^+$-cycle. And a vertex is *simplicial* if its neighborhood is a clique. Show that every chordal graph $G$ contains a simplicial vertex. Thus, $\mathrm{col}(G) = \chi(G) = \omega(G)$. (b) A graph is *interval* if each vertex can be represented by an interval on the real line so that two vertices are adjacent precisely when their intervals intersect. (Every interval graph is a chordal graph.) Given an interval graph $G$, find a simple description of a vertex order $\sigma$ such that coloring $G$ greedily using $\sigma$ produces an optimal coloring.

1.4. Prove that $\chi(G) + \chi(\overline{G}) \leqslant |G|$ for every graph $G$. [325]

1.5. Let $G$ be a plane graph. Fix a constant $g > 0$. Assign to each face $f$ charge $\ell(f) - g$ and to each vertex $v$ charge $\frac{g-2}{2}d(v) - g$. Show that the sum of these charges is negative. For a planar graph with girth at least $g$, show that this implies the result of Lemma 1.6, that $\mathrm{mad}(G) < \frac{2g}{g-2}$.

1.6. For each $g \geqslant 3$ construct an infinite family of examples for which the bound on $\|G\|$ in Lemma 1.6 holds with equality. [97]

1.7. Construct an efficient algorithm that, given a graph $G$, computes $\mathrm{mad}(G)$. [229]

1.8. Prove Lemma 1.10.

1.9. A graph $G$ is $k$-*critical* if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph $H$ of $G$. In particular, this implies $\delta(G) \geqslant k - 1$. Prove that for every $k \geqslant 8$ and surface $S$, the number of $k$-critical graphs embeddable in $S$ is finite.

1.10. For each $k \geqslant 9$, construct a planar graph $G_k$ with $\Delta(G_k) = k$ and $\delta(G_k^2) = \lceil \frac{9}{5}k \rceil$.

1.11. Provide the details needed to prove Theorem 1.35.

1.12. Extend the proof we gave for Theorem 1.24 to prove the analogous result for list-coloring. [426] (Essentially, the same proof works for correspondence coloring, which we will study in Section 4.4.)

1.13. Characterize degree-paintable graphs and degree-AT graphs. [219]

1.14. Strengthen Theorem 1.41 by requiring only $\mathrm{mad}(G) \leqslant \frac{5}{2}$. [94]

1.15. Give a more careful analysis of the proof of Lemma 1.43, to show that if $G$ is planar with no cycles of lengths 4 to 9, then $G$ contains at least twelve 10-suns. This is best possible, as witnessed by the truncated dodecahedron.

1.16. Give an alternate proof of Theorem 1.43 by using face charging, rather than balanced charging, as in the text. [105, Lemma 3.10]

1.17. Let G be a graph with $\Delta \geqslant 4$. Show that $\chi^i(G) \leqslant \Delta + 1$ when $\text{mad}(G) \leqslant \frac{5}{2}$ and $\chi^i(G) = \Delta$ when $\text{mad}(G) \leqslant \frac{9}{4}$. [93]

1.18. Modify the proof of Theorem 1.45 to avoid using a bank. Instead, explicitly assign each 3-vertex two $\Delta$-neighbors that it takes charge from, so that each $\Delta$-vertex loses charge to at most one 3-neighbor.

1.19. Prove that $\chi'_p(G) = \Delta$ for every planar graph G with $\Delta \geqslant 14$, by modifying the proof of Theorem 1.45. [50]

1.20. A *total coloring* assigns colors to edges and vertices so that elements receive distinct colors when they are adjacent or incident. The *total chromatic number*, $\chi''(G)$, is the smallest number of colors in a total coloring of G. Adapt the proofs of Theorem 1.45 and the previous exercise to prove that $\chi''_\ell(G) = \Delta + 1$ when G is planar with $\Delta \geqslant 14$ and $\chi''_\ell(G) \leqslant \Delta + 2$ when G is planar with $\Delta \geqslant 9$. [50]

1.21. Are the upper bounds on $\chi'_p$ in Theorem 1.45 and on $\chi'_p(G)$ in Exercise 19 also upper bounds on $AT'$? Explain why or why not? Do the bounds on $\chi''_\ell$ in the previous exercise extend to $AT''$?

# Chapter 2

# Gadgets: Constructions for Lower Bounds

gadget: a small device or machine with a particular purpose
—Cambridge Dictionary

What is so brilliant about the gadgets is their simplicity.
—Desmond Llewelyn

In the previous chapter we proved upper bounds on $\chi(G)$ and $\chi_\ell(G)$ for all G in various sparse graph classes. These included planar graphs, planar graphs with bounded girth, graphs on surfaces, and graphs with bounded maximum average degree. To understand how strong these upper bounds really are, we now seek lower bounds.

The simplest coloring lower bound is the fact that $\chi(K_n) \geqslant n$. This is true because, among the $n$ vertices, each pair must get distinct colors. A *gadget* in a graph G is a subgraph that ensures that every coloring of G satisfies some property, often that some specific set of vertices are not colored in a prescribed way. In $K_n$ the gadgets are edges, and each edge ensures that its endpoints get distinct colors. Many constructions of lower bounds for coloring and list-coloring problems are best understood from this viewpoint. Typically the graph classes that we study impose sparseness conditions that forbid large cliques. This leads us to search for more interesting gadgets. gadget

## 2.1 Girth 6 Planar Graphs with $\chi(G^2) \geqslant \Delta(G) + 2$

Since a $\Delta$-vertex and its neighbors in G induce a clique in $G^2$, every graph G satisfies $\chi(G^2) \geqslant \Delta(G)+1$. In Section 1.4 we proved that $\chi_\ell(G^2) \leqslant \Delta(G)+2$ for every planar graph G with girth at least 6 and $\Delta$ sufficiently large. It is natural to ask whether we can strengthen this upper bound to match the trivial lower one. The answer is yes when G is planar with girth at least 7 and $\Delta \geqslant 30$, as we mentioned in the Notes of Chapter 1. But for girth 6, our upper bound is best possible, as we show next.

**Theorem 2.1.** *For all* $k \geqslant 3$*, some planar graph* $G_k$ *has girth* 6*,* $\Delta = k$*, and* $\chi(G_k^2) \geqslant k+2$*.*

*Proof.* Consider graph $G'_k$ on the left in Figure 2.1, where $d(w) = k$. Let $w'$ denote the neighbor of $v$. For any $(k+1)$-coloring $\varphi$ of $(G'_k)^2$, we have $\varphi(v) \neq \varphi(x)$, as follows. By symmetry, assume that $\varphi(w) = 1$, $\varphi(w') = k+1$ and the other $k-1$ neighbors of $w$ each get a distinct color from $\{2, \dots, k\}$. So $\varphi(x) \in \{1, k+1\}$, but $\varphi(v) \notin \{1, k+1\}$; thus, $\varphi(v) \neq \varphi(x)$. Now consider $G_k$, on the right in Figure 2.1. Let $S := \{x_1, \dots, x_{k-1}, v, y, z\}$. Since $S \setminus \{v\}$ induces $K_{k+1}$ in $G_k^2$, if $\varphi$ is a $(k+1)$-coloring of $G_k^2$, then $\varphi(v) = \varphi(x_i)$ for some $i \in [k-1]$. But this contradicts our earlier analysis of $G'_k$. Thus, $\chi(G_k^2) \geqslant k+2$. $\square$

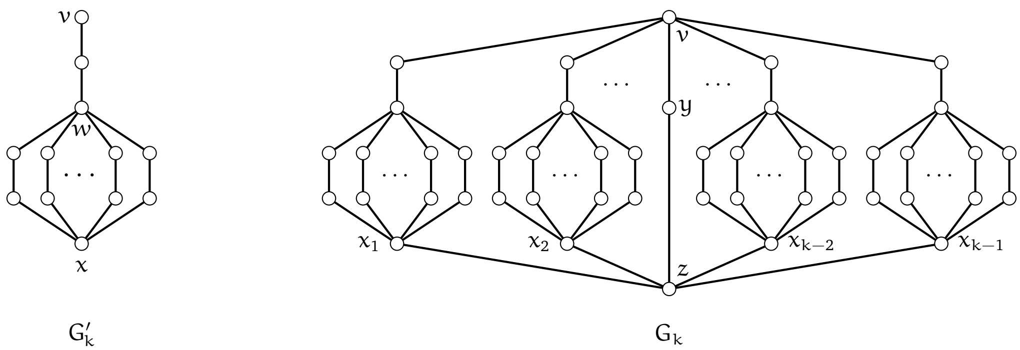



Figure 2.1: Left: In any $(k+1)$-coloring of $(G'_k)^2$, the 1-vertex $v$ and the $(k-1)$-vertex $x$ must receive distinct colors. Right: Hence, no $(k+1)$-coloring of $G_k^2$ is possible; so $\chi(G_k^2) \geqslant k+2$.

In the proof of Theorem 2.1, the gadgets are the copies of $G'_k$. Each gadget ensures that our coloring of $G_k^2$ uses distinct colors on $v$ and $x_i$, for some specific value of $i$. Together, the gadgets force the color on $v$ to differ from the colors used on all vertices in $S \setminus \{v\}$.

## 2.2 Girth 6 Graphs with Arbitrary Chromatic Number

In the 1940s and 1950s, various authors constructed triangle-free graphs with chromatic number arbitrarily large. But do there exist graphs with both chromatic number and girth arbitrarily large? Erdős answered this question affirmatively, using a probabilistic argument. However, for many years no explicit construction was known.

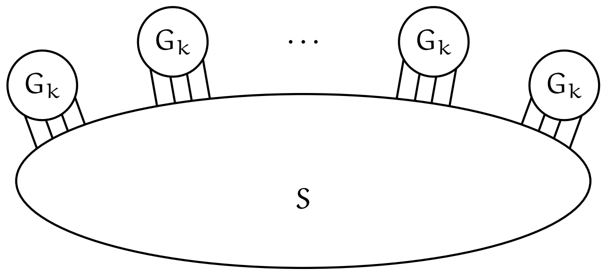



Figure 2.2: Each subset of S of size $|G_k|$ has some perfect matching to a copy of $G_k$.

Eventually, Lovász did construct such graphs deterministically, and we will do so at the end of this chapter. As a warmup, here we construct graphs with chromatic number arbitrarily large and with girth 6.

**Theorem 2.2.** *For each integer $k$, there is a graph $G_k$ with girth at least 6 and $\chi(G_k) > k$.*

*Proof.* We use induction on $k$, with base case $G_1 = K_2$. To form $G_{k+1}$, start with an independent set $I$ of size $(k+1)|G_k| - k$ and $\binom{(k+1)|G_k|-k}{|G_k|}$ disjoint copies of $G_k$. For each $|G_k|$-element subset $S$ of $I$, add any perfect matching between $S$ and the vertex set of a distinct copy of $G_k$. See Figure 2.2.

First we check the girth of $G_{k+1}$. By hypothesis, $G_k$ has girth at least 6, so any short cycle in $G_{k+1}$ must use vertices of $I$. Since $I$ is independent, and each of its vertices has at most one neighbor in each copy of $G_k$, every cycle $C$ through $I$ must visit at least two copies of $G_k$. Each vertex not in $I$ has exactly one neighbor in $I$, so $C$ must use at least four edges incident to vertices of $I$, and at least one edge within each of two copies of $G_k$. So, $G$ has girth at least 6.

Now we check that $\chi(G_{k+1}) > k+1$. Suppose instead that $G_{k+1}$ has a $(k+1)$-coloring $\varphi$. By Pigeonhole, some set of $|G_k|$ vertices of $I$ get the same color. Call such a set $T$. Now consider a copy $H$ of $G_k$ that is matched into $T$. By hypothesis, $\chi(H) \geqslant k+1$, so $\varphi$ uses every color in $[k+1]$ on $H$. However, the single color used on all vertices of $T$ must not appear on $H$, a contradiction. □

This proof uses a new idea: gadgets within gadgets. When we constructed $G_{k+1}$, each gadget was a copy of $G_k$, together with a pendent edge at each vertex. Each gadget $H$ ensured that no $(k+1)$-coloring of $G$ could use the same color on all 1-vertices of $H$. The notion of gadgets within gadgets reappears often in the rest of this chapter.

## 2.3 Non-4-Choosable Planar Graphs

When Erdős, Rubin, and Taylor introduced the notion of list-coloring [152], they conjectured that both (a) all planar graphs are 5-choosable and (b) some planar graphs are not 4-choosable. We see the proof of (a) in Theorem 11.1. Below we prove (b).

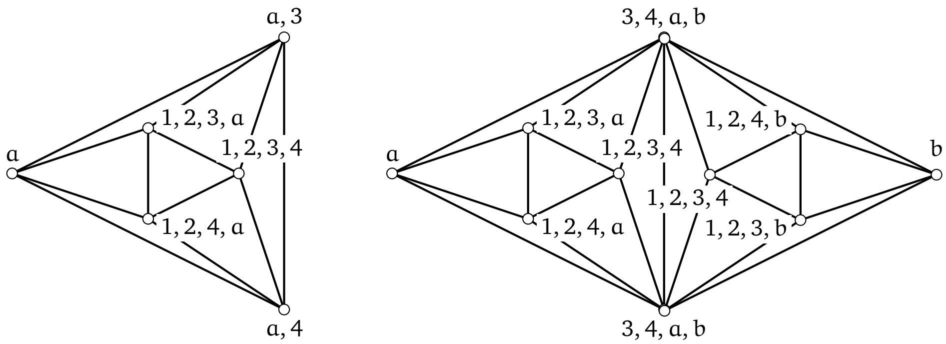



Figure 2.3: Gadget $G_1$ on the left and Gadget $G_2$ on the right.

**Theorem 2.3.** *There exist planar graphs that are not* 4*-choosable.*

*Proof.* Gadget $G_1$, on the left in Figure 2.3, cannot be colored from its lists. This is because the vertices on the outer face have only a single allowable coloring. Once we remove from the remaining lists the colors used by vertices on the outer face, we have only a triangle in which each vertex has the same list of size 2. Clearly, no coloring is possible.

Similarly, Gadget $G_2$, on the right in Figure 2.3, has no coloring from its lists. This is because $G_2$ is formed by identifying two vertices in two copies of $G_1$; in the right copy (which is rotated), color $a$ is replaced by color $b$. So, if the top vertex is colored 3 and the bottom vertex is colored 4, then the left copy of $G_1$ has no coloring. If instead the top vertex is colored 4 and the bottom vertex 3, then the right copy of $G_1$ has no coloring.

To form our non-4-choosable graph $G$, we begin with 16 copies of $G_2$, one for each choice of $a \in \{5, 6, 7, 8\}$ and $b \in \{9, 10, 11, 12\}$. We identify the leftmost vertex in each copy, giving this new vertex the list $\{5, 6, 7, 8\}$, and also identify the rightmost vertex in each copy, giving the new vertex the list $\{9, 10, 11, 12\}$. Now for any coloring of the leftmost vertex and rightmost vertex, some copy of Gadget $G_2$ has no coloring. □

The graph $G$ constructed in Theorem 2.3 has order $16(10 - 2) + 2 = 130$. Exercise 11 constructs a planar graph $H$ that is not 4-choosable and has order 63. Graph $H$ is currently the smallest known planar graph that is not 4-choosable.

## 2.4 Non-3-Choosable Girth 4 Planar Graphs

By Grötzsch's Theorem (Theorem 4.5), every planar graph with girth at least 4 is 3-colorable. We show next that the analogous statement for 3-choosability is false.

**Theorem 2.4.** *There exist planar graphs with girth* 4 *that are not* 3*-choosable.*

*Proof.* We first prove that gadget $G_1$, shown in Figure 2.4, cannot be colored from its lists. Let $f_1$ denote the 5-face incident to a vertex with list $\{a, 1, 2\}$, and $f_2$ the 5-face incident to one

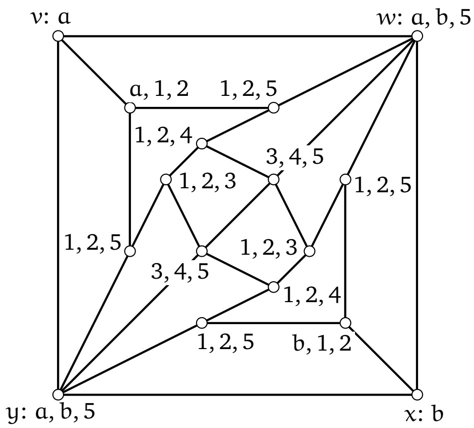



Figure 2.4: Gadget $G_1$ is not colorable from the lists shown.

with list $\{b, 1, 2\}$. Suppose $G_1$ has a coloring. Consider the diagonal path P between vertices $y$ and $w$. If the vertices of P are colored 5, 3, 4, 5, then each vertex of $f_1$ must be colored 1 or 2, a contradiction since $f_1$ is a 5-face. Similarly, if the vertices of P are colored 5, 4, 3, 5, then each vertex of $f_2$ must be colored 1 or 2, again a contradiction.

To form G we start with nine copies of $G_1$, one for each choice of $a \in \{6, 7, 8\}$ and $b \in \{9, 10, 11\}$. We identify all nine copies of $v$, and also identify all nine copies of $x$. Finally, we give the merged vertex $v$ the list $\{6, 7, 8\}$ and we give the merged vertex $x$ the list $\{9, 10, 11\}$. (The resulting graph G has order $9(16 - 2) + 2 = 128$.) Now for each coloring of $v$ and $x$, there exists a copy of $G_1$ that has no coloring from its lists. $\square$

## 2.5 Steinberg's Conjecture is False

In 1976, Steinberg conjectured that every planar graph with no 4-cycles and no 5-cycles is 3-colorable. This conjecture remained open for 40 years, being disproved only in 2016. The counterexample we will see below is similar to those in the previous two sections, but more involved, since it avoids both 4-cycles and 5-cycles.

**Theorem 2.5 (Steinberg's Conjecture is False).** *There exists a planar graph with no* 4*-cycles and no* 5*-cycles that is not* 3*-colorable.*

*Proof.* Our proof consists of four claims.

**Claim 1.** $G_1$*, shown in Figure 2.5, has no* 3*-coloring that uses the same color on all of* $v_1$*,* $v_2$*,* $v_3$.

Proof. Suppose to the contrary that $G_1$ has a 3-coloring $\varphi$ with $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = 1$. Now the path $w_1, w_2, \ldots, w_6$ must alternate colors 2 and 3. This implies that $\varphi(x_1) = \varphi(x_2) = \varphi(x_3) = 1$. However, now the 3-coloring cannot be completed, since color 1 is forbidden on each vertex of triangle $y_1y_2y_3$. $\diamondsuit$

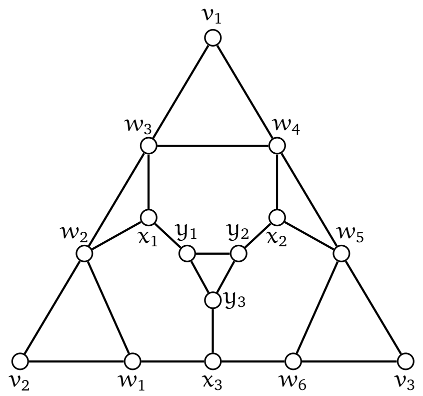



Figure 2.5: No 3-coloring of $G_1$ uses the same color on its three corner vertices: $v_1$, $v_2$, $v_3$.

**Claim 2.** *We form* $G_2$*, shown in Figure 2.6, from* 3 *copies of* $G_1$*, by identifying the instance of* $v_2$ *in each copy of* $G_1$ *with the instance of* $v_3$ *in the next* $G_1$ *(in clockwise order); we also add edges joining the three copies of* $v_1$ *(at the center of Figure 2.6). Now* $G_2$ *has no* 3*-coloring that uses the same color on all* 3 *corner vertices.*

Note that $G_1$ has one "side" of length 4 (the $v_2, v_3$-path on its outer face) and its two other sides have length 3. The point of $G_2$ is that all three sides have length 4. So we can use $G_2$ in a larger construction without any fear of creating a 5-cycle that uses edges in some copy of $G_2$.

Proof. Suppose to the contrary that $G_2$ has such a 3-coloring, with color 1 used on each corner vertex. Since the three copies of $v_1$ induce a triangle, one of them must receive color 1; call this vertex $v_1'$. However, now the copy of $G_1$ containing $v_1'$ has color 1 on all of its corner vertices, which contradicts Claim 1. $\diamondsuit$

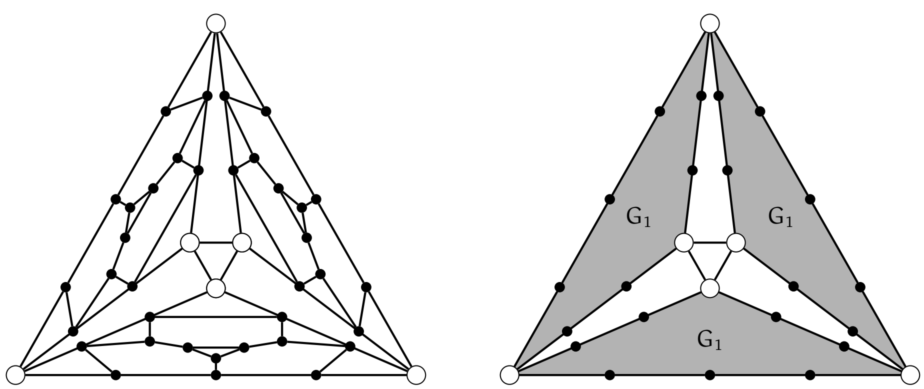



Figure 2.6: $G_2$ is shown on the left. Its abstract structure is shown on the right.

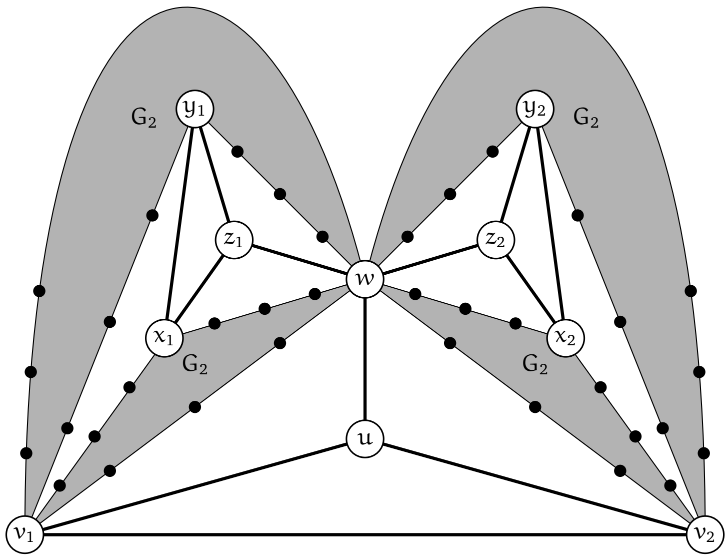



Figure 2.7: Graph G is formed from identifying corner vertices in 4 copies of $G_2$, and also adding 3 new vertices and 12 new edges.

**Claim 3.** *Form graph $G$ by identifying corner vertices in four copies of $G_2$, and also adding 3 new vertices and 12 new edges, as shown in Figure 2.7. Every proper 3-coloring of the subgraph of $G$ induced by its white vertices (the corner vertices in the four copies of $G_2$, as well as the 3 new vertices) has a copy of $G_2$ in which all three corner vertices use the same color. Thus, $G$ has no proper 3-coloring.*

Proof. Let $\varphi$ be a proper 3-coloring of the subgraph of G induced by its white vertices. Consider the subgraph of G induced by $u, w, v_1, v_2$. By symmetry between $v_1$ and $v_2$, we assume that $\varphi(w) = \varphi(v_1)$. Consider the subgraph of G induced by $w, x_1, y_1, z_1$. By symmetry between $x_1$ and $y_1$, we assume that $\varphi(w) = \varphi(x_1)$. But now $\varphi(v_1) = \varphi(w) = \varphi(x_1)$. Since $v_1$, $w$, and $x_1$ are the corners of a copy of $G_2$, this proves the first statement. By Claim 2, $\varphi$ does not extend to a proper 3-coloring of the copy of $G_2$ with $v_1$, $w$, and $x_1$ as its corners. Thus, G has no proper 3-coloring. $\diamondsuit$

To complete the proof of Theorem 2.5, we need only to verify the following claim.

**Claim 4.** *Graph $G$ has no 4-cycle and no 5-cycle.*

Proof. Suppose to the contrary that G has a cycle C of length 4 or 5. First we show C is not contained in a copy of $G_1$; suppose the contrary. Note that C contains no vertex $v_i$ (as in Figure 2.5), since for each $v_i$, the shortest path joining its neighbors, other than the edge, has length 5. Now, since C contains no $v_i$, a similar argument shows that C contains neither $w_1$ nor $w_6$, which in turn implies that C does not contain $x_3$. In this way, we show that C contains neither $w_2$, $w_5$, nor $y_3$. Now the only possible vertices of C are $y_1, y_2, x_2, w_4, w_3, x_1$, a contradiction. Thus, C is not contained in a copy of $G_1$.

Now we show C is not contained in a copy of $G_2$; suppose the contrary. The only cycle in $G_2$ with no edge in a copy of $G_1$ is the central triangle (among large vertices). Since C is not contained in a copy of $G_1$, assume C uses edges within at least two copies of $G_1$. The distance in each copy of $G_1$ between each pair of corner vertices is at least 3. So in each copy of $G_1$ where C has edges, it has at least 3 such edges. Thus, C has length at least 6.

Finally, we show that C cannot exist. Note that C must use some edge in a copy of $G_2$, but C is not contained in a copy of $G_2$. Since the distance between each pair of corner vertices in $G_2$ is 4, in each copy of $G_2$ where C has edges, it has at least 4 such edges. If C uses edges from only one copy of $G_2$, then it uses at least 2 additional edges, so has length at least 6. If C uses edges from at least two copies of $G_2$, then it has length at least 8. $\diamondsuit$

This completes the proof that Steinberg's Conjecture is false. $\square$

## 2.6 $K_3$-free Planar Graphs: Subexponentially Many 3-colorings

In most sections of this chapter, we use gadgets to construct graphs that have no coloring of a given type. Here we go in a slightly different direction. Often if a graph has one coloring of a given type, then it has many. For example, every $n$-vertex planar graph G has at least $5^{n/4}$ 5-colorings and if G has girth at least 5, then G has at least $3^{n/6}$ 3-colorings; see Section 8.5. Grötzsch's Theorem guarantees that every triangle-free planar graph has a 3-coloring. So along these line above, Thomassen conjectured that every triangle-free planar graph has exponentially many 3-colorings. Here we disprove this conjecture.

**Theorem 2.6.** *There exist infinitely many positive integers* $n$ *and* $n$*-vertex triangle-free planar graphs* $G_n$ *such that the number of* 3*-colorings of* $G_n$ *is at most* $32^{n^{\lg_{9/2} 3}} < 32^{n^{0.731}}$.

$G(v,w,k,\ell)$ *Proof.* The graph we construct will be $G(v,w,k,\ell)$, as defined recursively (using $\ell$) on the right
$V_{k,\ell}$ in Figure 2.8. For short, let $V_{k,\ell} := V(G(v,w,k,\ell))$. The base case in the recursion is $P(v,w,k)$ on the left in the figure;[1] that is, $G(v,w,k,0) := P(v,w,k)$. So the graph will consist of many vertex-disjoint copies of $P_k$ (in fact, $3^\ell$ of them), connected by some "linking" vertices, denoted by $v, w, x_1, \ldots, x_5$ in the figure. Later we will specify $k$, as a function of $\ell$. The number of vertices in the whole graph is dominated by the number of vertices in these $3^\ell$ copies of $P_k$; in particular, $|V_{k,\ell}| \geqslant k3^\ell$.

[1]The reason that we reuse vertex names from the left on the right is that the figure on the right can be formed by adding 3 copies of $G(x_i, x_{i+2}, k, \ell-1)$ each within a face of $P(v,w,5)$.

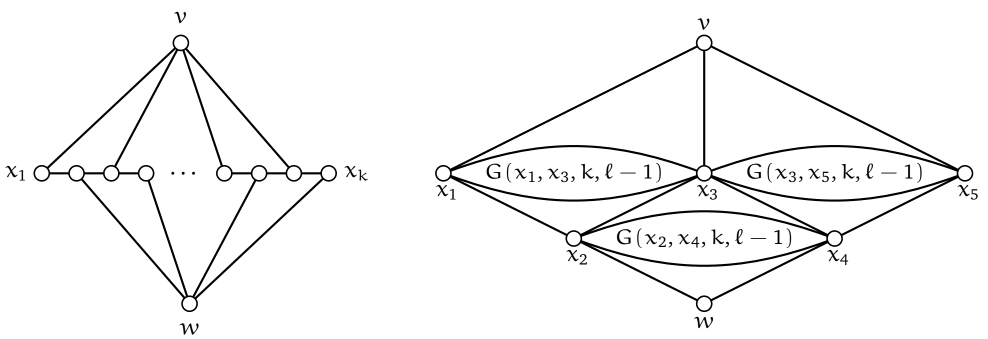



Figure 2.8: Left: The gadget $P(v, w, k)$; we also have $G(v, w, k, 0) := P(v, w, k)$. Right: $G(v, w, k, \ell)$ is defined recursively. Vertices $v$ and $w$ in each copy of $G(x_i, x_{i+2}, k, \ell - 1)$ are identified with $x_i$ and $x_{i+2}$.

Note that if $v$ and $w$ in $P(v, w, k)$ get a common color, then their coloring extends to precisely two 3-colorings of $P(v, w, k)$. A key fact is that in every 3-coloring $\varphi$ of $G(v, w, k, \ell)$, either $\varphi(x_1) = \varphi(x_3)$ or $\varphi(x_2) = \varphi(x_4)$ or $\varphi(x_3) = \varphi(x_5)$. Further, if $\varphi(v) = \varphi(w)$, then all 3 of these equalities hold. We prove both statements below, in Claim 1.

Let $V^P_{k,\ell}$ denote the subset of vertices in $V_{k,\ell}$ that appear in one of these $3^\ell$ copies of $P_k$; and let $V^L_{k,\ell} := V_{k,\ell} \setminus V^P_{k,\ell}$. (Here P and L are for "path" and "linking".) For each copy of $P(v, w, k)$ we call $v$ and $w$ the *linking vertices* for its induced subgraph $P_k$. We will count the number of ways to 3-color $G[V^L_{k,\ell}]$, and show that for each such 3-coloring, we have only a small number of ways to extend it to $V^P_{k,\ell}$. We observe that at most $2^\ell$ of the $3^\ell$ copies of $P(v, w, k)$ have distinct colors on their two linking vertices. (This holds by induction on $\ell$. If $f(\ell)$ is the maximum number of these copies whose linking vertices have distinct colors, then $f(1) = 2$ and $f(\ell) \leqslant 2f(\ell - 1)$, by Claim 1.) Thus, each of the other copies of $P(v, w, k)$ can be colored in at most 2 ways. To optimize our upper bound, we let $k := \lceil (3/2)^\ell \rceil$. Below we justify each step in the following chain of inequalities.

$V^P_{k,\ell}$

$V^L_{k,\ell}$

$$\begin{aligned}
\#\,\text{3-colorings of } G_{v,w,k,\ell} &\leqslant (\#\,\text{3-colorings of } G[V^L_{k,\ell}]) \times (\#\,\text{3-extensions to } V^P_{k,\ell} \text{ per coloring}) \\
&\leqslant (3 \times 2^{|V^L_{k,\ell}|-1}) \times ((2^k)^{2^\ell} \times 2^{3^\ell - 2^\ell}) && (2.1) \\
&\leqslant (3 \times 2^{|V^L_{k,\ell}|-1}) \times ((2^{((3/2)^\ell+1)2^\ell} \times 2^{3^\ell - 2^\ell}) \\
&\leqslant (2^{2.5\times 3^\ell+1}) \times (2^{2\times 3^\ell}) && (2.2) \\
&\leqslant 2^{5\times 3^\ell} = 32^{3^\ell} \leqslant 32^{|V_{k,\ell}|^{\lg_{9/2} 3}} < 32^{|V_{k,\ell}|^{0.731}}. && (2.3)
\end{aligned}$$

For the first factor in (2.1), we note that $G[V^L_{k,\ell}]$ is connected; as we color along a spanning tree away from its root, at each step after the first we have at most 2 available colors. For the second factor, recall that at most $2^\ell$ copies of $P(v, w, k)$ have linking vertices with distinct colors; similar to above, at each step we have at most 2 available colors. For the next two lines, ending in (2.2), we need only that $|V^L_{k,\ell}| = 2.5 \times 3^\ell - 0.5$. To see this, recall that $|V^L_{k,0}| = 2$

and $|V^L_{k,\ell}| = 3|V^L_{k,\ell-1}| + 1$; now the equality can be easily verified by substitution. The proof of (2.3) is mainly algebra; we must also recall that $|V_{k,\ell}| = \lceil (3/2)^\ell \rceil \times 3^\ell \geqslant (9/2)^\ell$.

**Claim 1.** *(a) In every 3-coloring $\varphi$ of $G(v, w, k, \ell)$, we must have either $\varphi(x_1) = \varphi(x_3)$ or $\varphi(x_2) = \varphi(x_4)$ or $\varphi(x_3) = \varphi(x_5)$; see the right of Figure 2.8. (b) Furthermore, if $\varphi(v) = \varphi(w)$, then all 3 of these equalities hold.*

Proof. Suppose that (a) is false. By symmetry, assume that $\varphi(v) = 1$ and $\varphi(x_1) = 2$. Now $\varphi(x_3) \notin \{\varphi(x_1), \varphi(v)\} = \{1, 2\}$, so $\varphi(x_3) = 3$. Similarly, $\varphi(x_5) = 2$. But now $\varphi(x_2) \notin \{\varphi(x_1), \varphi(x_3)\} = \{2, 3\}$; so $\varphi(x_2) = 1$. Likewise, $\varphi(x_4) = 1$. So, $\varphi(x_2) = \varphi(x_4)$, contradicting that the claim is false. Suppose instead that $\varphi(v) = \varphi(w) = 1$. Now $x_1, x_2, x_3, x_4, x_5$ must alternate colors 2 and 3. So all 3 equalities hold, as claimed. $\diamondsuit$

The proof of Claim 1 finishes the proof of Theorem 2.6. $\square$

## 2.7 Edge-Coloring Regular Graphs is NP-Hard

As we mentioned in Chapter 1, much of this book's content is motivated by the fact that it is NP-hard to decide whether an arbitrary input graph is $k$-colorable, for each $k \geqslant 3$. More succinctly, $k$-coloring is NP-hard. Here we prove the stronger result that $k$-edge-coloring is also NP-hard, for every $k \geqslant 3$. The proof is not conceptually difficult, but it is longer than any we have seen, and it uses a variety of gadgets.

### 2.7.1 An Overview and Proof Sketch for $k = 3$

3-SAT

truth assignment

satisfying assignment

**Definition 2.7.** An instance of 3-*SAT* consists of a set of variables $\{a_1, \ldots, a_s\}$ and a set of clauses $\{C_1, \ldots, C_t\}$, where each clause $C_i$ consists of 3 literals $\ell_{i,1}, \ell_{i,2}, \ell_{i,3}$ and each literal is either a variable $a_j$ or its negation $\overline{a_j}$; call these sets $A$ and $\mathcal{C}$. A *truth assignment* $f : A \to \{T, F\}$ assigns to each variable the value $T$ (true) or $F$ (false). A clause $C_i$ is *satisfied* by a truth assignment $f$ if at least one of its literals is true, i.e., if $\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3}$ is true. A truth assignment $f$ is a *satisfying assignment* for an instance of 3-SAT if it satisfies every clause in $\mathcal{C}$.

k-regular-EC

Given a satisfying assignment $f$ for an instance of 3-SAT, it is easy to check that $f$ is indeed satisfying. However, it is NP-Hard to determine whether such an assignment exists. Let $k$-*regular-EC* denote the problem of determining whether a given $k$-regular input graph is $k$-edge-colorable. The goal of Section 2.7 is to prove the following theorem.

**Theorem 2.8.** *For each $k \geqslant 3$, the problem $k$-regular-EC is NP-hard (even for simple graphs).*

To prove this, we reduce 3-SAT to $k$-regular-EC. More precisely, fix an integer $k \geqslant 3$ and an instance $S$ of 3-SAT. We construct an instance $G(S)$ of $k$-regular-EC that has a Yes answer if and only if $S$ has a Yes answer. Further, the time it takes to construct $G(S)$, and hence its size, is bounded by a polynomial in the size of $S$.

Our general approach is to construct gadgets in $G(S)$ corresponding to each clause in $S$, to each variable in $S$, and to each negation of a variable in a clause in $S$. The gadgets for each clause and each negation have constant size. The gadget for each variable has size linear in the number of clauses where it appears. Thus, $|G(S)|$ is in fact linear in $|S|$.

Before proving the theorem for general $k$, we sketch the proof for $k = 3$, which is the simplest case. Information is transmitted from one gadget to another by a pair of edges incident to both gadgets. If the edges receive the same color (in some, possibly partial, 3-edge-coloring), then the variable is "true"; otherwise, it is "false". This idea is central to the construction, and keeping it firmly in mind while reading this section will aid the reader greatly. We need the following three lemmas, the proofs of which we omit.[2] Intuitively, these lemmas say that the gadgets behave as we expect. (Note that we are coloring edges, so each instance of $v, w, \ldots$ refers to an edge, rather than a vertex.)

"true", "false"

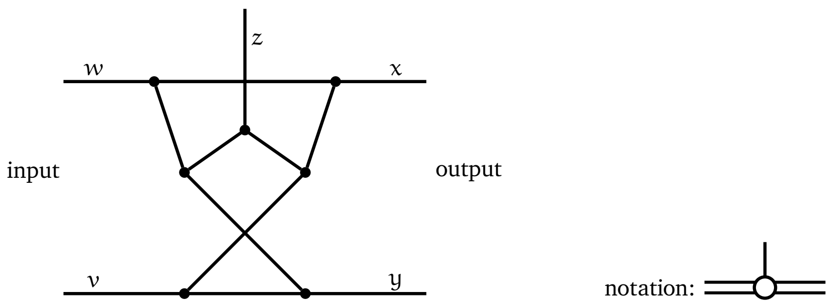



Figure 2.9: Left: A negation gadget. Right: The way we draw the negation gadget in later constructions.

**Lemma 2.9.** *In any 3-edge-coloring $\varphi$ of the negation gadget $N$ in Figure 2.9, the following hold:*

*(1)* $\varphi(v) = \varphi(w)$ *or* $\varphi(x) = \varphi(y)$*;*

*(2)* $\varphi(v) = \varphi(w)$ *implies* $\varphi(x) \neq \varphi(y)$ *and* $\varphi(x) \neq \varphi(z)$ *and* $\varphi(y) \neq \varphi(z)$*;*

*(3)* $\varphi(x) = \varphi(y)$ *implies* $\varphi(v) \neq \varphi(w)$ *and* $\varphi(v) \neq \varphi(z)$ *and* $\varphi(w) \neq \varphi(z)$.

*Every 3-edge-coloring $\varphi$ of $v, w, x, y, z$ that satisfies (1), (2), (3) above can be extended to a 3-edge-coloring of $N$.*

**Lemma 2.10.** *In any 3-edge-coloring of a variable-setting gadget (Figure 2.10 shows an example of such a gadget), either all output pairs are "true" or all output pairs are "false". Furthermore, both settings are possible.*

[2]But the astute reader will notice that these lemmas are all generalized in Section 2.7.2, and the proofs of those generalizations imply these lemmas as special cases.

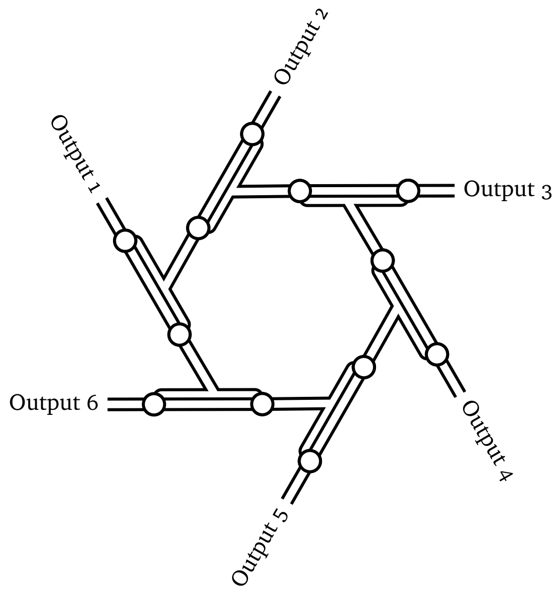



Figure 2.10: A variable-setting gadget with 6 output pairs, built from 12 negation gadgets.

**Lemma 2.11.** *In any* 3*-edge-coloring of a clause-testing gadget, shown in Figure 2.11, at least one pair of input edges must be "true". Further, any* 3*-edge-coloring of the input edges with at least one pair "true" can be extended to a* 3*-edge-coloring of the whole gadget.*

Now we specify how to link up the gadgets. Suppose that variable $a_i$ appears as the second literal in clause $C_j$ and this is the $r$th clause in which $a_i$ appears. Now we identify the pair of edges in the $r$th output of the variable-setting gadget $U_i$ for $a_i$ with the pair of edges in the second input for the clause-testing gadget for $C_j$ (similarly, if $a_i$ is the first or third literal in $C_j$). If instead $\overline{a_i}$ is the literal in $C_j$, then we do the same thing, but first insert a negation gadget. Specifially, we identify the output edges of $U_i$ with input edges of the negation gadget, and identify the output edges of the negation gadget with the input edges of the gadget for $C_j$.

To ensure $G(S)$ is 3-regular, we must handle all its pendent edges. We simply take two copies of the graph thus far constructed, $G'(S)$, and identify each pair of corresponding pendent edges. The resulting graph $G(S)$ is clearly 3-regular. And $G(S)$ is 3-edge-colorable if and only if $G'(S)$ is. Finally, $G(S)$ is 3-edge-colorable if and only if $S$ is satisfiable.

### 2.7.2 $k$-Edge-Coloring $k$-Regular Graphs is NP-Hard

In this section we prove Theorem 2.8, for all $k \geqslant 3$. The proof is not difficult, but allowing $k > 3$ introduces numerous complications. Although the proofs of the three lemmas from the previous section (which we generalize here) essentially amount to case-checking, there are many cases to check, so at times we omit details to focus on the more interesting ideas.

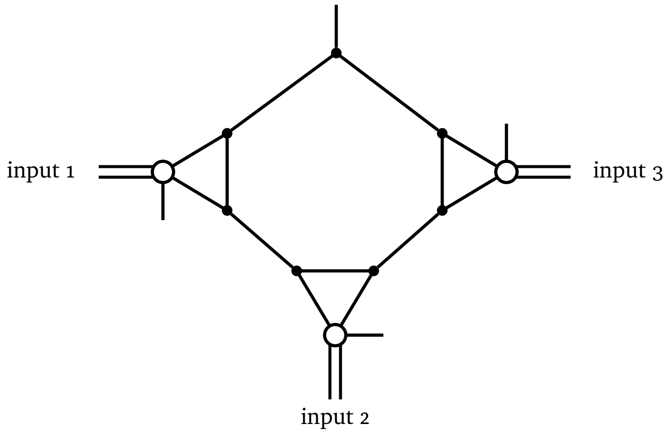



Figure 2.11: Each clause-testing gadget has a 3-edge-coloring that extends the coloring of its input edges if and only if at least one pair of input edges receives the same color. Thus, it checks to see if at least one of its inputs is "true".

When reducing an instance of 3-SAT, we prefer that the resulting instance of k-regular-EC be a simple graph. But for most steps of the reduction it is more convenient to work with multigraphs. So we begin the section by showing how to transform a multigraph instance of k-regular-EC to an equivalent simple graph instance of the same problem (keeping k unchanged). We use the following two lemmas; their proofs are the only places in this section that we explicitly name vertices.

**Lemma 2.12.** *There exists a gadget $M_k$ with $k$ vertices of degree $1$ and all other vertices of degree $k$ such that a $k$-edge-coloring of its pendent edges extends to a $k$-edge-coloring of $M_k$ if and only if the pendent edges all receive distinct colors. (Figure 2.12 shows $M_5$.)*

*Proof.* All subscript addition throughout this proof is modulo $k$. Let $V(M_k) := \{v_1, \ldots, v_k, w_1, \ldots, w_k, x_1, \ldots, x_{k-3}\}$ and $E(M_k) := \{v_i w_i, w_i w_{i+1}, w_i x_j \mid i \in [k], j \in [k-3]\}$. $M_k$ $v_i, w_j, x_\ell$

We first prove that if $\varphi$ is a coloring of the pendent edges in $M_k$ that gives each edge a distinct color, then $\varphi$ extends to a $k$-edge-coloring of $M_k$. By permuting colors, we assume that $\varphi(v_i w_i) = i \pmod k$. To extend this to $M_k$, let $\varphi(w_i w_{i-1}) := i + k - 2 \pmod k$ and $\varphi(w_i x_j) := i + j \pmod k$ for all $i \in [k]$ and $j \in [k-3]$. For each $w_i$ and each $x_j$, we can check that all incident edges use distinct colors. So the resulting edge-coloring is proper.

Assume instead that $\varphi$ is a proper $k$-edge-coloring of $M_k$ and that some color $\alpha$ is used on at least two pendent edges. Either $\alpha$ is used on at least one edge $w_i w_{i+1}$ or else some color $\beta$ is used on at least two such edges. In each case, either $\alpha$ or $\beta$ is used on edges of the form $v_i w_i$ and/or $w_i w_{i+1}$ that are incident to at least four vertices $w_i$. However, each color in $[k]$ is used on exactly $k-3$ edges of the form $w_j x_\ell$. By Pigeonhole, some $w_i$ has at least two incident edges with the same color. This contradicts that $\varphi$ is proper, which completes the proof. $\square$

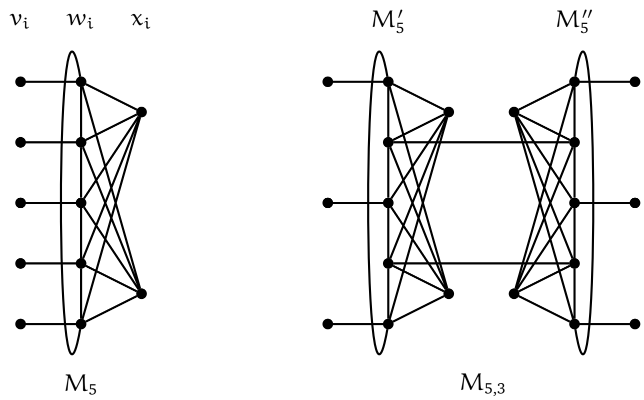



Figure 2.12: Left: $M_5$, a building block for simulating parallel edges in a 5-regular graph. Right: $M_{5,3}$, which simulates 3 parallel edges in a 5-regular graph.

**Lemma 2.13.** *For each* $k$*-regular multigraph* $G$*, there exists a* $k$*-regular simple graph* $G'$ *such that* $G'$ *has a* $k$*-edge-coloring if and only if* $G$ *does. Further,* $\|G'\|$ *is bounded by a polynomial in* $\|G\|$*.*

*Proof.* To simulate $i$ parallel edges in a $k$-regular graph, we start with two copies of $M_k$, say
$M_k', M_k''$ $M_k'$ and $M_k''$. We pair each of $k-i$ pendent edges in $M_k'$ with $k-i$ pendent edges in $M_k''$, and identify the two edges in each pair; call the resulting gadget $M_{k,i}$. (Figure 2.12 shows $M_{5,3}$.) A $k$-edge-coloring of the pendent edges in $M_{k,i}$ extends to all of $M_{k,i}$ if and only if the $i$ pendent edges from $M_k'$ use the same distinct colors as the $i$ pendent edges from $M_k''$. This follows directly from Lemma 2.12.

Starting from G, we repeatedly replace $i$ parallel edges (for some $i$ with $2 \leqslant i \leqslant k-1$) having endpoints $y$ and $z$ with a copy of $M_{k,i}$; we identify each degree 1 vertex inherited from $M_k'$ with $y$ and each degree 1 vertex inherited from $M_k''$ with $z$. We iteratively apply these replacements until the resulting graph $G'$ is simple. By induction on the number of replacements, we can prove that $G'$ has a $k$-edge-coloring if and only if G does. □

Throughout this section, we often use multigraphs. In actually constructing the reduction from an instance of 3-SAT to one of k-regular-EC, we would perform the transformation detailed in Lemmas 2.12 and 2.13 *after* applying the constructions in the remainder of the section. (We simply present the lemmas in the order that we do to aid the reader's understanding.)

Our gadgets now mainly generalize our earlier gadgets. More precisely, our gadgets here are multigraphs that have as their underlying simple graphs the gadgets in the previous section.
$(i)$ To denote that an edge has multiplicity $i$ we write $(i)$. We begin with the following easy counting result, which helps us understand more about proper $k$-edge-colorings.

**Lemma 2.14 (Parity Condition).** *Let* $H$ *be a graph with every vertex of degree either* $k$ *or* $1$ *(and no isolated edges), and let* $E'$ *denote the set of edges in* $H$ *with an endpoint of degree* $1$. *Let* $\varphi$ *denote a* $k$*-edge-coloring of* $H$. *If* $b_i$ *denotes the number of edges in* $E'$ *colored* $i$*, for each* $i \in [k]$*, then* $b_1 \equiv \cdots \equiv b_k \pmod 2$. $b_i$

*Proof.* Choose arbitrary distinct $i, j \in [k]$ and let $H_{i,j}$ denote the subgraph of $H$ induced by the edges colored $i$ and $j$. Each component of $H_{i,j}$ is either an even cycle or a path (with both endpoints in $E'$), so each component contributes either $0$ or $2$ to the sum $b_i + b_j$. This sum is thus even, so $b_i \equiv b_j \pmod 2$. Since $i$ and $j$ are arbitrary, the lemma follows. □

**Lemma 2.15.** *Fix* $i \in [k-2]$. *If* $\varphi$ *is a proper* $k$*-edge-coloring of the generalized negation gadget* $N_i$*, shown in Figure 2.13, then each of the* $k$ *colors must appear at* $v$*,* $w$*,* $x$*,* $y$*, or* $z$*. Further,* $k-1$ *colors each appear once and one color appears* $3$ *times.*

*Proof.* Suppose, to the contrary, that some color appears on none of $v$, $w$, $x$, $y$, or $z$. By symmetry, say color 1 is missing. In a $k$-edge-coloring, every color appears at (incident to) every $k$-vertex. Since $N - \{v, w, x, y, z\}$ has 7 vertices, its matching number is (at most) 3. Hence, no proper edge-coloring uses color 1 at each of its 7 vertices, which is a contradiction. Thus, each color appears on at least one of $v, w, x, y, z$. This proves the first statement.

The second statement follows directly from the first, together with the Parity Condition. For the sets of parallel edges $v$ and $y$, we can also view each set of edges as having one endpoint in common (the one of degree $k$) and having all other endpoints distinct. This does not change whether or not an edge-coloring is proper. Now the Parity Condition applies. By the first statement, $b_i \geqslant 1$ for all $i \in [k]$. If some $b_i$ is even, then all $b_i$ are positive and even, so $|E'| \geqslant k(2) > k+2$, a contradiction. Thus, each $b_i$ is odd. So one $b_i$ is 3, and the others are 1. This proves the second statement. □

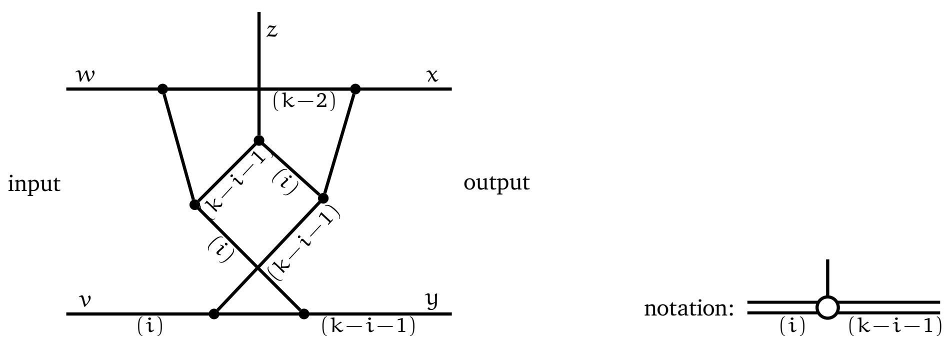



Figure 2.13: Left: A generalized negation gadget, $N_i$. We write $(i)$ to denote an edge of multiplicity $i$; this notation is omitted for edges with multiplicity 1. Right: The way we draw $N_i$ in later constructions.

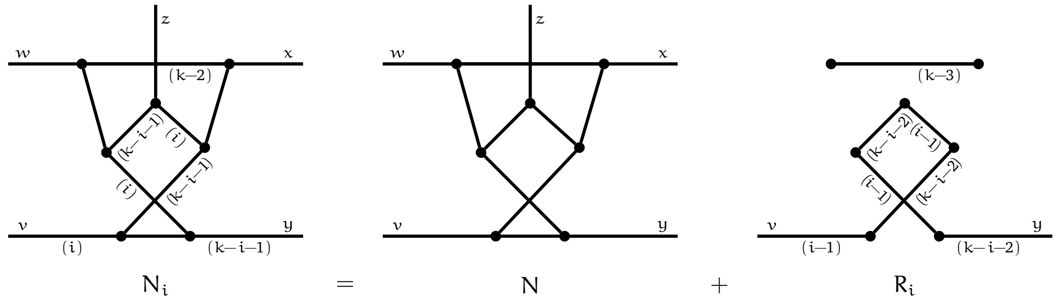



Figure 2.14: Left: A generalized negation gadget, $N_i$, decomposed into a negation gadget $N$ and a $(k-3)$-regular subgraph, $R_i$.

For an edge-coloring $\varphi$ and multi-edges $v$ and $w$ (possibly with multiplicity 1), we write
$\varphi(v) \equiv \varphi(w)$ $\varphi(v)$ and $\varphi(w)$ to denote the sets of colors used by $\varphi$ on $v$ and $w$. We write $\varphi(v) \equiv \varphi(w)$ when at least one color used on $v$ is also used on $w$, i.e., when $\varphi(v) \cap \varphi(w) \neq \emptyset$; otherwise,
$\varphi(v) \not\equiv \varphi(w)$ we write $\varphi(v) \not\equiv \varphi(w)$.

**Lemma 2.16 (Generalization of Lemma 2.9).** *In any $k$-edge-coloring $\varphi$ of the generalized negation gadget $N_i$ in Figure 2.13, the following 3 conditions hold:*

*(1)* $\varphi(v) \equiv \varphi(w)$ *or* $\varphi(x) \equiv \varphi(y)$*;*

*(2)* $\varphi(v) \equiv \varphi(w)$ *implies* $\varphi(x) \not\equiv \varphi(y)$ *and* $\varphi(x) \not\equiv \varphi(z)$ *and* $\varphi(y) \not\equiv \varphi(z)$*;*

*(3)* $\varphi(x) \equiv \varphi(y)$ *implies* $\varphi(v) \not\equiv \varphi(w)$ *and* $\varphi(v) \not\equiv \varphi(z)$ *and* $\varphi(w) \not\equiv \varphi(z)$.

*Every $k$-edge-coloring $\varphi$ of $v, w, x, y, z$ that satisfies conditions (1), (2), and (3) above can be extended to a $k$-edge-coloring of $N_i$.*

*Proof.* To prove (1), assume the contrary, that $\varphi(v) \not\equiv \varphi(w)$ and $\varphi(x) \not\equiv \varphi(y)$. By Lemma 2.15 some color appears three times at $v$, $w$, $x$, $y$, and $z$. By symmetry, we assume this color is 1. Since $\varphi(v) \not\equiv \varphi(w)$ and $\varphi(x) \not\equiv \varphi(y)$, we must have $\varphi(z) = 1$. We consider three cases.

**Case 1: $\boldsymbol{\varphi(w) \equiv \varphi(x)}$.** Let $u$ denote the multi-edge adjacent to both $w$ and $x$. Consider the $k-1$ edges that share an endpoint with $z$. Since $\varphi(z) = 1$, none of these edge can use color 1, and none can use the other color absent from $u$. Thus, these $k-1$ edges use at most $k-2$ colors, which is a contradiction.

**Case 2: $\boldsymbol{\varphi(w) \equiv \varphi(y)}$ or $\boldsymbol{\varphi(v) \equiv \varphi(x)}$.** By horizontal symmetry (and replacing $i$ with $k-i-1$), assume that $\varphi(w) \equiv \varphi(y)$. Now consider the vertex at distance one from one endpoint of each of edges $w$, $y$, and $z$. This vertex has degree $k$, but no incident edge uses color 1, which is a contradiction.

**Case 3: $\boldsymbol{\varphi(v) \equiv \varphi(y)}$.** Let $u$ denote the edge adjacent to both $w$ and $x$. Since color 1 is used incident to each vertex, and it is used on $v$, $y$, and $z$, color 1 is used on the edges other than $u$ that share an endpoint with $w$ and $x$. However, now $u$ cannot use 1 or $\varphi(w)$ or $\varphi(x)$. So at most $k-3$ colors are available to use on $u$, which has multiplicity $k-2$; this contradiction proves condition (1).

Now we prove conditions (2) and (3). By Lemma 2.15, one color appears three times at $v$, $w$, $x$, $y$, and $z$, and every other color appears once. So if $\varphi(v) \equiv \varphi(w)$, then $\varphi(x)$, $\varphi(y)$, and $\varphi(z)$ are disjoint. Similarly, if $\varphi(x) \equiv \varphi(y)$, then $\varphi(v)$, $\varphi(w)$, and $\varphi(z)$ are disjoint. This proves conditions (2) and (3).

To prove the final statement, we assume that color 1 appears three times at $v$, $w$, $x$, $y$, and $z$. Further, we assume that $\varphi(v) \equiv \varphi(w)$. The case $k = 3$ is easy to check. Either $\varphi(x) = 1$ or $\varphi(y) = 1$ or $\varphi(z) = 1$; by permuting colors 2 and 3, we have only three possibilities for $\varphi$ on $v$, $w$, $x$, $y$, and $z$. In each instance we have only one way to extend the matching colored 1, and the remaining uncolored edges induce a path. We omit the details, but the interested reader should be able to recreate them quickly. Now we reduce the case $k \geqslant 3$ to the case $k = 3$.

We decompose $N_i$ into two subgraphs, as in Figure 2.14: one is $N$, from the case $k = 3$, and the other is $R_i$, a regular graph of degree $k-3$. At each of $v$, $w$, $x$, $y$, and $z$ where color 1 does not appear, we choose a color that does appear there; by permuting colors, we assume that the other colors we choose are 2 and 3. The $k$-edge-coloring of $v$, $w$, $x$, $y$, and $z$ naturally induces a 3-edge-coloring $\varphi'$ of the same edges in $N$ and a proper edge-coloring $\varphi''$ of $v$ and $y$ in $R_i$ with colors $4, \ldots, k$. We extend $\varphi'$ to $N$ by the case $k = 3$ above. To extend $\varphi''$ to $R_i$, we color greedily. (This graph is two paths with each vertex of degree $k-3$, so we have no choice in this process.) $\square$

The *double negation gadget* is formed from two copies of the generalized negation gadget $N_1$, by identifying their output edges, as shown in Figure 2.15 (where the identified edges on the left are $x$ and $y$ $(k-2)$). The *generalized variable-setting gadget* is formed from multiple copies of the generalized negation gadget in exactly the same way as the variable-setting gadget is formed from the negation gadget; again, see the example in Figure 2.10.

**Lemma 2.17 (Generalization of Lemma 2.10).** *In every $k$-edge-coloring of the generalized variable-setting gadget, either all outputs are "true" or all outputs are "false". Further, both settings are possible in such a way that every output edge is colored* 1, 2, *or* 3.

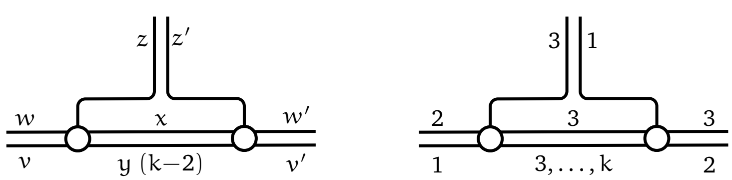



Figure 2.15: Left: A double negation gadget. Right: The gadget properly $k$-edge-colored, with all outputs "false".

*Proof.* Consider the double negation gadget shown in Figure 2.15, and fix some $k$-edge-coloring $\varphi$ of it. If $\varphi(v) \equiv \varphi(w)$, then $\varphi(x)$, $\varphi(y)$, and $\varphi(z)$ are disjoint, by Lemma 2.16. Since $\varphi(x) \not\equiv \varphi(y)$, we get $\varphi(v') \equiv \varphi(w')$. Thus, $\varphi(z') \not\equiv \varphi(x)$ and $\varphi(z') \not\equiv \varphi(y)$. This implies that $\varphi(z) \equiv \varphi(z')$, since a total of $k-1$ colors appear on $x$ and $y$. By induction, if one output pair of the variable-setting gadget is true, then so are all output pairs.

Now we prove the second statement of the lemma. Consider the $k$-edge-coloring of the double negation gadget that uses color 1 on each of edges $v$, $w$, $z$, $v'$, $w'$, and $z'$, uses color 2 on $x$, and uses colors $3, \ldots, k$ on $y$. It is easy to overlap copies of this coloring to properly $k$-edge-color every generalized variable-setting gadget so that each output is true (in fact, every output edge is colored 1). To get a $k$-edge-coloring with all output pairs set to false, we start from the coloring on the right in Figure 2.15. By permuting colors, and reflecting horizontally, we get a total of 12 colorings. It is easy to check that these twelve can be combined to yield the desired $k$-edge-coloring. □

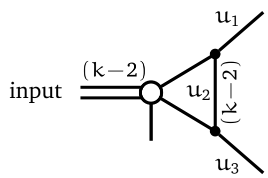



Figure 2.16: An "extended" negation gadget.

Figure 2.16 shows an *extended negation gadget*. We need the following easy observation.

**Lemma 2.18.** *Let $\varphi$ be a $k$-edge-coloring of the two input edges of an extended negation gadget, as well as $u_1$ and $u_3$. This $k$-edge-coloring of these four edges extends to a $k$-edge-coloring of the whole extended negation gadget if and only if either (a) the input is true and $\varphi(u_1) \not\equiv \varphi(u_3)$ or (b) the input is false and $\varphi(u_1) \equiv \varphi(u_3)$.*

*Proof.* The lemma is implied by Lemma 2.16. □

Figure 2.17 shows a *generalized clause-testing gadget*. Perhaps unsurprisingly, its underlying simple graph is the clause-testing gadget that we saw earlier, from the case $k = 3$. We show that a coloring of its input edges extends to a coloring of the whole gadget if and only if at least one input is true. Proving necessity is quite easy. To prove sufficiency, we decompose the gadget into a copy of its underlying simple graph and a matching in which each edge has multiplicity $k-3$. We use colors 1, 2, and 3 on the simple graph, and use the remaining colors on the multi-matching.

**Lemma 2.19 (Generalization of Lemma 2.11).** *(1) In any $k$-edge-coloring of a generalized clause-testing gadget, shown in Figure 2.17, at least one of the inputs must be true (a common color appears on the two edges of the input pair). (2) If $\varphi$ is a $k$-edge-coloring of the input edges satisfying (1), then $\varphi$ can be completed to a $k$-edge-coloring of the gadget.*

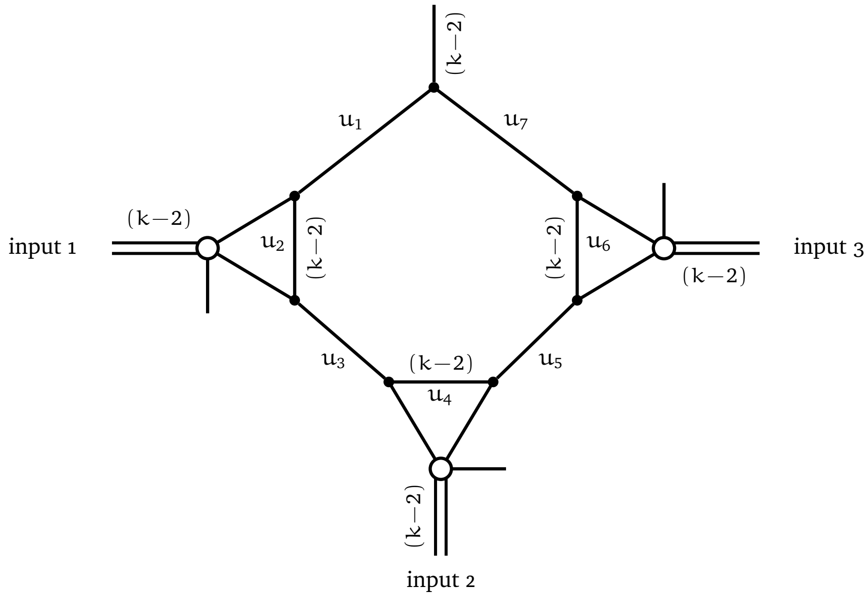



Figure 2.17: Each generalized clause-testing gadget has a k-edge-coloring that extends the coloring of its input edges if and only if at least one pair of input edges receives the same color. Thus, it checks to see if at least one of its inputs is "true".

*Proof.* First, we prove (1). Suppose instead that each input is false. By Lemma 2.18, $\varphi(u_1) \equiv \varphi(u_3) \equiv \varphi(u_5) \equiv \varphi(u_7)$, contradicting that $u_1$ and $u_7$ share an endpoint.

Now we prove (2). We first reduce the general case to the case $k = 3$. To begin, we restrict to a 3-edge-coloring of the input edges by ignoring multiple edges. For each input that is true, we pick the repeated color on its two edges. For each input that is false, we arbitrarily pick two colors, so that the total number of colors we pick is at most three. By symmetry, assume these colors are in $\{1, 2, 3\}$. We write *simple clause-testing gadget* to denote the clause-testing gadget when $k = 3$. If we can handle the case $k = 3$, then we can extend the resulting 3-edge-coloring of the simple clause-testing gadget to a k-edge-coloring of the whole clause-testing gadget: we just use colors $4, \ldots, k$ on each set of parallel edges that are not yet colored. Thus, we need only to consider the case $k = 3$.

Note that whether or not we can extend a 3-edge-coloring of the inputs of a simple clause-testing gadget to the whole gadget depends only on whether each input is true or false, not on the colors used on the edges of the input. This follows directly from Lemma 2.18, with $k = 3$.

To extend a coloring of its inputs to the whole simple clause-testing gadget, it suffices to extend it to $u_1$, $u_3$, $u_5$, $u_7$ in a way consistent with the hypotheses of Lemma 2.18, for each of the three copies of the extended negation gadget contained within. This amounts to 3-coloring the vertices of a 4-cycle, $u_1u_3u_5u_7$, where each successive pair of vertices is required to have either the same color or distinct colors (depending on whether the corresponding input is false or true). To color this 4-cycle, we contract each edge joining successive vertices that must use the same color. This results in a 2-cycle, 3-cycle, or 4-cycle that we must properly 3-color, since at least one input is true and also $u_7$ and $u_1$ must always use distinct colors. Such a coloring always exists, which proves the lemma. □

The way we connect (generalized) variable-setting gadgets with (generalized) clause-testing gadgets is similar to what we did for $k = 3$. However, now each input in a clause-testing gadget has one edge of multiplicity 1 and one edge of multiplicity $k-2$. If a variable appears negated in a clause, then we connect the output of the variable-setting gadget to the generalized negation gadget $N_1$, which has an output where one edge has multiplicity 1 and one edge has multiplicity $k-2$. This matches the desired input of the clause-testing gadget, so we identify the relevant edges and we are done. However, if the variable appears in the clause, but is not negated, then we must "add" $k-3$ parallel edges to the output of the variable-setting gadget before we can identify the edges of that output with the input edges of the clause testing gadget. So we need one final gadget.

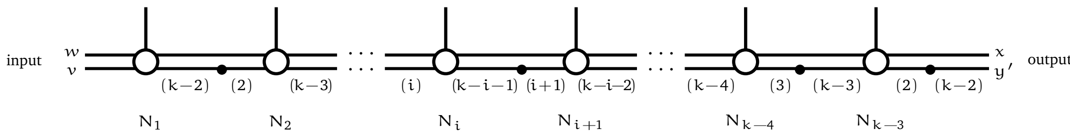



Figure 2.18: An edge-adding gadget.

An *edge-adding gadget* is formed from generalized negation gadgets $N_1, \ldots, N_{k-3}$ by identifying, for each $i \in [k-4]$, the output edge $x$ in $N_i$ with the input edge $w$ in $N_{i+1}$ and identifying one endpoint of $y$ in $N_i$ with one endpoint of $v$ in $N_{i+1}$. Finally, we add $k-2$ parallel edges incident to an endpoint of $y$ in $N_{k-3}$; the other endpoint of these new edges is $y'$. See Figure 2.18.

**Lemma 2.20.** *Each* $k$*-edge-coloring* $\varphi$ *of an edge-adding gadget satisfies* $\varphi(v) \equiv \varphi(w)$ *if and only if* $\varphi(x) \equiv \varphi(y')$.

*Proof.* More generally, the $i$-*edge-adding gadget* (for each $i \in [k-3]$) connects and identifies the edges of $N_1, \ldots, N_i$ as in the edge-adding gadget, and then adds $i+1$ parallel edges incident to an endpoint of edge $y$ in $N_i$. We let $v$ and $w$ denote the pendent edges (with multiplicity) in $N_1$ and $x$ and $y'$ denote the pendent edges (with multiplicity) in $N_i$.

We prove the more general statement that each $k$-edge-coloring $\varphi$ of the $i$-edge-adding gadget has $\varphi(v) \equiv \varphi(w)$ if and only if $\varphi(x) \equiv \varphi(y')$. We use induction on $i$. The base case follows from Lemma 2.16, with $i = 1$. This shows that $\varphi(v) \equiv \varphi(w)$ if and only if $\varphi(x) \not\equiv \varphi(y)$. Since $y$ and $y'$ together use each color in $[k]$ exactly once, $\varphi(x) \equiv \varphi(y')$ if and only $\varphi(x) \not\equiv \varphi(y)$. Thus, $\varphi(v) \equiv \varphi(w)$ if and only if $\varphi(x) \equiv \varphi(y')$, as desired. The induction step is similar. By hypothesis, we have $\varphi(v) \equiv \varphi(w)$ if and only if $\varphi(x) \equiv \varphi(y')$, in the $(i-1)$-edge-adding gadget. By Lemma 2.16, in $N_i$ we also have $\varphi(v) \equiv \varphi(w)$ if and only if $\varphi(x) \equiv \varphi(y')$. Since $x$ and $y'$ in the $(i-1)$-edge-adding gadget are also $v$ and $w$ in $N_i$, the more general statement is proved. □

We have now finished all of the ingredients needed to prove our main result (which we restate below). At this point, the reader will likely find it clear how they fit together. But, for easy reference, we gather most of the details in one place.

**Theorem 2.21.** *For each integer $k \geqslant 3$, the problem $k$-regular-EC is NP-hard. This remains true, even when the input graph must be simple.*

*Proof.* We reduce 3-SAT to $k$-regular-EC. Let $S$ be a given instance of 3-SAT, with variables $a_1, \ldots, a_s$, and with clauses $C_1, \ldots, C_t$. Fix an integer $k \geqslant 3$. For each clause, we create a generalized clause-testing gadget. For each variable $a_i$, let $n_i$ denote the number of clauses in which $a_i$ appears. For each $a_i$, we create a generalized variable-setting gadget with $n_i$ outputs. We "connect" the edges in output $j$ of the variable-setting gadget with the input edges in the $j$th clause where $a_i$ appears. If $a_i$ is negated in that clause, then we identify the output edges with the input edges of a generalized negation gadget $N_1$, and identify its output edges with the edges in one input of the clause-testing gadget. If the variable does not appear negated in the clause, then we create an edge-adding gadget, identify its input edges with the output edges of the variable-setting gadget, and identify its output edges with the input edges of the clause-testing gadget.

The resulting graph is neither simple nor regular; but each degree is either $k$ or $1$. To fix these degrees, we take two copies of the graph and identify each pendent edge in one copy with its corresponding edge in the other; this yields a $k$-regular multigraph. To reach a simple graph, we repeatedly substitute gadget $M_{k,i}$ for a set of $i$ parallel edges. Ultimately, this yields a simple $k$-regular graph $G(S)$. It is straightforward, though tedious, to check that $S$ has a satisfying assignment if and only if $G(S)$ has a proper $k$-edge-coloring. □

## 2.8 Chromatic Number and Girth both Arbitrarily Large

To conclude this chapter, we explicitly construct graphs with chromatic number and girth both arbitrarily large. By modifying this construction a bit, we also get *bipartite* graphs with girth and choice number arbitrarily large.

**Definition 2.22.** A *complete $d$-ary tree of height $h$* is a rooted tree in which each internal vertex has $d$ children and every leaf is distance $h$ from the root. An *$r$-augmented tree* consists of a rooted tree, called the *underlying tree*, plus edges from each leaf to $r$ of its ancestors (these are *augmenting edges*). For integers $d$, $r$, and $g$ a *$(d, r, g)$-graph* is a bipartite $r$-augmented complete $d$-ary tree with girth at least $g$. Let $h(d, r, g)$ denote the minimum height of a $(d, r, g)$-graph. See Figure 2.19.

complete $d$-ary tree of height $h$
$r$-augmented tree
underlying tree
$(d, r, g)$-graph
$h(d, r, g)$

The goal of this section is to prove the following three theorems.

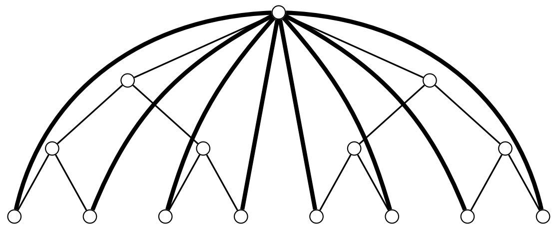


Figure 2.19: A (2,1,4)-graph, with augmenting edges in bold; hence, $h(2, 1, 4) = 3$. The underlying tree is a complete 2-ary tree of height 3.

**Theorem 2.23.** *For all* $d$*,* $r$*, and* $g$*, the value of* $h(d, r, g)$ *is finite. In other words,* $(d, r, g)$*-graphs always exist.*

**Theorem 2.24.** *For all integers* $g$ *and* $k$*, there exists a graph* $G_{g,k}$ *with girth at least* $g$ *and chromatic number greater than* $k$*. Further, we can construct* $G_{g,k}$ *explicitly.*

**Theorem 2.25.** *For all integers* $g$ *and* $k$ *there exists a bipartite graph* $B_{g,k}$ *with girth at least* $g$ *such that mad*$(H) \leqslant 2(k-1)$ *for every proper subgraph* $H$ *of* $B_{g,k}$*, but* $\chi_\ell(B_{g,k}) > k$*.*

The proof of Theorem 2.23 uses a double induction, primarily on $g$ and secondarily on $r$ (handling all $d$ at once). To emphasize the role of gadgets in our constructions, we defer the proof of Theorem 2.23 to Section 2.8.2. In Section 2.8.1, we assume that Theorem 2.23 holds and we use it to prove Theorems 2.24 and 2.25.

In Theorem 2.2, we constructed graphs with girth 6 and chromatic number arbitrarily large. Given that $(d, r, g)$-graphs exist, the proof of Theorem 2.24 is similar. The main difference is that we use a $(d, r, g)$-graph in place of the independent set $I$, to ensure that our girth is large. Starting from the $(d, r, g)$-graph, we replace each leaf $v$ in the underlying tree $T$ with a (recursive) gadget that is not $(k-1)$-colorable, and each vertex of the gadget inherits one augmenting edge from $v$. For each $k$-coloring $\varphi$ of the internal vertices of the underlying tree, at least one copy of the gadget has the same color used by $\varphi$ on the endpoints of all of its augmenting edges. So each $k$-coloring $\varphi$ of the internal vertices of $T$ results in some copy of the gadget that has no $k$-coloring extending $\varphi$. The construction for Theorem 2.25 is similar.

### 2.8.1 The Coloring Results

full path
[d]-coloring
reference coloring
φ-path

**Definition 2.26.** In a complete $d$-ary tree, a *full path* is a path from the root to a leaf. A $[d]$-*coloring* is a $d$-coloring using the colors in $[d]$. Given an order of the children at each internal vertex, define an edge-coloring as follows: if $v$ is the $i$th child of $w$ in the order, then color edge $vw$ with $i$. This is the *reference coloring*; note that it is *not* proper; see Figure 2.20. For a $[d]$-coloring $\varphi$ of the vertices of $T$, a full path $P$ is a $\varphi$-*path* if for each non-leaf vertex $w$ on $P$ the color $\varphi(w)$ matches the reference color on the edge from $w$ to its child in $P$. Every $[d]$-coloring $\varphi$ of $V(T)$ has a unique $\varphi$-path. Likewise, every full path is a $\varphi$-path for some $[d]$-coloring $\varphi$.

A *descending edge* at a vertex $v$ in a rooted tree is any edge from $v$ to one of its children. Let $G$ be a $(d, r, g)$-graph with a specified vertex order of the children at each internal vertex of its underlying tree $T$. In the reference coloring of $G$, each internal non-root vertex $v$ has exactly one descending edge colored the same as the edge from $v$ to its parent. To form the *reduced $(d, r, g)$-graph* $H$ corresponding to $G$, for each such $v$ we delete from $G$ the subtree under this descending edge with repeated color. Each non-leaf vertex of $H$ has degree $d$ in $H \cap T$, and the reference coloring is a proper edge-coloring of $H \cap T$.

descending edge

reduced $(d, r, g)$-graph

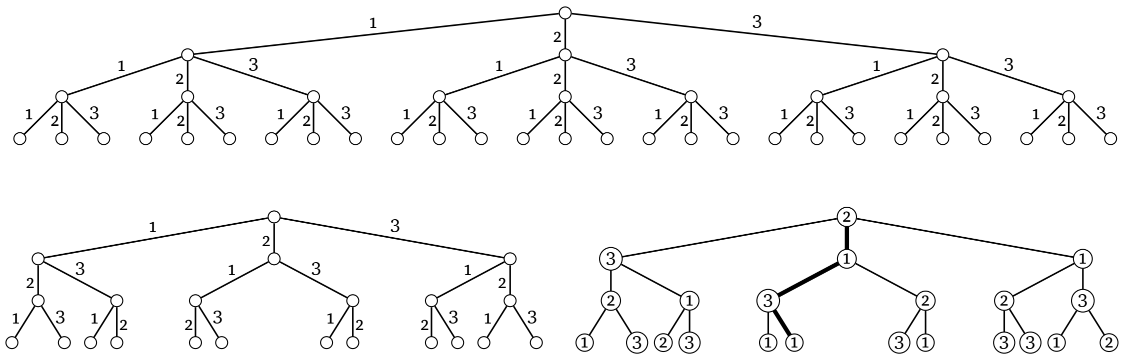


Figure 2.20: Top: A complete 3-ary tree of height 3, with the reference coloring. Bottom Left: The tree $T$ underlying a reduced $(3, 1, 4)$-graph $H$; to form $H$ we add an edge from each leaf to the root. Bottom Right: An arbitrary proper $[3]$-coloring $\varphi$ of $T$; the $\varphi$-path is marked in bold.

*Proof of Theorem 2.24.* For fixed $g$, we use induction on $k$. Let $G_{g,2}$ be an odd cycle of length at least $g$. For the induction step, let $p := \left|G_{g,k-1}\right|$. Start with a reduced $(k, (p-1)k+1, g)$-graph, with underlying tree $T$ and edges colored by the reference coloring. Fix a leaf $v$ of $T$ and let $P$ be the full path ending at $v$.

$v$, $T$

$P$

By Pigeonhole, at least $p$ neighbors of $v$ (along augmenting edges) have the same color on their descending edges in $P$. Let $v$ keep its augmenting edges to $p$ of these, and delete all other augmenting edges from $v$. Repeat this process for each leaf $v$ of $T$; the resulting graph $H$ is a reduced $(k, p, g)$-graph. Finally, replace each leaf $v$ of $T$ by a copy of $G_{g,k-1}$, with each of its vertices inheriting one augmenting edge from $v$; this is $G_{g,k}$. See Figure 2.21.

First we prove $\chi(G_{g,k}) > k$. Let $\varphi$ be a $[k]$-coloring of $V(T)$, let $P$ be a $\varphi$-path in $T$ ending at a leaf $v$, and let $S$ be the set of $p$ neighbors of $v$ along augmenting edges. By construction, each vertex of $S$ gets the same color $\alpha$ in $\varphi$. By hypothesis $\chi(G_{g,k-1}) > k-1$, so $\varphi$ does not extend to a $[k]$-coloring of the copy of $G_{g,k-1}$ substituted at $v$, since each of its vertices has a neighbor in $S$ colored $\alpha$.

Now we check that $G_{g,k}$ has girth at least $g$. By hypothesis, every cycle contained in a single copy of $G_{g,k-1}$ has length at least $g$. So suppose that $C$ is a cycle that uses edge $e$ corresponding to an augmenting edge in $H$. Contracting each copy of $G_{g,k-1}$ to a single vertex yields $H$. Let $C'$ be a closed walk in $H$ corresponding to $C$ in $G_{g,k}$. Since each augmenting

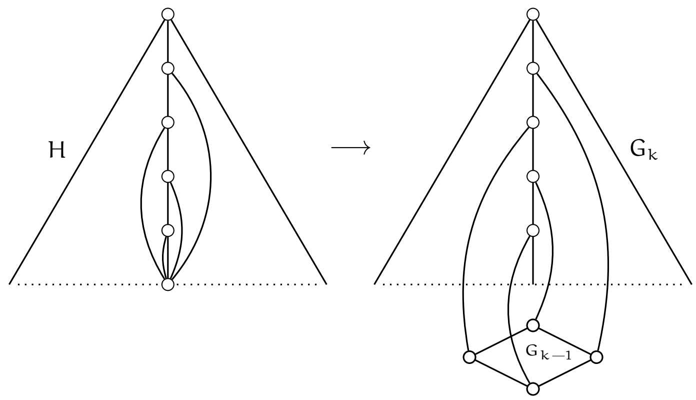



Figure 2.21: The final step constructing $G_k$ in the proof of Theorem 2.24.

edge of $H$ is inherited by a distinct vertex in a copy of $G_{g,k-1}$, the image of $e$ appears exactly once in $C'$. So $C'$ contains a cycle $C''$ in $H$ that includes $e$. Since $H$ has girth at least $g$, cycle $C''$ has length at least $g$; thus, so does $C$. □

It is interesting to note that $\mathrm{mad}(G_{g,k}) \leqslant 2(k-1)$; see Exercise 11(a). If we do not care about this property, then we can give an even simpler construction; see Exercise 11(b).

*Proof of Theorem 2.25.* Fix an even $g \geqslant 4$; we use induction on $k$. For each proper subgraph $H$, we prove that $\overline{d}(H) \leqslant 2(k-1)$. We specify a root, and orient the edges so that the root has outdegree $k$ and each other vertex has outdegree $k-1$; further, every vertex is reachable from the root. Thus, if a proper subgraph $J$ contains the root, then it also contains the tail of some deleted edge, so $\|J\| \leqslant |J|\,(k-1)$, as desired.

We build our base case $B_{g,2}$ from two $g$-cycles by identifying a vertex of each to form the root. Next we direct the edges of each $g$-cycle cyclically (so each vertex has outdegree 1, except the root). Finally, we can easily verify that $\chi_\ell(B_{g,2}) > 2$; see Exercise 6.

For the induction step, fix $k \geqslant 3$ and assume that $B_{g,k-1}$ has the desired properties. Let $p := \left|B_{g,k-1}\right| - 1$. Let $H$ be a reduced $(k,p,2g)$-graph with underlying tree $T$. Let $(U,W)$ be a bipartition of $B_{g,k-1}$ with $U$ containing the root. For each leaf $v$ in $T$, add to $H$ a copy of $B_{g,k-1}$, identifying its root with $v$ and letting each of its non-root vertices inherit from $v$ exactly one augmenting edge. Now for each vertex $w \in W$, shift the other end of its augmenting edge to be one vertex closer to $v$ on the full path to $v$. Since $H$ is bipartite, by definition, this ensures that the resulting graph $B_{g,k}$ is also bipartite, since every closed walk in $B_{g,k}$ corresponds to a walk in $H$ of the same parity. (This construction is similar to the final step constructing $G_k$, shown in Figure 2.21. The two differences are that (i) we identify $v$ with the root of $B_{g,k}$, rather than deleting it, and (ii) we shift the endpoints of augmenting edges ending in $W$.)

Orient each edge of $T$ from parent to child, each edge in a $B_{g,k-1}$ recursively, and each augmenting edge away from its endpoint in a $B_{g,k-1}$. Since $H$ is a reduced $(k,p,2g)$-graph, the root has outdegree $k$ and each other internal vertex of $T$ has outdegree $k-1$. By hypothesis, each vertex in a copy of $B_{g,k-1}$ has outdegree $k-2$ in that copy, so its augmenting edge gives it outdegree $k-1$.

Now consider the girth of $B_{g,k}$. By hypothesis, $B_{g,k-1}$ has girth at least $g$. Each cycle $C$ that uses an augmenting edge corresponds to a closed walk $C'$ in $H$, of length at most $2|C|$. Since $H$ has girth at least $2g$, the desired girth bound holds for $B_{g,k}$.

Finally, we construct a $k$-assignment $L$ such that $B_{g,k}$ has no $L$-coloring. For each non-leaf vertex $v$ in $T$, let $L(v) := [k]$. Let $L'$ be a $(k-1)$-assignment such that $B_{g,k-1}$ has no $L'$-coloring; further, we assume that $L'$ uses no colors in $[k]$. For each leaf $v$ of $T$ and vertex $x$ in $B_{g,k-1}$, let $x_v$ denote vertex $x$ in the copy of $B_{g,k-1}$ rooted at $v$. Let $P$ be the full path ending at $v$. For each $x_v$, let $L(x_v) := L'(x) \cup \{\alpha\}$, where $\alpha$ is the color on the edge of $P$ descending from the neighbor of $x_v$ in $V(P)$. Each $L$-coloring $\varphi$ of $V(T)$ has a $\varphi$-path $P$, ending at a leaf $v$. Now each vertex $x_v$ loses the color used on its neighbor in $V(P)$. By hypothesis, the copy of $B_{g,k-1}$ has no $L'$-coloring. Thus, $B_{g,k}$ has no $L$-coloring. □

### 2.8.2 Construction of $(d, r, g)$-Graphs

In this section, we prove Theorem 2.23, which we restate for convenience. The theorem follows easily from the three lemmas below, so we prove the theorem first, assuming the lemmas, and prove the lemmas thereafter. (Recall that, by definition, every $(d, r, g)$-graph is bipartite.)

**Theorem 2.6.2.** For all $d$, $r$, and $g$, the value of $h(d, r, g)$ is finite. In other words, $(d, r, g)$-graphs always exist.

**Lemma 2.27.** *For all positive integers* $d$ *and* $r$*, we have* $h(d, r, 4) = 2r + 1$.

**Lemma 2.28.** *For all positive integers* $d$ *and* $g$*, with* $g$ *at least* $4$ *and even,* $h(d, 1, g+2) \leqslant 2 + h(d, d^2, g)$.

**Lemma 2.29.** *For all positive integers* $d$*,* $r$*, and* $g$*, with* $g$ *at least* $4$ *and even,* $h(d, r+1, g) \leqslant h_1 + h_2$*, where* $h_1 := h(d, 1, g) + 1$ *and* $h_2 := h(d^{h_1}, r, g)$.

*Proof of Theorem 2.23.* Let $P(r, g)$ denote the claim that $h(d, r, g)$ is finite for all $d$. We prove that $P(r, g)$ holds for all $r$ and $g$ by induction; the primary induction is on $g$, and the secondary on $r$. It suffices to consider $g$ even and at least 4. The base case is Lemma 2.27. For the primary induction, $P(1, g+2)$ holds by Lemma 2.28, since $P(r, g)$ holds for all $r$. For the secondary induction, $P(r+1, g+2)$ holds by Lemma 2.29, since $P(r, g+2)$ holds by hypothesis. □

*Proof of Lemma 2.27.* Given a complete $d$-ary tree of height $2r+1$, add augmenting edges from every leaf $v$ to each of its ancestors $w$ such that $\mathrm{dist}_T(v, w)$ is at least 3 and odd. □

*Proof of Lemma 2.28.* Start with a $(d, d^2, g)$-graph $H$. For each leaf $v$ of the underlying tree $T$, identify with $v$ the root of a complete $d$-ary tree $T'$ of height 2; let each leaf $w$ of $T'$ inherit exactly one augmenting edge from $v$. (This is nearly the same as in Figure 2.21 but now $v$ is identified with the root of $T'$, not deleted.) Call this new graph $G$. Every cycle $C$ in $G$ maps to a cycle in $H$ that is shorter than $C$ and has the same parity. So $G$ is a $(d, 1, g+2)$-graph. □

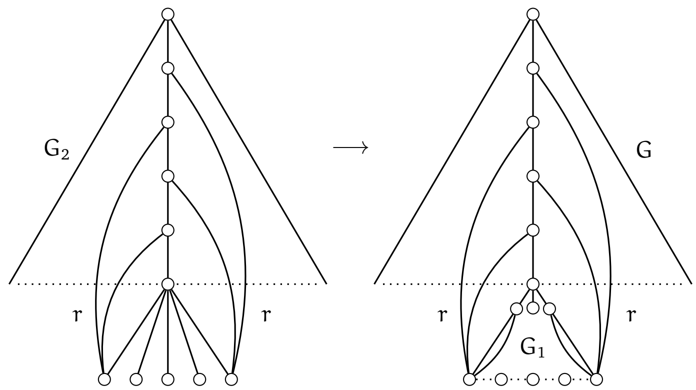



Figure 2.22: The proof of Lemma 2.28. To form G from $G_2$, we replace each star induced by leaves and their parent with a copy of $G_1$.

$h_1, h_2$ *Proof of Lemma 2.29.* Let $h_1 := 2\lfloor h(d,1,g)/2 \rfloor + 1$ and $h_2 := h(d^{h_1}, r, g)$. We prove the slightly stronger statement that $h(d, r+1, g) \leqslant h_1 + h_2 - 1$. We construct the desired graph G from two smaller graphs $G_1$ and $G_2$.

$G_1$ We want $G_1$ to be a $(d, 1, g)$-graph of height $h_1$. Note that $h_1$ is the smallest odd integer that is at least $h(d, 1, g)$. If $h(d, 1, g)$ is odd, then $G_1$ exists by definition. Otherwise, start with d copies of a $(d, 1, g)$-graph of minimum height, and add a new vertex with the roots of these copies as its d children. (This fussing about the parity of the height ensures that G is bipartite.)

$G_2$ Now we build $G_2$. Let $d' := d^{h_1}$, and let H be a $(d', r, g)$-graph with height $h_2$. We form $G_2$ from H by starting at the root and deleting all but d children of each vertex that we keep,
$T_2$ except that we keep all $d'$ children at the last level; let $T_2$ be the tree underlying $G_2$. (So deleting the leaves of $T_2$ would yield a complete d-ary tree of height $h_2 - 1$.) For each vertex $v$ that is a parent of leaves in $T_2$, replace the star in $T_2$ induced by $v$ and its children with a copy
$T_1$ of $G_1$; identify $v$ with the root of the tree $T_1$ underlying $G_1$ and let each leaf $w$ in $T_1$ inherit the r augmenting edges from one child of $v$. We call the resulting graph G.

To show that G is a $(d, r+1, g)$-graph, we consider its underlying tree T. Form $T_2'$ from $T_2$ by deleting its leaves. Now T has a top part, $T_2'$, coming from $G_2$, and many bottom parts, each isomorphic to $T_1$. We observe that T is a complete d-ary tree of height $h_1 + h_2 - 1$. The augmenting edges in G coming from $G_2$ are *long* and those coming from $G_1$ are *short*. Each leaf of T has r long edges and 1 short edge. So the vertices of G have the desired degrees.

For each vertex $v$ that is a parent of a leaf in $G_2$, each edge $e$ descending from $v$ in $G_2$ was replaced by a path of length $h_1$. Since $h_1 - 1$ is even, the resulting graph G is bipartite. By hypothesis, each cycle contained in a copy of $G_1$ has length at least $g$. So consider a cycle C in G that uses a long edge $e$; C corresponds to a closed walk $C'$ in $G_2$ (formed from G by contracting each copy of $G_1$ to a star centered at its root). Further $C'$ uses $e$ only once, since each leaf of $G_1$ maps to a distinct leaf of $G_2$. So $C'$ contains a cycle $C''$ that includes $e$. Since $G_2$ has girth at least $g$, so does G. Thus, G is a $(d, r+1, g)$-graph. $\square$

## Notes

Borodin et al. [56] first constructed planar graphs $G_k$ with girth 6 and $\Delta = k$ such that $\chi(G_k^2) \geqslant k+2$. Our proof of Theorem 2.1 follows Dvořák, Král', Nejedlý, and Škrekovski [132]. Triangle-free graphs with arbitrary chromatic number were found by Mycielski [315] and Zykov [439]; see Exercise 2. Kelly and Kelly [242] gave such graphs with girth 6. The construction of these in Theorem 2.2 follows Descartes[3] [394]. Kostochka and Nešetřil revisited Descartes' construction in [274], more than 40 years after its initial publication, and enumerated its beautiful properties.

Theorem 2.3 is due to Voigt [403]. The smallest known non-4-choosable planar graph was found by Mirzakhani [300]; it has 63 vertices and is constructed in Exercise 11. The constructions of both Voigt and Mirzakhani are 3-colorable, so a 3-colorable planar graph need not be 4-choosable. Voigt [404] also proved Theorem 2.4, that some planar graphs with girth 4 are not 3-choosable; her construction has 166 vertices. By modifying this construction, we gave a graph in Section 2.4 with order 128. Glebov, Kostochka, and Tashkinov [177] further improved this bound by constructing such graphs with orders 109 and 97. Their graph with order 109 has each list a subset of $\{1, \ldots, 5\}$ and their graph with order 97 has each list a subset of $\{1, \ldots, 6\}$.

In contrast to Voigt's result for planar graphs of girth 4, Thomassen [378, 380] proved that all planar graphs with girth at least 5 are 3-choosable. His proof is similar to those of Theorems 4.5 and 11.1, but more detailed, and we present it in Section 11.4. For planar graphs with girth at least 6, proving 3-choosability is easy: they are 2-degenerate. Erdős, Rubin, and Taylor also conjectured that bipartite planar graphs are 3-choosable. Alon and Tarsi [20] confirmed this, as we show in Proposition 5.6.

Grötzsch proved that every triangle-free planar graph has a 3-coloring. Thomassen [382] then conjectured that such a graph G has exponentially many 3-colorings; here we mean 'exponential' in the order of G. This conjecture was motivated by analogous results for 5-list-colorings of planar graphs and for 3-list-colorings of planar graphs with girth at least 5; see Section 8.5. (Section 6.3 presents similar results for graphs with exponentially many nowhere-zero $\mathbb{Z}_k$-flows.) As evidence in support of this conjecture, Thomassen [382] proved that such an $n$-vertex graph has at least $c^{n^{1/12}}$ 3-colorings, where $c = 2^{1/20000}$. This lower bound was improved to $2^{\sqrt{n/212}}$ by Asadi, Dvořák, Postle, and Thomas [29]. However, Thomassen ultimately disproved his own conjecture [385], constructing $n$-vertex triangle-free planar graphs with at most $2^{15n/\lg_2 n}$ 3-colorings. This upper bound was improved to $64^{n^{\lg_{9/2} 3}}$ by Dvořák and Postle [140]. Their construction is what we presented in Section 2.6. In the same paper, Dvořák and Postle conjectured this upper bound is best possible.

In 1976, Steinberg conjectured [7] that every planar graph without 4-cycles and 5-cycles

[3]Blanche Descartes was a pseudonym of R. Leonard Brooks, Arthur H. Stone, Cedric Smith, and William T. Tutte [364]. The four met in 1935, when they were students at Cambridge, and became close friends. While undergraduates, they worked on a number of mathematical research problems. To get the name Blanche, they combined the initials of their first names, Bill, Leonard, Arthur, Cedric, to form BLAC, which they then extended. They chose the last name Descartes to play on the phrase *Carte Blanche*.

is 3-colorable. This problem received lots of attention, and led to many partial results (which are discussed further in the Chapter 1 Notes). Steinberg's Conjecture was strengthened to the Strong Bordeaux Conjecture [70] and still further to the Novosibirsk 3-Color Conjecture [57]. But these were all disproved by Cohen-Addad, Hebdige, Král', Li, and Salgado [89], as we saw in Section 2.5. Their construction is striking for its simplicity.

The use of gadgets certainly predates the notion of NP-hardness (this term is formally defined in Section A.1; informally, a problem is NP-hard if it seems impossible to optimally solve every instance of it efficiently). But it was the widespread work in this area, particularly during the 1970s and 1980s, that elevated gadget use to a fine art. The classic reference on this topic is [173], and we still highly recommend it.

The most famous NP-hard coloring problem is 3-edge-coloring 3-regular graphs. Its NP-hardness was proved by Holyer [220], and Section 2.7.1 essentially reproduces his paper. Shortly thereafter, Leven and Galil [282] extended the result to Theorem 2.8: For every integer $k \geqslant 3$, it is NP-hard to decide if a $k$-regular graph is $k$-edge-colorable. Section 2.7.2 follows their presentation. An immediate corollary is that $k$-coloring is NP-hard for every $k \geqslant 3$; simply take the line graph of a graph we are trying to edge-color. Perhaps the next most well-known NP-hard coloring problem is 3-coloring planar graphs [174]. To prove such a result, we simply draw an arbitrary graph in the plane (allowing edge-crossings) and then substitute a planar "crossover gadget" for each edge-crossing in our drawing. Exercise 10 sketches the proof.

The quest for graphs with both girth and chromatic number arbitrarily large has attracted many researchers over many years. Erdős [153] was the first to prove that such graphs exist. He constructed a random graph, adding each edge with equal probability $p$; for the right choice of $p$, with high probability the graph has few short cycles and few large independent sets. So deleting a few specific vertices yields a graph with no short cycles and no large independent sets. We always have $\chi(G) \geqslant |G|/\alpha(G)$, where $\alpha$ is the independence number; so the result follows. A nice exposition is given by Aigner and Ziegler [5, Chapter 35]. But what about *explicit* constructions?

Lovász [287] first constructed these graphs in 1968; a key step was generalizing the problem from graphs to hypergraphs. Subsequent proofs were given by Nešetřil and Rödl [323], Kříž [278], and Kostochka and Nešetřil [274]. Section 2.8 follows Alon, Kostochka, Reiniger, West, and Zhu [17]. The $(d, r, g)$-graphs we construct have height given by a version of the Ackerman function; Alon [13] showed that this is essentially best possible. In Theorem 2.25 we construct bipartite graphs $B_k$ with $\chi_\ell(B_k) > k$ and $\mathrm{mad}(H) \leqslant 2(k-1)$ for every proper subgraph $H$. This edge bound is very sharp. In Theorem 5.5, we prove that every bipartite graph $G$ with $\mathrm{mad}(G) \leqslant 2(k-1)$ is $k$-choosable.

To close this section we mention two beautiful constructions using gadgets that are too long to include here. In the same paper where they introduced list-coloring, Erdős, Rubin, and Taylor [152] mentioned the generalization to list-multicoloring. A graph is $(a, b)$-choosable if, given any $a$-list assignment $L$ to its vertices there exists a function $\varphi$ such that $\varphi(v) \subseteq L(v)$ and $|\varphi(v)| = b$ and $\varphi(v) \cap \varphi(w) = \emptyset$ for each edge $vw \in E(G)$. (When $b = 1$, we recover standard list-coloring.) They asked whether every graph that is $a$-choosable is also $(am, m)$-choosable, for every positive integer $m$. This question remained open for more than 30 years, before being answered negatively by Dvořák, Hu, and Sereni [130]. For each $a \geqslant 4$, they construct a graph that is $a$-choosable, but not $(2a, 2)$-choosable.

Another longstanding open problem was Jaeger's Circular Flow Conjecture. Posed in 1981, it was disproved only in 2018 [201]. We omit the necessary definitions, but provide more details on this problem in the Notes sections of Chapters 4 and 6. Again, the construction was remarkably simple for a problem that stood open for more than 35 years. Now research in this area focuses on finding the weakest hypotheses (necessarily stronger than Jaeger originally conjectured) under which the desired conclusion holds.

## Exercises

2.1. Show that $\chi_\ell(K_{k,k^k}) > k$. [Remark: This is best possible, since $\chi_\ell(K_{k,k^k-1}) \leqslant k$.]

2.2. (a) Let $G_1 := K_1$. For each $k \geqslant 1$, to form $G_{k+1}$, start with the disjoint union of $G_1, \ldots, G_k$. For each set $S$ consisting of one vertex from each $G_j$ with $j \in [k]$, add a new vertex adjacent precisely to the vertices in $S$. Show that $G_{k+1}$ is triangle-free and $\chi(G_{k+1}) = k + 1$. [439] (b) Let $G_k$ be a triangle-free graph such that $\chi(G_k) \geqslant k$. Let $v_1, \ldots, v_t$ denote the vertices of $G_k$. To form $G_{k+1}$ from $G_k$, add new vertices $w_1, \ldots, w_t$ such $w_i \leftrightarrow v_j$ in $G_{k+1}$ whenever $v_i \leftrightarrow v_j$ in $G_k$; finally add a vertex $x$ adjacent to all $w_i$. Show that $G_{k+1}$ is triangle-free and $\chi(G_{k+1}) \geqslant k + 1$. This is known as Mycielski's construction. [315]

2.3. Show that no Gallai tree is degree-choosable.

2.4. A *clustered coloring* with clustering at most $C$ is a, possibly improper, coloring such that each color class induces a subgraph where each component has order at most $C$. (So a proper coloring has clustering 1.) Let $P_t''$ denote the join of a path $P_t$ with two isolated vertices. For every constant $C$, show that there exists $t$ and a 3-assignment $L$ for $P_t''$ such that $P_t''$ does not admit any $L$-coloring with clustering at most $C$. [128]

2.5. (a) For every $k \geqslant 2$, construct a planar graph $G_k$ such that $\Delta = k$ and $\chi(G_k^2) \geqslant \lfloor \frac{3}{2}k \rfloor$. (b) Modify this graph slightly to get $H_k$ with $\Delta = k$ and $\chi(H_k^2) = 1 + \lfloor \frac{3}{2}k \rfloor$. [Remark: Hell and Seyffarth [215] showed that if $G$ is planar with $\Delta \geqslant 8$ and $G^2$ is a clique, then $|G| \leqslant 1 + \lfloor \frac{3}{2}\Delta \rfloor$. Cranston [92] extended their ideas to show that if $G$ is planar and $\Delta \geqslant 36$, then $\omega(G^2) \leqslant 1 + \lfloor \frac{3}{2}\Delta \rfloor$, even if $G^2$ is not a clique.]

2.6. A *barbell* is formed from two vertex disjoint cycles by either (i) identifying one vertex from each cycle with the distinct endpoints of a path or (ii) identifying one vertex from one cycle with one vertex from the other. Show that no barbell is 2-choosable. Similarly, show that the $\Theta$-graph $\Theta_{a,b,c}$ is not 2-choosable when $a \neq 2$ and $b \neq 2$. [152]

2.7. Show that every graph with $\delta \geqslant 2$ that is neither an even cycle nor $\Theta_{2,2,2t}$ (for some $t \geqslant 1$) contains as a subgraph one of the following: (i) an odd cycle, (ii) $K_{2,4}$, (iii) a barbell (as in the previous exercise), or (iv) a $\Theta$-graph $\Theta_{a,b,c}$ with $a \neq 2$ and $b \neq 2$. [Together with Lemma 1.34 and the previous exercise, this completes the characterization of 2-choosable graphs. [152]]

2.8. For the construction of triangle-free planar graphs with few 3-colorings, in Section 2.6, show that our choice $k := \lceil (3/2)^{\ell} \rceil$ is asymptotically best possible. More formally, show that no value of $k$ leads to an upper bound on the number of 3-colorings of the form $a^{n^c}$ where $n := |V(G(v, w, k, \ell))|$, $a$ is constant, and $c < \log_{9/2} 3$. [140]

2.9. (a) At the top of Figure 2.23, we write $S$ to denote $\{1, 2, 3, 4\}$ and we write $\bar{i}$ for $S \setminus \{i\}$. Prove that the graph shown has no coloring from its lists. (b) At the bottom of Figure 2.23, we write $\bar{i}$ to denote $\{1, 2, 3, 4, 5\} \setminus \{i\}$. Form a graph $G$ from the graph shown by adding a vertex adjacent to all vertices on the outer face and giving it the list $\{2, 3, 4, 5\}$. Show that $G$ has no coloring from its lists. [300]

2.10. Figure 2.24 shows a "crossover gadget". The four vertices at its top, bottom, right, and left are called external connectors. (a) Prove that every 3-coloring of the crossover gadget uses the same color on each pair of external connectors at distance 4. Prove that if a coloring of the external connectors uses the same colors on each pair at distance 4, then it extends to a 3-coloring of the whole crossover gadget. (b) Use this to show that it is NP-hard to 3-color planar graphs. [174]

2.11. (a) Prove that the graph $G_{g,k}$ constructed in Theorem 2.24 have $\mathrm{mad}(G_{g,k}) \leqslant 2(k-1)$. [17] (b) By dropping the criteria that $\mathrm{mad}(G_{g,k}) \leqslant 2(k-1)$, give an even simpler construction than we did in Theorem 2.24. [17]

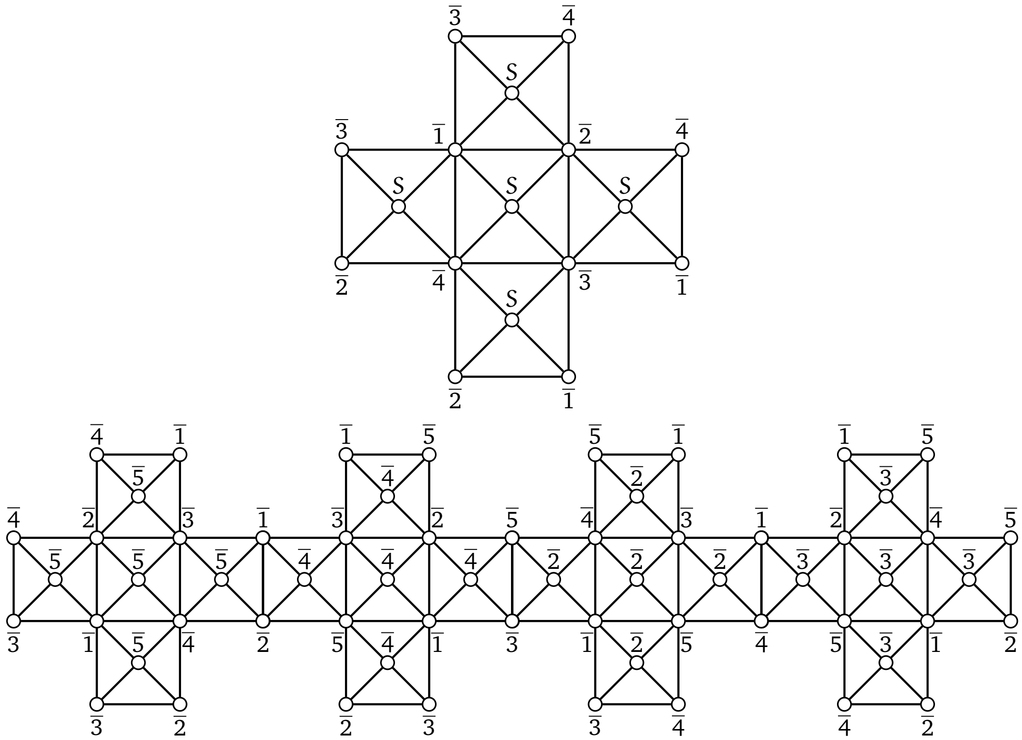



Figure 2.23: Mirzakhani's construction of a non-4-choosable planar graph with 63 vertices.

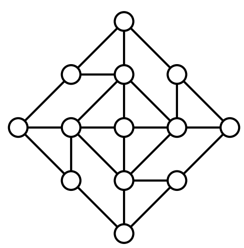


Figure 2.24: A crossover gadget, which is used in the proof that 3-coloring planar graphs is NP-hard. See Exercise 10.

# Chapter 3

# Recoloring

> ... we should continually be striving to transform every art into a science: in the process, we advance the art.
>
> —Donald Knuth

In Chapter 1, we studied greedy coloring. We can view this process as repeatedly extending a partial coloring. To reduce the number of colors we need to finish, at each step we now try to modify our partial coloring, so the vertex colored next has many neighbors with the same color. Specifically, suppose an uncolored vertex $v$ has colors $\alpha$ and $\beta$ both used once on its neighborhood, with $\beta$ appearing only once, on some vertex $w$. We try to recolor $w$ with $\alpha$, so that $\beta$ is no longer used on $N(v)$. But if $w$ already has neighbors colored $\alpha$, then we must recolor them with $\beta$, which may lead to further swaps of $\alpha$ and $\beta$ throughout the graph.

These recoloring techniques are among the most versatile tools that we will study. Kempe swaps, introduced in 1879, were the central idea in the original proof[1] of the 5 Color Theorem, and also played a key role in the eventual proof of the 4 Color Theorem. They yield a nice proof of Brooks' Theorem, and also underlie nearly all of the best results in edge coloring.

## 3.1 Kempe Chains: Edge-coloring Simple Graphs

We begin by studying edge-coloring, which we will see is a special case of vertex coloring. Up 'til now we have mainly considered simple graphs, where each pair of vertices is joined by at most one edge. This is natural, since parallel edges rarely effect vertex coloring results[2]. Now we also study multigraphs, which allow parallel edges.

[1]This was the only proof known until 1974, when Kainen found the proof we present in Section 4.1.

[2]A notable exception is correspondence coloring, which we study in Section 4.4, where parallel edges do matter.

### 3.1.1 Definitions and König's Theorem

parallel edges; multigraph; simple graph; multiplicity; maximum edge multiplicity; line graph

**Definition 3.1.** Edges with the same pair of endpoints are *parallel edges*; now the edge set of the graph is replaced by an edge multiset. When we explicitly allow a graph $G$ to have parallel edges, we call $G$ a *multigraph*. If $G$ has no parallel edges, then $G$ is a *simple graph*. The *multiplicity*, $\mu(vw)$, of $vw$ in a multigraph $G$ is the number of edges with the pair of endpoints $(v, w)$. The *maximum edge multiplicity*, $\mu(G)$, is given by $\mu(G) := \max_{v,w \in V(G)} \mu(vw)$. The *line graph*, $L(G)$, of a graph $G$ has $E(G)$ as its vertices and two vertices of $L(G)$ are adjacent if their corresponding edges share an endpoint.

proper edge-coloring; edge-coloring

A *proper edge-coloring* of a graph $G$ assigns a color to each edge of $G$ so that any two edges with a common endpoint receive distinct colors; when the context is clear, we usually just write *edge-coloring*. Note that edge-colorings of a graph are in bijection with vertex colorings of its line graph. So edge-coloring is a special case of vertex coloring, since not all graphs are line graphs. Figure 3.1 illustrates this bijection.

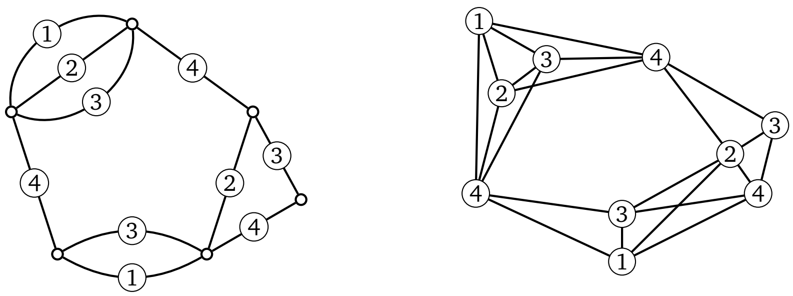


Figure 3.1: A proper edge-coloring of a multigraph, and the corresponding vertex coloring of its line graph.

partial edge-coloring; edge-chromatic number; chromatic index; edge-critical; seen at vertex $v$; missed at $v$

A *partial edge-coloring* of $G$ is a proper edge-coloring of some subgraph of $G$, allowing edges to be uncolored. A $k$*-edge-coloring* is an edge-coloring that uses at most $k$ colors. The *edge-chromatic number* of $G$, denoted $\chi'(G)$ and also called its *chromatic index*, is the smallest $k$ such that $G$ has a $k$-edge-coloring. A graph $G$ is *edge-critical*[3] if $\chi'(G - e) < \chi'(G)$ for every edge $e$. (For example, each graph in Figure 3.2 is edge-critical.) In a partial edge-coloring $\varphi$, a color $\alpha$ is *seen at vertex* $v$ if $\alpha$ is used on some edge incident to $v$; otherwise $\alpha$ is *missed at* $v$. Let $\varphi(v)$ and $\overline{\varphi}(v)$ denote the sets of colors seen and missed at $v$, respectively.

$\alpha, \beta$-chain; $\alpha, \beta$-swap; recoloring

Given a partial $k$-edge-coloring $\varphi$ of a graph and two colors $\alpha, \beta \in [k]$, an $\alpha, \beta$*-Kempe chain*, or simply an $\alpha, \beta$*-chain*, $H$ is a component of the subgraph induced by the edges colored $\alpha$ and $\beta$. If vertex $v$ sees exactly one of colors $\alpha$ and $\beta$, then $P_v(\alpha, \beta)$ denotes the $\alpha, \beta$-chain starting at $v$. An $\alpha, \beta$*-swap* recolors the edges of an $\alpha, \beta$-chain, using $\alpha$ in place of $\beta$, and $\beta$ in place of $\alpha$. We call this *recoloring* $H$.

Edge-critical graphs are useful because every graph $G$ contains an edge-critical subgraph $H$

[3] To be precise, we also require that $G$ has no isolated vertices.

with $\Delta(H) = \Delta(G)$, but edge-critical graphs have more structure than general graphs, which may facilitate a proof. As a result, many edge-coloring conjectures are known to be true if they are true for the class of edge-critical graphs.

**Remark 3.2.** If $\varphi$ is a proper partial $k$-edge-coloring of a graph $G$ and we form $\varphi'$ from $\varphi$ by performing a Kempe swap, then $\varphi'$ is also a proper partial $k$-edge-coloring of $G$. This observation is the fundamental property of Kempe swaps, and we often use it without remark. Every Kempe chain has maximum degree at most 2, so must be a path or an even cycle. When viewed in the context of line graphs, this is because $K_{1,3}$ is not an induced subgraph of any line graph. In fact, many coloring results for line graphs extend to larger classes of $K_{1,3}$-free graphs.

With definitions done, we turn to proofs. We start with the case when $G$ is bipartite.

**Theorem 3.3 (König's Theorem).** *If $G$ is a bipartite multigraph, then $\chi'(G) = \Delta$.*

*Proof.* We use induction on $\|G\|$. The base case $\|G\| = 1$ is easy. So suppose that $\|G\| > 1$. Choose an arbitrary edge $vw$ and let $G' := G - vw$. By hypothesis, $\chi'(G') = \Delta(G') \leqslant \Delta(G)$. So let $\varphi$ be a $\Delta(G)$-edge-coloring of $G'$. We want to extend $\varphi$ to $vw$. If $v$ and $w$ miss a common color $\alpha$, then we color $vw$ with $\alpha$, so assume they do not. Since $v$ and $w$ each see at most $\Delta(G) - 1$ colors in $\varphi$, there exist distinct colors $\alpha \in \overline{\varphi}(v)$ and $\beta \in \overline{\varphi}(w)$. Let $P := P_v(\alpha, \beta)$. Clearly $w$ is not an internal vertex of $P$, since $\beta \in \overline{\varphi}(w)$. Also $w$ is not an endpoint of $P$, since then $P + vw$ would be an odd cycle, which is forbidden since $G$ is bipartite. So $w \notin V(P)$. Now an $\alpha, \beta$-swap at $v$ yields a new $\Delta(G)$-edge-coloring $\varphi'$ of $G'$, with $\beta \in \overline{\varphi'}(v) \cap \overline{\varphi'}(w)$. To finish the $\Delta(G)$-edge-coloring of $G$, we color $vw$ with $\beta$. □

We might hope that $\chi' = \Delta$ for all graphs, but this is false.

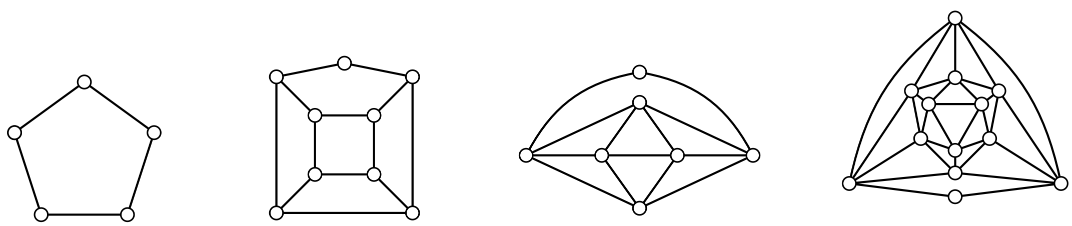


Figure 3.2: Simple planar graphs with $\chi' = \Delta + 1$, when $\Delta \in \{2, 3, 4, 5\}$.

**Example 3.4 (Simple graphs with $\chi' > \Delta$).** For every $k \geqslant 2$ there exists a simple graph $G$ with $\Delta(G) = k$ and $\chi'(G) > k$. When $2 \leqslant k \leqslant 5$, there exist such graphs that are planar.

*Proof.* Let $H$ be a $k$-regular graph with $|H|$ even, say $|H| = 2t$ for some integer $t$. Form $G$ from $H$ by subdividing a single edge. Now $\|H\| = tk$, so $\|G\| = tk + 1$. However, each matching of $G$ has size at most $\lfloor |G|/2 \rfloor = t$, so $\chi'(G) \geqslant \lceil \|G\|/t \rceil = k + 1$. When $k$ is 2, 3, 4, or 5, choose $H$ to be an even cycle, the cube, the octahedron, or the icosahedron (respectively). Subdividing an edge preserves planarity, as shown in Figure 3.2; so $G$ is planar. □

### 3.1.2 Vizing's Theorem and Kierstead Paths

Vizing used Kempe swaps to show that every simple graph $G$ satisfies $\chi'(G) \leqslant \Delta + 1$. We present 3 proofs of this result. All 3 of the proofs rely on Kempe swaps to modify partial colorings. But each of the 3 uses induction in its own way.

**Lemma 3.5.** *Let $G$ be a simple graph and $\varphi$ a $k$-edge-coloring of all of $G$ except for some edges $e_1, \ldots, e_s$ incident to a specified vertex $w$, where $e_i = v_i w$ for each $i \in [s]$. If $|\overline{\varphi}(w)| \geqslant s$, $|\overline{\varphi}(v_1) \cap \overline{\varphi}(w)| \geqslant 1$, and $|\overline{\varphi}(v_i) \cap \overline{\varphi}(w)| \geqslant 2$ for all $i \geqslant 2$, then $G$ has a $k$-edge-coloring.*

*Proof.* We use induction on $s$. For the base case, $s = 1$, color $v_1 w$ from $\overline{\varphi}(v_1) \cap \overline{\varphi}(w)$. So assume $s \geqslant 2$. Choose $A_i \subseteq \overline{\varphi}(v_i) \cap \overline{\varphi}(w)$, for each $i \in [s]$, so that $|A_1| = 1$ and $|A_i| = 2$ for all $i \geqslant 2$. Let $B = \overline{\varphi}(w)$. Suppose there exists $j \in [s]$ and a color $\alpha$ such that $\alpha \in A_j \cap B$ and $\alpha \notin A_i$ for all $i \neq j$. Now color $e_j$ with $\alpha$ and proceed by induction on $s$. So assume instead that each $\alpha \in \cup_{i=1}^{s} A_i$ appears in $A_i$ for at least two values of $i$. Since $\sum_{i=1}^{s} |A_i| = 2s - 1$ and $|B| \geqslant s$, there exists a color $\beta \in B \setminus (\cup_{i=1}^{s} A_i)$. Choose $\gamma \in A_1$ and recolor $P_{v_1}(\beta, \gamma)$. Now $\beta$ is available at $v_1$ and at most one other $v_i$ (since $P_{v_1}(\beta, \gamma)$ is a path). Use $\beta$ on $e_1$ and proceed by induction on $s$, taking $v_i$ as the new instance of $v_1$. □

**Theorem 3.6 (Vizing's Theorem).** *If $G$ is a simple graph, then $\chi'(G) \leqslant \Delta + 1$.*

*Proof.* Let $k := \Delta + 1$. We show that $G$ has a $k$-edge-coloring, by induction on $|G|$ (the base case $|G| = 1$ is trivial). Choose an arbitrary $w \in V(G)$. By hypothesis, $G - w$ has a $k$-edge-coloring $\varphi$. Clearly $|\overline{\varphi}(w)| = k \geqslant d(w)$. For each neighbor $v_i$ of $w$, we have $|\overline{\varphi}(v_i)| \geqslant 2$ and $\overline{\varphi}(v_i) \subseteq \overline{\varphi}(w)$. By Lemma 3.5, we can extend the coloring $\varphi$ to $G$. □

Kierstead path

**Definition 3.7.** For a multigraph $G$, let $\varphi$ be a partial $k$-edge-coloring, for some integer $k \geqslant \Delta + 1$. For an uncolored edge $v_0 v_1$, a *Kierstead path* is a path $v_0, \ldots, v_s$ such that for each $i \in [s]$ there exists $j \in \{0, \ldots, i-1\}$ such that $\varphi(v_i v_{i-1}) \in \overline{\varphi}(v_j)$. That is, each color used on an edge of the path is missed at an earlier vertex on the path. By definition, every prefix of a Kierstead path is again a Kierstead path. See Figure 3.3.

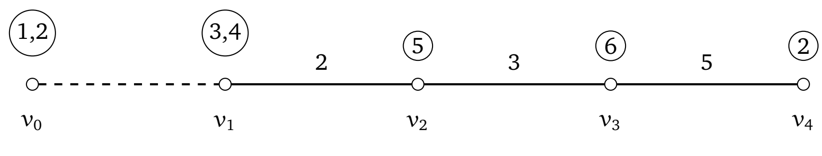



Figure 3.3: A Kierstead path, where circled labels denote colors missed at a vertex; we use this convention throughout the chapter. Since color 2 is missed at vertices $v_0$ and $v_4$, Lemma 3.8 ensures that, after some Kempe swaps, we can extend the coloring to $v_0 v_1$.

**Lemma 3.8 (Kierstead's Lemma).** *Let* $G$, $\varphi$, $k$, *and* $v_0, \ldots, v_s$ *satisfy Definition 3.7. Let* $H$ *denote the subgraph induced by edges colored by* $\varphi$, *and let* $H' := H + v_0v_1$. *If there exist distinct* $v_i$ *and* $v_j$ *missing a common color, i.e.,* $\overline{\varphi}(v_i) \cap \overline{\varphi}(v_j) \neq \emptyset$, *then* $H'$ *has a* $k$*-edge-coloring.*

For a color $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_j)$, We will "push" the two vertices where $\alpha$ is missed down the path toward $v_0$, so that $\alpha$ (or some other common color) is eventually missed by $v_0$ and $v_1$, at which point we can clearly extend the coloring. This pushing process consists of repeated Kempe swaps at vertices on the path.

*Proof.* We use double induction, first on $s$, and second on $|i - j|$. Choose $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_j)$. By symmetry, we assume $i < j$. We also assume $j = s$, since otherwise we apply the induction hypothesis to a prefix of the path. Similarly, we assume $\alpha \notin \overline{\varphi}(v_h)$ for all $h \notin \{i, j\}$; also for all distinct $q, r \in \{0, \ldots, s-1\}$, we have $\overline{\varphi}(v_q) \cap \overline{\varphi}(v_r) = \emptyset$. For the base case, $s = 1$, we have $i = 0$ and $j = 1$. So we color edge $v_0v_1$ with $\alpha$.

Now consider the induction step, when $s > 1$. First suppose that $i = j - 1 = s - 1$. Let $\beta := \varphi(v_iv_{i+1})$. Recolor edge $v_iv_{i+1}$ with $\alpha$, and call this new coloring $\varphi'$. Now $\beta \in \overline{\varphi'}(v_i)$. Also, since $\varphi(v_iv_{i+1}) = \beta$, there exists $h \in \{0, \ldots, i-1\}$ such that $\beta \in \overline{\varphi}(v_h)$. We apply the induction hypothesis to $v_i$ and $v_h$, which shows that $H'$ has a $k$-edge-coloring.

So assume instead that $j - i \geqslant 2$. Choose $\beta \in \overline{\varphi}(v_{i+1})$ and let $P := P_{v_{i+1}}(\alpha, \beta)$. Form $\varphi'$ from $\varphi$ by recoloring $P$. If $P$ does not end at $v_i$, then we invoke the (primary) induction hypothesis, keeping $i$ unchanged and letting $j := i + 1$. We must check that $v_0, \ldots, v_{i+1}$ remains a Kierstead path after the Kempe swap. Note that $\{\alpha, \beta\} \subseteq \varphi(v_h) = \varphi'(v_h)$ for all $h < i$; now the definition of Kierstead path implies that $\varphi'(v_hv_{h+1}) = \varphi(v_hv_{h+1}) \notin \{\alpha, \beta\}$ for all $h < i$. So assume that $P$ ends at $v_i$. Now we invoke the (secondary) induction hypothesis, with $j$ unchanged and $i$ increased by 1. Again, we can check that $v_0, \ldots, v_s$ remains a Kierstead path after the Kempe swap, since $\beta \in \overline{\varphi'}(v_i)$ and $\alpha \in \overline{\varphi'}(v_{i+1})$. Thus, by induction $H'$ has a $k$-edge-coloring. □

*Second proof of Vizing's Theorem.* We use induction on $\|G\|$; the base case $\|G\| = 1$ is easy, so assume that $\|G\| > 1$. Choose an arbitrary edge $v_0v_1$ and let $G' := G - v_0v_1$. Let $k := \Delta(G) + 1$. By hypothesis, $\chi'(G') \leqslant \Delta(G') + 1 \leqslant k$, so let $\varphi$ be a $k$-edge-coloring of $G'$. To extend the coloring to $v_0v_1$ by Lemma 3.8, it suffices to find a Kierstead path $P$ starting from $v_0v_1$ and a color $\alpha$ missed at distinct vertices of $P$. Suppose we have constructed a Kierstead path $v_0, \ldots, v_i$ for some $i \geqslant 1$. We can assume that at most one vertex misses each color.

Now we extend $P$ to $v_{i+1}$. Since $k > \Delta(G)$, each vertex misses at least one color; further $v_0$ and $v_1$ each miss at least two colors. Thus, $\left|\bigcup_{h=0}^{i-1} \overline{\varphi}(v_h)\right| \geqslant i+2$. Each color missed earlier on the path is seen at $v_i$, and we must extend the path by following an edge to a new vertex. Since $G$ is simple, each of $v_0, \ldots, v_{i-1}$ forbids at most one edge incident to $v_i$. Thus, we have at least 2 choices to extend the path. We can keep extending the path until two of its vertices miss a common color. By Pigeonhole, this happens before the path has length $k$. Now by Lemma 3.8, we can extend the $k$-edge-coloring to $G$. □

### 3.1.3 Vizing's Adjacency Lemma and 2 Applications

Vizing proved that in an edge-critical graph every vertex has at least two $\Delta$-neighbors, and if a vertex $v$ has a low-degree neighbor, then $v$ has even more $\Delta$-neighbors. Next we prove this so-called "adjacency lemma", and its generalization, the Fan Equation. We use the phrase *with respect to* frequently, so we often shorten it to *w.r.t.*

w.r.t.

multi-fan

**Definition 3.9.** Let $G$ be a multigraph with an edge $e$ with endpoints $v$ and $w$. Let $\varphi$ be a $k$-edge-coloring of $G - e$, for some integer $k \geqslant \Delta$. A *Vizing multi-fan*, or simply *multi-fan*, at $w$ w.r.t. $e$ and $\varphi$ is a sequence $(e_1, v_1, \ldots, e_s, v_s)$ such that $e_1 = e$, each $e_j$ is an edge, each $v_j$ is a vertex, and the following two conditions hold: (i) the edges are distinct, and for each $j \in [s]$ edge $e_j$ has endpoints $v_j$ and $w$, and (ii) for every $j \in \{2, \ldots, s\}$, there exists $h \in [j-1]$ such that $\varphi(e_j) \in \overline{\varphi}(v_h)$. That is, each color used on an edge of the fan is missed at an earlier vertex in the fan. (Note that (ii) is also required in the definition of Kierstead path.) When the multi-fan is simple, we may simply call it a *fan*. Figure 3.4 shows an example.

fan

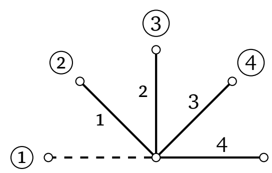



Figure 3.4: A simple Vizing fan.

k, e

φ, F, w

s

**Lemma 3.10.** *Let $G$ be a graph with $\chi'(G) = k + 1$ for some $k \geqslant \Delta$. Let $e$ be an edge such that $\chi'(G - e) = k$, and $w$ an endpoint of $e$. Let $\varphi$ be a $k$-edge-coloring of $G - e$, and let $F$ be a multi-fan $(e_1, v_1, \ldots, e_s, v_s)$ at $w$, w.r.t. $e$ and $\varphi$. The following two statements hold.*

*(a) The sets $\overline{\varphi}(w), \overline{\varphi}(v_1), \ldots, \overline{\varphi}(v_s)$ are pairwise disjoint.*

*(b) If $F$ is a maximal multi-fan at $w$ w.r.t. $e$ and $\varphi$, then*

$$\sum_{v_i \in V(F)} (d(v_i) + \mu_F(v_i w) - k) = 2. \tag{3.1}$$

Given a multi-fan $F$ and partial edge-coloring $\varphi$ satisfying the hypotheses, the key idea is that we can "move the uncolored edge" to be any edge of $F$, by shifting around the colors on the edges of $F$. This is easy to prove by induction on $|F|$. Suppose there exist $\alpha \in \overline{\varphi}(w) \cap \overline{\varphi}(v_i)$ for some $i$. Now we make $wv_i$ be the sole uncolored edge, and color it with $\alpha$. If there exists $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_j)$, then after a Kempe swap we can reduce to the previous case. This proves (a). Part (b) follows from (a) by a simple counting argument.

*Proof.* (a) First we prove that $\overline{\varphi}(w) \cap \overline{\varphi}(v_i) = \emptyset$, for all $i \in [s]$. Assume the contrary, and choose $i$ and $\alpha$ such that $\alpha \in \overline{\varphi}(w) \cap \overline{\varphi}(v_i)$. We get a $k$-edge-coloring of $G$ by induction on $i$, as follows. If $i = 1$, then we color $e_1$ with $\alpha$. Otherwise, let $\beta = \varphi(e_i)$. Now recolor $e_i$ with $\alpha$. Since $F$ is a multi-fan, there exists $j \in [i-1]$ with $\beta \in \overline{\varphi}(v_j)$. Now $\beta$ is unused at both $w$ and $v_j$. So by induction $G$ has a $k$-edge-coloring, a contradiction.

Now we show that $\overline{\varphi}(v_i) \cap \overline{\varphi}(v_j) = \emptyset$ for all distinct $i, j \in [s]$. The main step is proving the following claim. For all $i \in [s]$ and $\alpha \in \overline{\varphi}(v_i)$ and $\beta \in \overline{\varphi}(w)$, the path $P_{v_i}(\alpha, \beta)$ ends at $w$. Assume the contrary, and choose $i$ minimum such that the claim is false. Let $P := P_{v_i}(\alpha, \beta)$, and form $\varphi'$ from $\varphi$ by recoloring $P$. Since $\alpha \in \overline{\varphi}(v_i)$, $P$ is a path. By assumption, $P$ does not end at $w$; so $w \notin V(P)$, since $\beta \in \overline{\varphi}(w)$. Thus, none of $e_1, \ldots, e_i$ appears on $P$. So $\varphi'(e_j) = \varphi(e_j)$ for each $j \in \{2, \ldots, i\}$. Similarly, the colors seen at $w$ and at each $v_j$, with $j \in [i-1]$, are unchanged. Thus, $(e_1, v_1, \ldots, e_i, v_i)$ is a multi-fan at $w$ w.r.t. $e$ and $\varphi'$. But now $\beta \in \varphi'(w) \cap \varphi'(v_i)$, which contradicts the previous paragraph. This proves the claim. To complete the proof of (a), assume that it is false, and choose $i$, $j$, and $\alpha$ such that $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_j)$. Choose $\beta \in \overline{\varphi}(w)$. Now either $v_i$ or $v_j$ contradicts the claim, since the $\alpha, \beta$-chain starting at $w$ ends at exactly one vertex.

(b) Let $F$ be a maximal multi-fan at $w$ w.r.t. $e$ and $\varphi$. By definition, every color used on an edge of $F$ is missed at a vertex of $F$. Conversely, by (a) and the maximality of $F$, every color missed at a vertex $v_i$ of $F$ is used on a edge of $F$. Since $e_1$ is uncolored, this gives $s - 1 = \sum_{v_i \in V(F)} |\overline{\varphi}(v_i)| = 1 + \sum_{v_i \in V(F)} (k - d(v_i))$. Since $F$ is maximal, $s = \sum_{v_i \in V(F)} \mu_F(v_i w)$, so substituting and regrouping terms gives the desired equation. □

We call Equation (3.1) the Fan Equation. It yields two immediate corollaries.

**Lemma 3.11 (Vizing's Adjacency Lemma).** *Let $G$ be an edge-critical simple graph with maximum degree $\Delta$ and $\chi'(G) > \Delta$. If vertices $v$ and $w$ are adjacent, then $w$ has at least $\max\{\Delta + 1 - d(v), 2\}$ $\Delta$-neighbors.*

*Proof.* Let $e$ be an edge with endpoints $v$ and $w$, and let $\varphi$ be a $k$-edge-coloring of $G - e$, where $k = \Delta$. Let $F$ be a maximal multi-fan at $w$ with respect to $e$ and $\varphi$. Since $G$ is simple, note that every term in the sum of Equation (3.1) is at most 1, and the term for each vertex $v_i$ equals 1 precisely when $d(v_i) = \Delta$. Since the term for $v$ equals $d(v) + 1 - \Delta$, the sum must contain at least $\Delta + 1 - d(v)$ terms for $\Delta$-vertices, as desired. □

We use Vizing's Adjacency Lemma often, and typically abbreviate it as VAL.

**Theorem 3.12 (Vizing's Theorem for Multigraphs).** *If $G$ is a multigraph with maximum edge multiplicity $\mu$, then $\chi'(G) \leqslant \Delta + \mu$.*

*Proof.* Let $H$ be an edge-critical subgraph of $G$ with $\Delta(H) = \Delta$. When $k \geqslant \Delta + \mu$, every term on the left in Equation (3.1) is non-positive, so the Fan Equation fails to hold. Thus, $\chi'(G) = \chi'(H) \leqslant \Delta + \mu$. □

The proof of Lemma 3.10 also yields the following corollary, which we use in Section 3.3.

$k$ $\varphi$

**Corollary 3.13.** *Let* $G$ *be a simple graph, and* $k$ *be an integer with* $k \geqslant \Delta + 2$. *Given any* $k$*-edge-coloring* $\varphi$ *of* $G$*, we can form from* $\varphi$ *a* $(k-1)$*-edge-coloring of* $G$ *by a series of Kempe swaps.*

$m_k$ $M$ $e, G', \varphi'$

*Proof.* Let $\varphi$ be a $k$-edge-coloring of $G$. We use induction on $m_k$, the number of edges colored $k$ in $\varphi$. The base case, $m_k = 0$ is trivial. So assume $m_k \geqslant 1$. Let $M$ denote the set of edges colored $k$ and choose an arbitrary edge $e \in M$. Let $G' := G - M + e$. Let $\varphi'$ denote the restriction of $\varphi$ to colors $1, \ldots, k-1$. Note that $\varphi'$ is a $(k-1)$-edge-coloring of $G' - e$. Implicit in the proof of Lemma 3.10 is an algorithm to transform $\varphi'$ into a $(k-1)$-edge-coloring of $G'$, as follows.

Since $k - 1 \geqslant \Delta + 1$, substituting $k-1$ for $k$ in (3.1) (applied to $\varphi'$) yields a statement that is false. Recall that (3.1) was derived from Lemma 3.10(a) by an easy counting argument. So the falseness of (3.1), with $k-1$ in place of $k$, implies that the sets $\overline{\varphi'}(w), \overline{\varphi'}(v_1), \ldots, \overline{\varphi'}(v_s)$ are not all pairwise disjoint. Since the proof of Lemma 3.10(a) was constructive, we get an algorithm to transform $\varphi'$ to a $(k-1)$-edge-coloring of $G'$. Further, each step in this algorithm is either a Kempe swap or simply recoloring an edge. But the latter is also a Kempe swap. Specifically, consider an edge $wv_i$ that is colored $\beta$. If $\alpha \in \overline{\varphi}(w) \cap \overline{\varphi}(v_i)$, then to recolor $wv_i$ with $\alpha$, we simply perform an $\alpha, \beta$-swap at $w$ or at $v_i$.

Now we view the series of Kempe swaps above in the context of a $k$-edge-coloring of $G$. Since color $k$ is involved only in the final Kempe swap (recoloring $wv_1$ with a color other than $k$), this process does not effect any edge in $M - e$. Thus, the result is a $k$-edge-coloring of $G$ with $m_k - 1$ edges colored $k$. So by induction on $m_k$, the corollary holds. $\square$

**Proposition 3.14.** *Let* $G$ *be a simple,* $k$*-degenerate graph. If* $\Delta(G) \geqslant 2k$*, then* $\chi'(G) = \Delta(G)$*. In particular, if* $G$ *is planar with* $\Delta \geqslant 10$*, then* $\chi'(G) = \Delta(G)$*.*

*Proof.* The second statement follows from the first when $k := 5$; now we prove the first. Let $H$ be an edge-critical subgraph of $G$ with $\Delta(H) = \Delta(G)$; see Exercise 1. Let $J$ be the subgraph of $H$ induced by all vertices $v$ with $d_H(v) \geqslant k+1$. Since $G$ is $k$-degenerate, there exists $w \in J$ such that $d_J(w) \leqslant k$. Since $d_H(w) > d_J(w)$, vertex $w$ has some neighbor $x$ in $H$ such that $d_G(x) \leqslant k$. Now by VAL, the number of $\Delta(H)$-neighbors of $w$ is at least $\Delta(H) + 1 - d(x) \geqslant 2k + 1 - k = k + 1$; this contradicts that $d_J(x) \leqslant k$. Thus, we conclude that $\chi'(G) = \Delta(G)$, as claimed. $\square$

In our next theorem, we strengthen the previous result when $G$ is a planar graph. All of our reducible configurations come from Vizing's Adjacency Lemma (VAL).

**Theorem 3.15.** *If* $G$ *is planar with* $\Delta \geqslant 8$*, then* $\chi'(G) = \Delta$*.*

*Proof.* Let $G$ be a planar graph with $\Delta \geqslant 8$, and suppose that $G$ is a counterexample to the theorem, i.e., $\chi'(G) > \Delta$. We assume that $G$ is edge-critical; if not, then we take an edge-critical subgraph with the same maximum degree. Note that VAL implies that $\delta(G) \geqslant 2$.

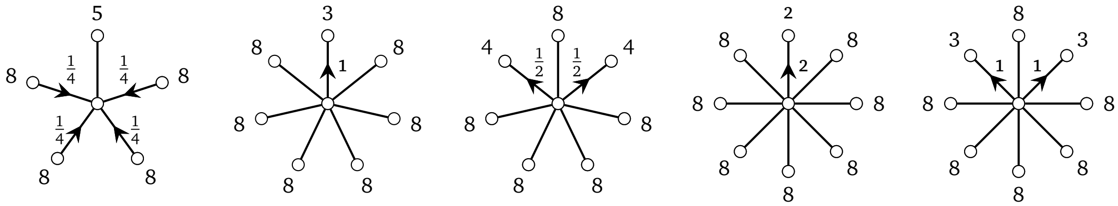


Figure 3.5: Some cases in the proof of Theorem 3.15 where vertices end with charge 0.

Further, VAL implies that the $4^-$-vertices induce an independent set. We use discharging with initial charge $d(v) - 6$ and the two discharging rules below. Figure 3.5 shows some examples. Since $\overline{d}(G) < 6$, to reach a contradiction we show that every vertex ends happy.

(R1) Each vertex $v$ with $d(v) \in \{2, 3, 4\}$ takes $\frac{6-d(v)}{d(v)}$ from each of its neighbors.

(R2) Each vertex $v$ with $d(v) \in \{5, 6\}$ takes $\frac{1}{4}$ from each of its $6^+$-neighbors.

Now we show that every vertex ends happy. For a vertex $v$, let $j$ equal the smallest degree of its neighbors. By VAL, $v$ has at least $\Delta + 1 - j$ neighbors of degree $\Delta$. Since $\Delta \geqslant 8$, if $v$ is a $6^-$-vertex, then $v$ takes charge from at least $9-j$ of its neighbors. So $d(v) \geqslant (9-j)+1$, which implies that $j \geqslant 10 - d(v)$. Similarly, $v$ gives charge to at most $d(v) - (\Delta + 1 - j)$ vertices, which is at most $d(v) + j - 9$. To show that each vertex ends happy, we consider the possible values of $d(v)$. When $v$ is a $6^+$-vertex, we also consider the possible values of $j$.

**Case 1: $d(v) \leqslant 4$.** Because the $4^-$-vertices induce an independent set, $v$ loses no charge and ends happy by (R1), since $d(v) - 6 + d(v)\frac{6-d(v)}{d(v)} = 0$.

**Case 2: $d(v) = 5$.** VAL implies that $v$ has at least four $6^+$-neighbors. So $v$ ends happy by (R2), since $5 - 6 + 4(\frac{1}{4}) = 0$.

**Case 3: $d(v) = 6$.** Now $j \geqslant 10 - 6 = 4$. We consider the three cases $j = 4$, $j = 5$, and $j \geqslant 6$. If $j = 4$, then $v$ gives at most $1(\frac{2}{4})$, by (R1), and gets at least $5(\frac{1}{4})$, by (R2). If $j = 5$, then $v$ gives at most $2(\frac{1}{4})$ and gets at least $4(\frac{1}{4})$. If $j \geqslant 6$, then $v$ gives at most $3(\frac{1}{4})$ and gets at least as much as it gives. So $v$ ends happy, since $0 + \min\{-\frac{2}{4} + 5(\frac{1}{4}), -2(\frac{1}{4}) + 4(\frac{2}{4}), 0\} = 0$.

**Case 4: $d(v) = 7$.** Now $j \geqslant 10 - 7 = 3$, so either $j = 3$, $j = 4$, $j = 5$, $j = 6$, or $j \geqslant 7$. Similar to the previous case, $v$ ends happy, since $7 - 6 - \max\{1(\frac{3}{3}), 2(\frac{2}{4}), 3(\frac{1}{4}), 4(\frac{1}{4}), 0\} = 0$.

**Case 5: $d(v) \geqslant 8$.** Now $j \geqslant 10 - 8 = 2$, so either $j = 2$, $j = 3$, $j = 4$, $j = 5$, $j = 6$, or $j \geqslant 7$. Here $v$ ends happy, since $8 - 6 - \max\{1(\frac{4}{2}), 2(\frac{3}{3}), 3(\frac{2}{4}), 4(\frac{1}{4}), 5(\frac{1}{4}), 0\} = 0$. $\square$

Hadwiger conjectured that every graph $G$ with chromatic number $k$ contains $K_k$ as a minor. That is, we can form $K_k$ from some subgraph of $G$ by contracting edges. This is the biggest open question in graph coloring, and we say more about it in the Notes. Below we prove Hadwiger's Conjecture for line graphs of multigraphs.

**Theorem 3.16.** *Hadwiger's Conjecture is true for line graphs of multigraphs. More formally, if* $G$ *is a multigraph with* $\chi'(G) = k+1$*, then there exist connected edge-disjoint subgraphs* $H_1, \ldots, H_{k+1}$ *such that* $V(H_i) \cap V(H_j) \neq \emptyset$ *whenever* $1 \leqslant i < j \leqslant k+1$.

The proof is a nice application of the Fan Equation and Menger's Theorem. For reference, we state the latter below and defer the proof (see Theorem A.6(c)).

**Theorem 3.17 (Menger's Theorem).** *Fix a graph* $G$*. For every pair of distinct vertices* $v$ *and* $w$ *in* $G$*, the maximum number of edge-disjoint* $v, w$*-paths equals the minimum size of an edge-cut that disconnects* $v$ *from* $w$.

*Proof of Theorem 3.16.* Assume the theorem is false. Let $G$ be a counterexample that minimizes
$k$ $|G|$. We can assume that $G$ is edge-critical. Let $k := \chi'(G) - 1$. Clearly $\Delta \leqslant k$, since otherwise
$w, e, \varphi$ we can take as $H_1, \ldots H_{k+1}$ distinct edges incident to a $\Delta$-vertex. Let $w$ be any $\Delta$-vertex, $e$ be
$F$ any edge incident to $w$, and $\varphi$ be any $k$-edge-coloring of $G - e$. Finally, let $F$ be a maximal multi-fan $(e_1, v_1, \ldots, e_s, v_s)$ at $w$ with respect to $e$ and $\varphi$. Since the Fan Equation holds for $F$,

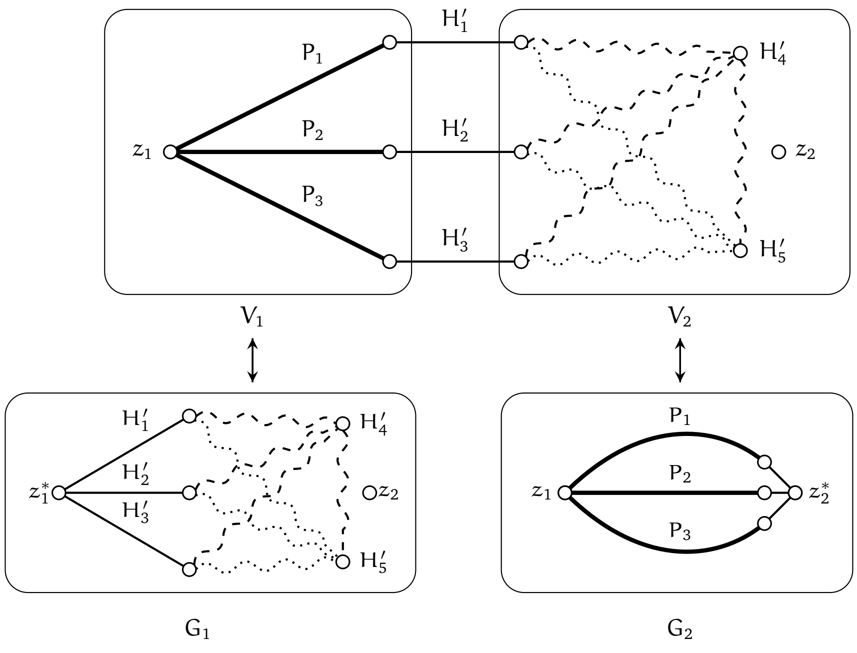



Figure 3.6: Form a partition $V_1, V_2$ of $V(G)$ that minimizes $|E(V_1, V_2)|$ such that $z_1 \in V_1$ and $z_2 \in V_2$. Form $G_1$ and $G_2$ from $G$ by contracting $V_1$ and $V_2$, respectively. Given the desired subgraphs $H'_1, \ldots, H'_{k+1}$ in $G_1$ and $P_1, \ldots, P_{|E(V_1,V_2)|}$ in $G_2$, we lift them all to $G$ and combine them to form the desired subgraphs in $G$.

we have

$$2 = \sum_{i=1}^{s}(d(v_i) + \mu(wv_i) - k) = \sum_{i=1}^{s}(d(v_i) + \mu(wv_{i+1}) - k), \tag{3.2}$$

where we write $v_{s+1}$ for $v_1$. Equation (3.2) implies there exists $i$ such that $d(v_i) + \mu(wv_{i+1}) \geqslant k+1$. To reach a contradiction, we show that $v_i$ has $d(v_i)$ edge-disjoint paths to $\{w, v_{i+1}\}$, which we can choose to avoid edges between $w$ and $v_{i+1}$. To get our desired edge-disjoint subgraphs $H_1, \ldots, H_{k+1}$, we take all parallel edges $wv_{i+1}$ and this set of edge-disjoint paths from $v_i$ to $\{w, v_{i+1}\}$. This is at least $k+1$ subgraphs since $d(v_i) + \mu(wv_{i+1}) \geqslant k+1$. Now it suffices to show that there exist $d(v_i)$ edge-disjoint $v_i, w$-paths (we truncate each path when it first reaches $\{w, v_{i+1}\}$). In fact, something stronger holds: For all distinct pairs $z_1, z_2 \in V(G)$, the maximum number of edge-disjoint $z_1, z_2$-paths is $\min\{d(z_1), d(z_2)\}$.

Suppose to the contrary that $G$ has vertices $z_1$ and $z_2$ without $\min\{d(z_1), d(z_2)\}$ edge-disjoint $z_1, z_2$-paths. We partition $V(G)$ into $V_1$ and $V_2$ such that $z_1 \in V_1$ and $z_2 \in V_2$ and subject to this $|E(V_1, V_2)|$ is minimized; see the top of Figure 3.6. Let $r := |E(V_1, V_2)|$. By Menger's Theorem, $r < \min\{d(z_1), d(z_2)\}$. This implies that $|V_1| \geqslant 2$ and $|V_2| \geqslant 2$. Form $G_1$ from $G$ by identifying all vertices in $V_1$, deleting any resulting loops; call the new vertex $z_1^*$. Form $G_2$ and $z_2^*$ analogously, by identifying vertices in $V_2$; see the bottom of Figure 3.6. If both $G_1$ and $G_2$ are $k$-edge-colorable, then so is $G$: we permute the color classes for $G_2$ to agree with those for $G_1$ on $E(V_1, V_2)$. So assume $\chi'(G_1) > k$ or $\chi'(G_2) > k$.

$r$

$G_1, G_2$

$z_1^*, z_2^*$

Note that $|G_1| < |G|$ and $|G_2| < |G|$. So by minimality, the theorem holds for both $G_1$ and $G_2$. By symmetry, we assume that $\chi'(G_1) > k$. So there exist edge-disjoint subgraphs $H_1', \ldots, H_{k+1}'$ in $G_1$ such that $V(H_i') \cap V(H_j') \neq \emptyset$ whenever $1 \leqslant i < j \leqslant k+1$. When viewed as subgraphs of $G$, these $H_1', \ldots, H_{k+1}'$ are still edge-disjoint. However, some pair $H_i'$ and $H_j'$ may not share a vertex in $G$ if in $G_1$ they shared only the vertex $z_1^*$. We fix this problem as follows. By the minimality of our choice of the partition $(V_1, V_2)$, in $G_2$ no edge-cut of size less than $r$ disconnects $z_1$ and $z_2^*$. So, by Menger's Theorem, $G_2$ contains $r$ edge-disjoint $z_1, z_2^*$-paths, $P_1, \ldots, P_r$. Now we are nearly done. We order the subgraphs $H_1', \ldots, H_{k+1}'$ of $G_1$ such that each one including an edge incident to $z_1^*$ (in $G$, these are the edges of $E(V_1, V_2)$) is among the first $r$. Now the subgraphs $H_1' \cup P_1, \ldots, H_r' \cup P_r, H_{r+1}', \ldots, H_{k+1}'$ of $G$ are edge-disjoint, and each pair shares a common vertex; see the top of Figure 3.6. So $G$ is not a counterexample, and this contradiction completes the proof. □

## 3.2 A Glimpse of the 4 Color Theorem

In this section we apply the recoloring ideas above in the more general context of vertex coloring. One notable difference here is that the subgraph induced by any two colors classes, $\alpha$ and $\beta$, need not be a path or an even cycle. As a result, the theory of Kempe swaps for vertex coloring is far less developed than for edge-coloring.

Although the proof of the 4 Color Theorem is long[4], the central idea is simple: reducibility and unavoidability. One thorny complication is showing that the unavoidable subgraphs appear not just as subgraphs, but as induced subgraphs. Appel and Haken lacked a consistent way of doing this, which lead to many "immersion" difficulties in their proof. To explain a solution to this problem, we need a few definitions.

internally 6-connected

2-neighborhood

$\alpha, \beta$-component

$\alpha, \beta$-swap

**Definition 3.18.** A minimal counterexample to the 4 Color Theorem is a counterexample $G$ that minimizes $|G|$. A graph $G$ is *internally* 6*-connected* if every cutset $X$ of $G$ has $|X| \geqslant 5$, and for every cutset $X$ of size 5, the subgraph $G \setminus X$ has exactly two components, one of which is an isolated vertex. (We will show that each minimal counterexample is internally 6-connected.) The 2*-neighborhood*, $N^2(v)$, of each vertex $v$ is the set of vertices at distance at most 2 from $v$.

Given a partial vertex coloring of a graph $G$ and colors $\alpha$ and $\beta$, an $\alpha, \beta$*-component* is a component of the subgraph induced by vertices colored $\alpha$ or $\beta$. An $\alpha, \beta$*-swap* at a vertex $v$ interchanges the colors used on vertices in the $\alpha, \beta$-component containing $v$.

In 1913 Birkhoff proved that every minimal counterexample to the 4 Color Theorem must be an internally 6-connected plane triangulation. A consequence is that the subgraph induced by each vertex $v$ and its 2-neighbors must be "well-behaved", which we make precise below. Informally, this means that every reducible subgraph appearing in the 2-neighborhood of a vertex must appear as an induced subgraph. So, to avoid immersion difficulties, it suffices to find a set of unavoidable configurations that must each appear in the 2-neighborhood of some vertex. This is exactly what Robertson, Sanders, Seymour, and Thomas did [344].

**Lemma 3.19.** *Let $G$ be an internally 6-connected plane triangulation. For each vertex $v$, its neighbors induce a chordless cycle; also the vertices at distance 2 from $v$ induce a chordless cycle.*

*Proof.* Let $G$ satisfy the hypothesis. This implies that $G$ has no separating 3-cycle and no separating 4-cycle; in particular, $\delta(G) \geqslant 5$. Choose an arbitrary $v \in V(G)$.

$w_1, \ldots, w_{d(v)}$

Let $w_1, \ldots, w_{d(v)}$ be the neighbors of $v$ in clockwise order (the neighbors of $w_1$, followed by those of $w_2$, etc.[5]). Since $G$ is a triangulation, $G$ contains the cycle $w_1 \cdots w_{d(v)}$. Suppose this cycle also contains a chord $w_i w_j$. Now the 3-cycle $v w_i w_j$ separates $w_{i-1}$ from $w_{i+1}$, a contradiction. So $N(v)$ induces a chordless cycle.

$x_1, \ldots, x_s$

Denote by $x_1, \ldots, x_s$ the vertices at distance two from $v$, in clockwise order. Since $G$ is a triangulation, each successive pair $x_i, x_{i+1}$ (subscripts modulo $s$) must have a common neighbor $w_j$ in $N(v)$, and $w_j$ is unique, since $G$ has no separating 4-cycle. Since $G$ is a triangulation, edge $x_i x_{i+1}$ is present. So $x_1, \ldots, x_s$ induces a cycle, $C$, possibly with chords. We now show that this cycle $C$ must be chordless.

Suppose that $C$ has a chord $x_i x_j$, with $i < j-1$. If $x_i$ and $x_j$ have a common neighbor $w_h$, with $w_h \in N(v)$, then the cycle $x_i x_j w_h$ separates $x_{i-1}$ from $x_{i+1}$, a contradiction. So assume

[4]There are at least three distinct proofs, one of which was also encoded to be verified by a formal proof checker. But all three follow the same outline, and all three are long.

[5]Since $G$ has no separating 4-cycle, each pair $w_i$ and $w_j$ have at most one common neighbor other than $v$, and if they have one, then $|i-j| = 1$. So this ordering can be made precise.

that $x_i$ and $x_j$ have no such common neighbor. Let $P_1$ and $P_2$ denote a shortest $v, x_i$-path and a shortest $v, x_j$-path. And let $D$ denote the 5-cycle induced by $P_1, P_2, x_ix_j$. Note that $D$ separates $x_{i-1}$ from $x_{i+1}$. Further, $x_{i+1}$ has a neighbor inside $D$, since $d(x_{i+1}) \geqslant 5$ and $x_{i+1} \not\leftrightarrow v$. So $D$ has at least two vertices in its interior. Likewise, $D$ has at least two vertices in its exterior, which contradicts that $G$ is internally 6-connected. Thus, $x_1, \ldots, x_s$ induces a chordless cycle. □

**Theorem 3.20.** *If $G$ is a minimal counterexample to the 4 Color Theorem, then $G$ is an internally 6-connected plane triangulation.*

*Proof.* Let $G$ be a minimal counterexample to the 4 Color Theorem. If $G$ has a cut-vertex, then we can 4-color each block by minimality and permute color classes to agree on the cut-vertices. So assume $G$ is 2-connected. Suppose $G$ has a $4^+$-face $f$, say with boundary $v_1 \cdots v_{\ell(f)}$. By planarity, either $v_1 \not\leftrightarrow v_3$ or $v_2 \not\leftrightarrow v_4$. So we can identify some non-adjacent pair on $f$ to form a smaller plane graph $G'$. By minimality, $G'$ has a 4-coloring $\varphi$. But $\varphi$ also gives a 4-coloring of $G$, a contradiction. Thus, $G$ is a plane triangulation.

Since $G$ is a plane triangulation, every minimal cutset must induce a cycle, $C$, with one component inside $C$ and another component outside. (This is intuitively clear, but making it precise requires the Jordan Curve Theorem, so we omit the details.) Thus, it suffices to show that every separating $5^-$-cycle in $G$ is a 5-cycle that separates only a single vertex from the rest of the graph. For a separating cycle $C$, let $C_{in}$ denote the subgraph induced by $C$ and the vertices of $G$ inside $C$. Similarly, define $C_{out}$ for $C$ and the vertices outside. $C_{\text{in}}$ $C_{\text{out}}$

Suppose that $G$ has a separating 3-cycle, $C$. By minimality, $C_{\text{in}}$ has a 4-coloring $\varphi_{\text{in}}$; likewise, $C_{\text{out}}$ has a 4-coloring $\varphi_{\text{out}}$. In each coloring, the vertices of $C$ get distinct colors. So we can permute the color classes of $\varphi_{\text{out}}$ to agree on $V(C)$ with those of $\varphi_{\text{in}}$. Together, these colorings give a 4-coloring of $G$, a contradiction. So $G$ has no separating 3-cycle.

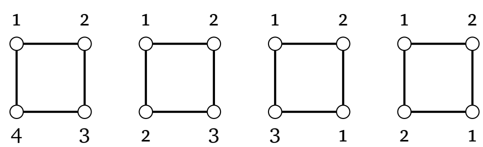



Figure 3.7: The 4 ways to 4-color a 4-cycle.

Suppose that $G$ has a separating 4-cycle, $v_1v_2v_3v_4$; call it $C$. In every 4-coloring of $V(C)$, the colors around $C$ in order of increasing index, up to permuting colors, are either 1234, 1232, 1213, or 1212 (we omit the commas within each type of coloring); see Figure 3.7. $v_1, \ldots, v_4$ $C$

Form $C'_{\text{in}}$ and $C''_{\text{in}}$ from $C_{\text{in}}$ by adding edges $v_1v_3$ and $v_2v_4$, respectively (if they are not yet present). Form $C'_{\text{out}}$ and $C''_{\text{out}}$ analogously; see Figure 3.8. Note that every 4-coloring of $C'_{\text{out}}$ (and $C'_{\text{in}}$) has type 1234 or 1232. If any 4-colorings of $C_{\text{in}}$ and $C_{\text{out}}$ have the same type, then they combine to give a 4-coloring of $G$, a contradiction. So by symmetry we assume that $C'_{\text{out}}$ and $C''_{\text{out}}$ have only 4-colorings of type 1234, while $C'_{\text{in}}$ and $C''_{\text{in}}$ have only 4-colorings of types 1232 and 1213, respectively. Let $\varphi$ be a 4-coloring of $C_{\text{out}}$ of type 1234. By planarity, either $v_1$

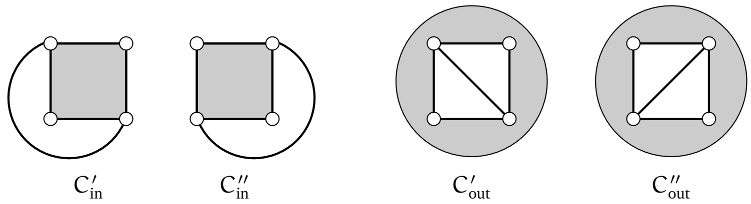



Figure 3.8: The 4 graphs $C'_{\text{in}}$, $C''_{\text{in}}$, $C'_{\text{out}}$, $C''_{\text{out}}$.

and $v_3$ are in different $1,3$-components of $\varphi$ or else $v_2$ and $v_4$ are in different $2,4$-components (if not then these $1,3$- and $2,4$-components intersect, a contradiction). By either a $1,3$-swap at $v_1$ or a $2,4$-swap at $v_2$, we form from $\varphi$ a 4-coloring $\varphi'$ of $C_{\text{out}}$ of type 1232 or type 1213. In each case, we combine $\varphi'$ with a 4-coloring of the same type of $C_{\text{in}}$, after possibly permuting color classes. This gives a 4-coloring of G, a contradiction. So G has no separating 4-cycle.

$v_1, \dots, v_5$ Finally, suppose that G has a separating 5-cycle, $v_1v_2v_3v_4v_5$, with at least two vertices inside and at least two outside. Define $C_{\text{in}}$ and $C_{\text{out}}$ as above. Up to permuting colors, the restriction of a 4-coloring to $v_1 \cdots v_5$ that uses only 3 colors on $V(C)$ has one of the five types shown in the top row of Figure 3.9 (in clockwise order from $v_1$, on the top left): 12123, 31212, 23121, 12312, 21231. If a 4-coloring uses 4 colors on $V(C)$, then it has one of the five types shown in $P_i, Q_i$ the bottom row of Figure 3.9: 12134, 41213, 34121, 13412, 21341. Let $P_i(C_{\text{in}})$ be a boolean function that is true when $C_{\text{in}}$ has a 4-coloring of the $i$th type in the top row; similarly for $C_{\text{out}}$. We define $Q_i(C_{\text{in}})$ and $Q_i(C_{\text{out}})$ analogously for the 5 types in the bottom row. To reach a $H$ contradiction, we prove two claims. In each, we always assume that $H \in \{C_{\text{in}}, C_{\text{out}}\}$.

**Claim 1.** $P_i(H) \wedge \neg P_{i+1}(H) \Rightarrow Q_i(H)$.

<u>Proof.</u> By rotational symmetry, we assume that $i = 1$; that is, $\varphi(v_1) = 1$, $\varphi(v_2) = 2$, $\varphi(v_3) = 1$, $\varphi(v_4) = 2$, and $\varphi(v_5) = 3$. Form $\varphi'$ from $\varphi$ by a $1,3$-swap at $v_1$. Now $\varphi'(v_1) = 3$, $\varphi'(v_2) = 2$, $\varphi'(v_4) = 2$, and $\varphi'(v_5) = 1$. If $\varphi'(v_3) = 1$, then $P_2(H)$. So we assume that $\varphi'(v_3) = 3$. This means that $v_1$ and $v_3$ are in the same $1,3$-component, which implies that $v_2$ and $v_4$ are in different $2,4$-components. Now a $2,4$-swap at $v_4$, starting from $\varphi$, shows that $Q_1(H)$. ♢

**Claim 2.** $\neg P_{i+1}(H) \wedge \neg Q_{i+1}(H) \Rightarrow P_i(H)$.

<u>Proof.</u> By symmetry, we assume $i = 1$. Form $H'$ from $H$ by identifying $v_2$ and $v_4$. By minimality, $H'$ has a 4-coloring, and it induces a 4-coloring $\varphi$ of $H$ with $\varphi(v_2) = \varphi(v_4)$. By symmetry, we assume that $\varphi(v_2) = 1$ and $\varphi(v_3) = 2$. Either (a) $\varphi(v_5) = \varphi(v_3)$, (b) $\varphi(v_1) = \varphi(v_3)$, or (c) $\varphi(v_3) \notin \{\varphi(v_5), \varphi(v_1)\}$. Each case implies $P_2(H)$, $P_1(H)$, or $Q_2(H)$. Thus, the claim holds. ♢

Now we show that G has a 4-coloring. Form $C'_{\text{in}}$ and $C'_{\text{out}}$ from $C_{\text{in}}$ and $C_{\text{out}}$ by adding a single vertex adjacent to every $v_i$. By minimality $C'_{\text{in}}$ and $C'_{\text{out}}$ each have a 4-coloring that uses only 3 colors on $V(C)$. That is $P_i(C_{\text{in}})$ and $P_j(C_{\text{out}})$ for some $i, j \in [5]$. Clearly $i \neq j$, or

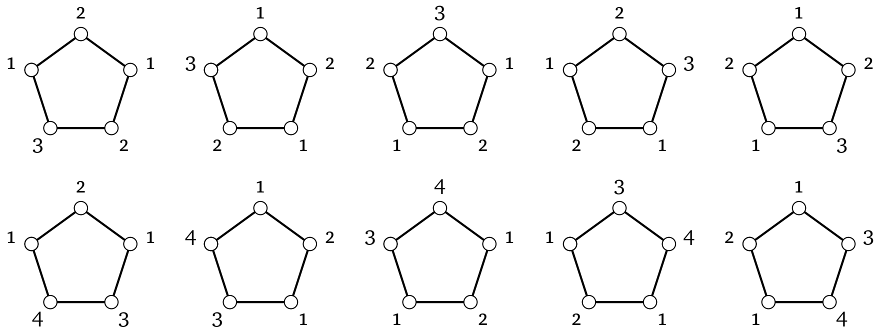


Figure 3.9: The 10 ways to 4-color a 5-cycle. The 3-colorings are on top.

else G has a 4-coloring. By symmetry, assume that $P_1(C_{in})$ and $\neg P_2(C_{in})$. By Claim 1, we get $Q_1(C_{in})$. Since $P_1(C_{in})$ and $Q_1(C_{in})$, we assume $\neg P_1(C_{out})$ and $\neg Q_1(C_{out})$. By Claim 2, this implies $P_5(C_{out})$, so by Claim 1, we get $Q_5(C_{out})$. So we assume that $\neg P_5(C_{in})$ and $\neg Q_5(C_{in})$. By Claim 2, this gives $P_4(C_{in})$. Now repeating the same arguments, starting from $P_4(C_{in})$ and $\neg P_5(C_{in})$, implies $P_2(C_{in})$, a contradiction. □

## 3.3 Kempe Equivalence of Colorings

In this section we are motivated by a question of Vizing [399, 401] from the 1960s (answered affirmatively in 2022, as we discuss in the Notes). Recall that a coloring is *optimal* if it uses as few colors as possible.

**Question 3.21** (Now answered)**.** Starting from any proper edge-coloring of a graph G, can we reach an optimal proper edge-coloring by a sequence of Kempe swaps, suppressing empty color classes (and never introducing more colors than in the initial edge-coloring)?

We are interested more generally in when we can reach one proper coloring from another, by repeated Kempe swaps. (Figure 3.10 shows an example where we cannot.)

**Definition 3.22.** For a graph G and an integer k, with $k \geqslant \chi(G)$, two k-colorings $\varphi$ and $\varphi_0$ of G are k-*Kempe equivalent* if $\varphi_0$ can be obtained from $\varphi$ by a sequence of Kempe swaps (never using more than k colors). For short, we write that $\varphi \sim_k \varphi_0$. When the context is clear, we say the colorings are k-*equivalent* or simply *equivalent*.

$\varphi \sim_k \varphi_0$
k-equivalent

How can we show that all k-colorings of a graph G are k-equivalent? Given k-colorings $\varphi$ and $\varphi_0$, we start from $\varphi$ and "move toward" $\varphi_0$. We use Kempe swaps to reach an equivalent k-coloring that agrees with $\varphi_0$ on one color class, say I. Now we need only prove the result for $G - I$, by induction on $\chi$. Our next lemma formalizes this approach.

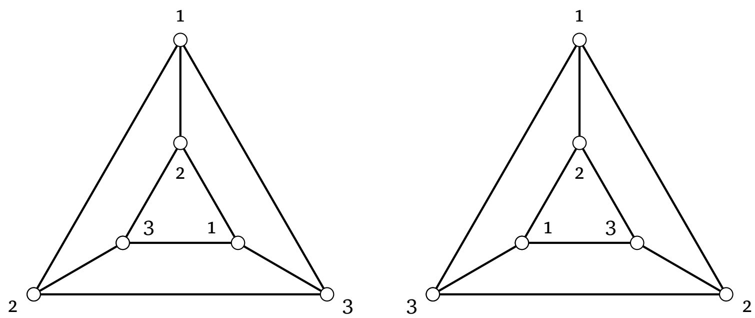



Figure 3.10: A 3-regular graph with two 3-colorings that are not 3-Kempe equivalent. This is the only connected, k-regular graph with k-colorings that are not k-Kempe equivalent.

**Lemma 3.23.** *Let* $k$ *be an integer and* $G$ *be a graph such that each* $k$*-coloring of* $G$ *is* $k$*-equivalent to some* $(k-1)$*-coloring of* $G$*. If* $I$ *is an independent set such that all* $(k-1)$*-colorings of* $G-I$ *are* $(k-1)$*-equivalent, then all* $k$*-colorings of* $G$ *are* $k$*-equivalent.*

*Proof.* Let $\varphi$ and $\varphi_0$ be $k$-colorings of $G$. By assumption, these colorings are $k$-equivalent (respectively) to $(k-1)$-colorings $\varphi^*$ and $\varphi_0^*$. We assume that $\varphi^*$ and $\varphi_0^*$ each use only $[k-1]$, as follows. If $\varphi^*$ uses $k$, then we replace color $k$ everywhere with the same unused color $\alpha$ (by a series of $k,\alpha$-swaps, which each recolor a single vertex); similarly for $\varphi_0^*$. Starting from $\varphi^*$ and from $\varphi_0^*$, we recolor each vertex of $I$ with $k$. This gives $k$-colorings $\varphi^{**}$ and $\varphi_0^{**}$ such that $\varphi^* \sim_k \varphi^{**}$ and $\varphi_0^* \sim_k \varphi_0^{**}$. Further, $\varphi^{**}$ and $\varphi_0^{**}$ each use $k$ only on $I$, so they both restrict to $(k-1)$-colorings of $G-I$. By assumption, these $(k-1)$-colorings of $G-I$ are $(k-1)$-equivalent. And the sequence of Kempe swaps that proves this $(k-1)$-equivalence also proves that $\varphi^{**} \sim_k \varphi_0^{**}$, since color $k$ is used only on $I$ throughout. So $\varphi \sim_k \varphi^* \sim_k \varphi^{**} \sim_k \varphi_0^{**} \sim_k \varphi_0^* \sim_k \varphi_0$, as desired. □

**Definition 3.24.** For edge-colorings, we abuse terminology. Two $k$-edge-colorings of a graph $G$ are $k$-*equivalent* when the corresponding $k$-colorings of the line graph of $G$ are $k$-equivalent.

**Theorem 3.25.** *If* $G$ *is a graph and* $k \geqslant \chi'(G)+2$*, then all* $k$*-edge-colorings of* $G$ *are* $k$*-equivalent.*

*Proof.* We use induction on $\chi'(G)$; the base case $\chi'(G) = 1$ is easy. So let $s := \chi'(G)$, and assume $s > 1$. We will apply Lemma 3.23 to the line graph. Fix an $s$-edge-coloring $\varphi$ of $G$, with color classes $M_1, \ldots, M_s$. Clearly $\chi'(G - M_s) = s - 1$. By the induction hypothesis, all $(k-1)$-edge-colorings of $G - M_s$ are $(k-1)$-equivalent. (Here $M_S$ is $I$.) To apply Lemma 3.23, it suffices to show that every $k$-edge-coloring of $G$ is $k$-equivalent to a $(k-1)$-edge-coloring of $G$. This follows from the Vizing fan results in Section 3.1, specifically, from Corollary 3.13. □

Now we return to vertex colorings.

**Theorem 3.26.** *If* $G$ *is* $d$*-degenerate and* $k > d$*, then all* $k$*-colorings of* $G$ *are* $k$*-equivalent.*

*Proof.* We use induction on $|G|$; the base case $|G| \leqslant 1$ is trivial. Fix $k$-colorings $\varphi$ and $\varphi_0$ of $G$. Let $v$ be a $d^-$-vertex, and let $G' := G - v$. Let $\varphi'$ and $\varphi'_0$ denote the restrictions to $G'$ of $\varphi$ $v, G'$ and $\varphi_0$. By hypothesis there exist $\varphi'_1, \ldots, \varphi'_r$ such that $\varphi'_0 \sim_k \varphi'_1 \cdots \varphi'_{r-1} \sim_k \varphi'_r = \varphi'$; in fact, $\varphi'_0, \ldots, \varphi'_r$ we can assume that $\varphi'_i$ differs from $\varphi'_{i-1}$ by only a single Kempe swap, for each $i \in [r]$. We construct a sequence of $k$-colorings $\varphi_1, \ldots, \varphi_r$ of $G$ such that $\varphi_i \sim_k \varphi_{i-1}$ and the restriction of $\varphi_i$ to $G'$ is $\varphi'_i$.

Choose $\alpha$ and $\beta$ such that $\varphi'_i$ is formed from $\varphi'_{i-1}$ by an $\alpha, \beta$-swap. If $\varphi_{i-1}(v) \notin \{\alpha, \beta\}$, then we form $\varphi_i$ from $\varphi_{i-1}$ using the same Kempe swap as in $G'$. This idea also works when $\varphi_{i-1}(v) \in \{\alpha, \beta\}$, as long as at most one neighbor of $v$ is colored from $\{\alpha, \beta\}$ (since the Kempe components of $G'$ are identical to those of $G$, except that one in $G$ includes $v$). So assume that $\varphi_{i-1}(v) = \alpha$ and at least two neighbors $w$ of $v$ have $\varphi_{i-1}(w) = \beta$. Since $d(v) < k$, some color $\gamma \in [k]$ is unused by $\varphi_{i-1}$ on $N(v) \cup \{v\}$. So we recolor $v$ with $\gamma$ (technically, an $\alpha, \gamma$-swap at $v$). Now we proceed as above.

So $\varphi_0 \sim_k \varphi_r$ for some $\varphi_r$ that agrees with $\varphi$ on every vertex except for $v$. If necessary, we recolor $v$ in $\varphi_r$ with its color in $\varphi$. This proves that $\varphi_0 \sim_k \varphi$, as desired. □

**Corollary 3.27.** *If* $G$ *is planar and* $k \geqslant 6$*, then all* $k$*-colorings of* $G$ *are Kempe equivalent.*

*Proof.* This is true because every planar graph is 5-degenerate. □

Next we improve by 1 the lower bound in Corollary 3.27.

**Theorem 3.28.** *If* $G$ *is planar and* $k \geqslant 5$*, then all* $k$*-colorings of* $G$ *are Kempe equivalent.*

The proof of Theorem 3.28 is by induction on $|G|$. For some $5^-$-vertex $v$, we assume the result for $G - v$ and use this to prove it for $G$. We need two lemmas. The first is proved in Section 11.1 (as Lemma 11.3), since it follows directly from Thomassen's proof that planar graphs are 5-choosable. The second captures key ideas that we use repeatedly in the induction step.

**Lemma 3.29.** *Let* $G$ *be a planar graph having a* $6^-$*-vertex* $v$ *with four neighbors* $w_1, w_2, y_1, y_2$*. (a) If* $w_1 \not\leftrightarrow w_2$*, then* $G$ *has a* $5$*-coloring* $\varphi$ *such that* $\varphi(w_1) = \varphi(w_2)$*. (b) If also* $y_1 \not\leftrightarrow y_2$*, then we can further require that* $\varphi(y_1) = \varphi(y_2)$*.*

**Lemma 3.30.** *Let* $G$ *be a planar graph such that, for each planar graph* $H$ *with* $|H| < |G|$*, all* $5$*-colorings of* $H$ *are Kempe equivalent. Fix a* $5^-$*-vertex* $v$ *and two* $5$*-colorings of* $G$*, say* $\varphi$ *and* $\varphi_0$*. If any of the following hold, then* $\varphi \sim_5 \varphi_0$*:*

(a) $d(v) \leqslant 4$; or

(b) $v$ has neighbors $w_1$ and $w_2$, with $\varphi(w_1) = \varphi(w_2)$ and $\varphi_0(w_1) = \varphi_0(w_2)$; or

(c) $v$ has neighbors $w_1, w_2, x_1, x_2$ with $\varphi(w_1) = \varphi(w_2)$ and $\varphi_0(x_1) = \varphi_0(x_2)$.

(In (b) and (c) we assume that all listed neighbors are distinct.)

*Proof.* The proof of (a) is the same as the proof of Theorem 3.26. We prove (b) similarly, but with a wrinkle. We form $G'$ from $G - v$ by identifying $w_1$ and $w_2$; call the new vertex $w_1 * w_2$. Let $\varphi'$ and $\varphi'_0$ be the restrictions to $G'$ of $\varphi$ and $\varphi_0$. We again basically copy the proof of Theorem 3.26. By assumption, we have $\varphi'_0, \ldots, \varphi'_r$ where $\varphi'_r = \varphi'$, and each successive pair $\varphi'_{i-1}, \varphi'_i$ differ by a single Kempe swap. Suppose the Kempe swap from $\varphi'_{i-1}$ to $\varphi'_i$ uses colors $\alpha$ and $\beta$, and that $\varphi_{i-1}(v) = \alpha$. When we perform the same Kempe swap in $G$, we only need to recolor $v$ beforehand if $\varphi_{i-1}$ uses $\beta$ on at least two neighbors of $v$, including one neighbor that is neither $w_1$ nor $w_2$; this is because $\varphi'_{i-1}(w_1) = \varphi'_{i-1}(w_2)$ and $\varphi'_i(w_1) = \varphi'_i(w_2)$. But when $v$ has two such neighbors, we can always recolor it first, since $\varphi'_{i-1}(w_1) = \varphi'_{i-1}(w_2)$. If the Kempe swap from $\varphi'_{i-1}$ to $\varphi'_i$ recolors $w_1 * w_2$, then we may need to use two Kempe swaps to get from $\varphi_{i-1}$ to $\varphi_i$, since $w_1$ and $w_2$ may be in different Kempe components of $G$.

Now we prove (c). By Lemma 3.29(b), $G$ has a 5-coloring $\varphi^*$ with $\varphi^*(w_1) = \varphi^*(w_2)$ and $\varphi^*(x_1) = \varphi^*(x_2)$. Now $\varphi \sim_5 \varphi^*$ and also $\varphi^* \sim_5 \varphi_0$, by (b). So $\varphi \sim_5 \varphi^* \sim_5 \varphi_0$. □

$\sim$ *Proof of Theorem 3.28.* By Corollary 3.27, we can assume that $k = 5$. So below we write $\sim$ to denote $\sim_5$. Suppose the theorem is false. Let $G$ be a counterexample minimizing $|G|$ and, subject to that, maximizing $\|G\|$. By minimality, Lemma 3.30(a) implies that $G$ has no $4^-$-vertex. So let $v$ be a 5-vertex. Fix two arbitrary non-equivalent 5-colorings of $G$, say $\varphi$ and $\varphi_0$. ($\varphi, \varphi_0$) In other words, $\varphi \not\sim \varphi_0$.

($w_1, w_2$) Let $w_1$ and $w_2$ be neighbors of $v$ such that $\varphi(w_1) = \varphi(w_2)$; such $w_i$ exist by Pigeonhole. ($x_1, x_2$) Let $x_1$ and $x_2$ be neighbors of $v$ such that $\varphi_0(x_1) = \varphi_0(x_2)$. By Lemma 3.30(b,c), we must have $|\{w_1, w_2\} \cap \{x_1, x_2\}| = 1$. So assume that $w_1 = x_1$ and $w_2 \neq x_2$, as in Figure 3.11. Let ($y_1, y_2$) $y_1$ and $y_2$ be the other two neighbors of $v$. Suppose that $y_1 \not\leftrightarrow y_2$. By Lemma 3.29(b), there exists a 5-coloring $\varphi^*$ with $\varphi^*(w_1) = \varphi^*(w_2)$ and $\varphi^*(y_1) = \varphi^*(y_2)$. Similarly, there exists a 5-coloring $\varphi^{**}$ with $\varphi^{**}(x_1) = \varphi^{**}(x_2)$ and $\varphi^{**}(y_1) = \varphi^{**}(y_2)$. Now using Lemma 3.30(b) three times gives $\varphi \sim \varphi^* \sim \varphi^{**} \sim \varphi_0$, as in Figure 3.11 Thus, we assume $y_1 \leftrightarrow y_2$.

**Case 1: G is a plane triangulation.** By symmetry, we assume the neighbors of $v$ in clockwise order are $w_1, y_1, w_2, x_2, y_2$ (recall that $w_1 = x_1$). Since $G$ is a triangulation, we have edges $w_1y_1, y_1w_2, w_2x_2, x_2y_2, y_2w_1$. Suppose that $y_1 \not\leftrightarrow x_2$ and $y_2 \not\leftrightarrow w_2$, as in Figure 3.12. We use the same idea as above, but with one more step. By Lemma 3.29(b), $G$ has three 5-colorings, $\varphi^*$, $\varphi^{**}$, and $\varphi^{***}$ such that $\varphi^*(w_1) = \varphi^*(w_2)$ and $\varphi^*(y_1) = \varphi^*(x_2)$; also $\varphi^{**}(w_2) = \varphi^{**}(y_2)$

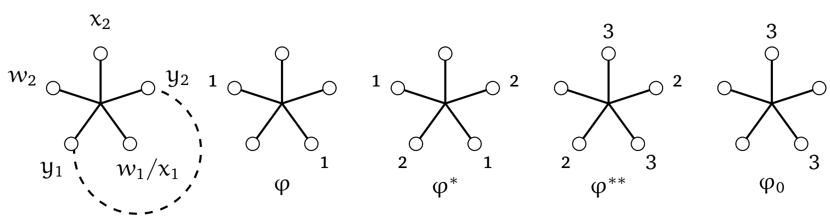



Figure 3.11: If $y_1 \not\leftrightarrow y_2$, then using Lemma 3.30 three times shows $\varphi \sim \varphi_0$.

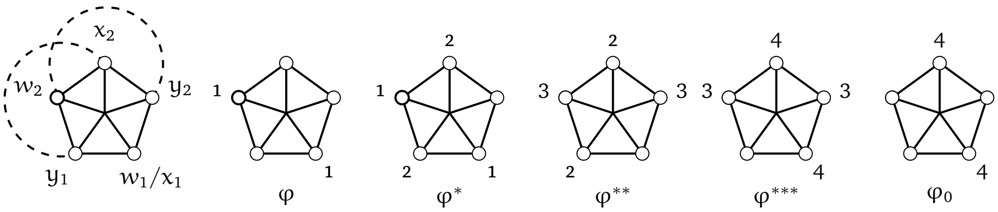



Figure 3.12: If $y_1 \not\leftrightarrow x_2$ and $w_2 \not\leftrightarrow y_2$, then using Lemma 3.30(b) four times shows $\varphi \sim \varphi_0$.

and $\varphi^{**}(y_1) = \varphi^{**}(x_2)$; and also $\varphi^{***}(w_2) = \varphi^{***}(y_2)$ and $\varphi^{***}(x_1) = \varphi^{***}(x_2)$. By four applications of Lemma 3.30(b), we get $\varphi \sim \varphi^* \sim \varphi^{**} \sim \varphi^{***} \sim \varphi$, as in Figure 3.12.

So assume that either $y_1 \leftrightarrow x_2$ or $y_2 \leftrightarrow w_2$; by planarity, we cannot have both. By symmetry between $\varphi$ and $\varphi_0$, and possibly relabeling vertices, we assume that $y_1 \leftrightarrow x_2$. Recall from above that $y_1 \leftrightarrow y_2$. Let $\alpha := \varphi(w_1)$ and $\beta := \varphi(x_2)$; recall that $\varphi(w_2) = \varphi(w_1)$. Now an $\alpha,\beta$-swap at $x_2$ gives a new 5-coloring $\varphi^*$ with $\varphi^*(x_2) = \alpha$ and $\varphi^*(w_2) = \beta$; also $\varphi^*(w_1) = \alpha$, since $w_1$ is separated from $w_2$ and $x_2$ by the cycle $y_1y_2v$, which does not use $\alpha$ or $\beta$. Clearly $\varphi \sim \varphi^*$. So we are done, since Lemma 3.30(b) gives $\varphi^* \sim \varphi_0$.

**Case 2: G is not a plane triangulation.** Let f be a $4^+$-face with boundary $w_1 \cdots w_{\ell(f)}$. By $w_1, \ldots, w_4$
planarity, either $w_1 \not\leftrightarrow w_3$ or $w_2 \not\leftrightarrow w_4$. By symmetry, we assume that $w_1 \not\leftrightarrow w_3$. Form $G'$ from $G'$
G by identifying $w_1$ and $w_3$, as in the center of Figure 3.13. Since $|G'| < |G|$, by hypothesis, every 5-coloring of $G'$ is Kempe equivalent. Note that the 5-colorings of $G'$ are in bijection with the 5-colorings of G that give $w_1$ and $w_3$ the same color. So all of these 5-colorings of G are Kempe equivalent. Further, let $G'' := G + w_1w_3$, as on the right of Figure 3.13. Since $G''$
$\|G''\| > \|G\|$, by minimality all 5-colorings of $G''$ are Kempe equivalent. So we need only show the equivalence of these two types of 5-colorings of G: those giving $w_1$ and $w_3$ the same color, and those giving $w_1$ and $w_3$ distinct colors.

Suppose that $v$ has non-adjacent neighbors $y_1$ and $y_2$ such that $w_1, w_3, y_1, y_2$ are all distinct. (We have no reason to expect that $w_1$ and/or $w_3$ is adjacent to $v$, but this is possible.) We apply Lemma 3.29(a) to $G'$ to get a 5-coloring $\varphi^*$ of G such that $\varphi^*(y_1) = \varphi^*(y_2)$ and

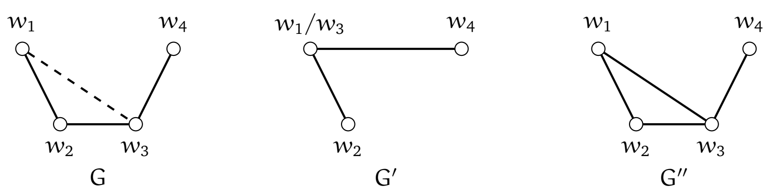



Figure 3.13: Graphs $G'$ and $G''$ are smaller than G, so the 5-colorings for each are Kempe equivalent, by the minimality of G.

$\varphi^*(w_1) = \varphi^*(w_3)$. Similarly, applying Lemma 3.29(a) to $G''$ gives a 5-coloring $\varphi^{**}$ of G such that $\varphi^{**}(y_1) = \varphi^{**}(y_2)$ and $\varphi^{**}(w_1) \neq \varphi^{**}(w_3)$. By Lemma 3.30(b), we have $\varphi^* \sim \varphi^{**}$, so all 5-colorings of G are Kempe equivalent.

Assume instead that no such $y_1$ and $y_2$ exist. Now $v$ and three of its neighbors, say $y_1, y_2, y_3$, induce $K_4$. Further, since $K_5$ is non-planar, the other two neighbors of $v$ must be $w_1$ and $w_3$. Since $w_1$ and $w_3$ both lie on the boundary of f, they must lie inside the same face of the $K_4$ induced by $v, y_1, y_2, y_3$. So $w_1$ and $w_3$ must both be non-adjacent to some $y_i$, say $y_1$. By Lemma 3.29(a), $G'$ has a 5-coloring that gives the same color to $w_1 * w_3$ and $y_1$, and $G''$ has a 5-coloring that gives the same color to $w_1$ and $y_1$, but not $w_3$. These colorings naturally induce a 5-coloring $\varphi^*$ of G with $\varphi^*(w_1) = \varphi^*(w_3) = \varphi^*(y_1)$ and a 5-coloring $\varphi^{**}$ of G with $\varphi^{**}(w_1) = \varphi^{**}(y_1) \neq \varphi^{**}(w_3)$. By Lemma 3.30(b), $\varphi^* \sim \varphi^{**}$. Thus, all 5-colorings of G are Kempe equivalent. □

## 3.4 Tashkinov Trees

### 3.4.1 The Goldberg–Seymour Conjecture is True Asymptotically

This section is about edge-coloring multigraphs. For every simple graph G, Vizing's Theorem states that $\Delta \leqslant \chi'(G) \leqslant \Delta + 1$. But when G is a multigraph, this upper bound can fail. For example, suppose G has 3 vertices and k parallel edges between each pair of vertices, as on the left in Figure 3.14. Now $\Delta = 2k$, but $\chi'(G) = 3k$. For every multigraph G, Claude Shannon proved that $\chi'(G) \leqslant \frac{3}{2}\Delta$, and Vizing proved that $\chi'(G) \leqslant \Delta + \mu(G)$ (recall that $\mu(G)$ denotes the maximum edge multiplicity in G). Both bounds hold with equality in the example above, which is known as Shannon's "fat triangle". But when our graph is not the fat triangle, we seek stronger upper bounds. As motivation, we begin with an easy lower bound, which holds for all multigraphs. We will aim to prove that this lower bound is always nearly sharp.

**Proposition 3.31.** *Every multigraph* G *satisfies*

$$\chi'(G) \geqslant \left\lceil \max_{H \subseteq G} \frac{\|H\|}{\lfloor |H|/2 \rfloor} \right\rceil. \tag{3.3}$$

*Proof.* Every multigraph H satisfies $\chi'(H) \geqslant \lceil \|H\| / \lfloor |H|/2 \rfloor \rceil$, since its $\|H\|$ edges must be partitioned into color classes of size[6] at most $\lfloor |H|/2 \rfloor$. If H is a subgraph of G, then every coloring of G induces a coloring of H; so $\chi'(G) \geqslant \chi'(H)$. Now maximizing over all subgraphs H proves the proposition. □

$\rho(G)$, density

For a graph G, let $\rho(G) := \max_{H \subseteq G} \|H\| / \lfloor |H|/2 \rfloor$; we call $\rho(G)$ the density[7] of G. So $\chi'(G) \geqslant \lceil \rho(G) \rceil$. Goldberg [178] and Seymour [360] each conjectured (independently) the following remarkable upper bound on $\chi'(G)$.

[6] To be precise, we require that $|H| \geqslant 2$.

[7] The density of G is closely related to its *fractional chromatic index*, which we discuss in Section A.12.

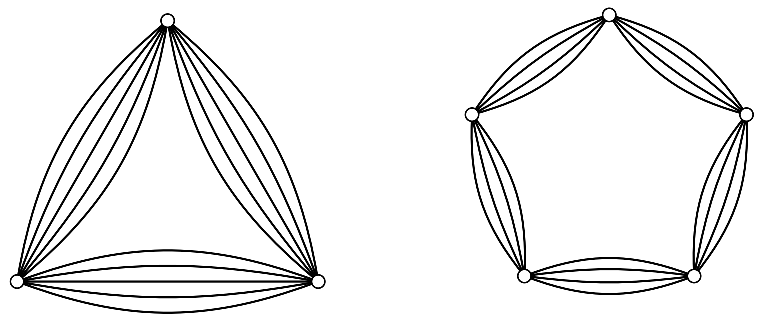


Figure 3.14: On the left, all edges are pairwise incident, so $\chi' = \|G\| = \lfloor \frac{3}{2}\Delta \rfloor$. On the right, each set of pairwise incident edges has size at most $\Delta$, but still $\chi' = \lceil \lfloor \frac{5}{2}\Delta \rfloor / 2 \rceil$.

**Conjecture 3.32 (Goldberg–Seymour Conjecture).** *Every multigraph* $G$ *satisfies*

$$\chi'(G) \leqslant \max\left\{\Delta + 1, \left\lceil \max_{H \subseteq G} \frac{\|H\|}{\lfloor |H|/2 \rfloor} \right\rceil \right\}.$$

Recall that for a simple graph $G$, it is NP-hard to determine whether $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$. Although it is far from obvious, given an arbitrary input graph $G$, we can compute $\rho(G)$ in polynomial time. Now the Goldberg–Seymour conjecture implies that if $\chi'(G) > \Delta + 1$, then $\chi'(G) = \lceil \rho(G) \rceil$. So in this case computing $\chi'$ becomes easy!

In a recent breakthrough, the Goldberg–Seymour conjecture was proved, by Chen, Jing, and Zang [81]! Their proof is about 60 pages, so we do not reproduce it here. Instead, we only show that the conjecture is true asymptotically. More precisely, in Theorem 3.42, we show that $\chi'(G) \leqslant \max\left\{\lceil \rho(G) \rceil, \Delta + \sqrt{\frac{\Delta}{2} - 1}\right\}$. The Notes discuss more of this problem's history.

**Definition 3.33.** For a $k$-edge-coloring $\varphi$ of a graph $G$, let $\varphi(e)$ denote the color used on $e$, let $\varphi(v)$ denote the set of all colors used on edges incident to $v$, and let $\overline{\varphi}(v)$ denote $[k] \setminus \varphi(v)$. We extend these definitions to a set of vertices $U$ or edges $F$ by $\varphi(U) := \cup_{u \in U} \varphi(u)$, $\overline{\varphi}(U) := \cup_{u \in U} \overline{\varphi}(u)$, and $\varphi(F) := \cup_{e \in F} \varphi(e)$. $\varphi(e)$, $\varphi(v)$ $\overline{\varphi}(v)$, $\varphi(U)$ $\varphi(F)$, $\overline{\varphi}(U)$

A multigraph $G$ is *elementary* if $\chi'(G) = \lceil \rho(G) \rceil$; that is, the trivial lower bound on $\chi'$ in Inequality (3.3) holds with equality. A set of vertices $U$ is *elementary*[8] w.r.t. an edge-coloring $\varphi$ if each color is missed at no more than a single vertex of $U$, that is, $\overline{\varphi}(v) \cap \overline{\varphi}(w) = \emptyset$ for each pair $v, w \in U$. We often abbreviate "with respect to" as *w.r.t.* Within Section 3.4.1, we say that a graph $G$ is $k$-*critical* if $\chi'(G) > k$ and $\chi'(G - e) = k$ for all $e \in E(G)$. elementary w.r.t. k-critical

Elementary sets are of interest because of the following lemma. In essence, it says that every $k$-critical graph with $k$ much larger than $\Delta$ cannot have large elementary sets.

[8] Admittedly, the use of this term for both graphs and sets of vertices can be a bit confusing. Perhaps the best explanation comes from Lemma 3.40.

**Lemma 3.34 (Tashkinov's Lemma).** *Let $s$ be an integer, with $s \geqslant 2$, and let $G$ be a $k$-critical graph with $k > \frac{s}{s-1}\Delta + \frac{-2}{s-1}$. Fix an arbitrary edge $e \in E(G)$ and $X \subseteq V(G)$ such that $X$ is an elementary set w.r.t. a $k$-coloring $\varphi$ of $G - e$. If $X$ contains both endpoints of $e$, then $|X| \leqslant s - 1$.*

The hypothesis on $k$ in terms of $s$ and $\Delta$ may look mysterious, but it arises naturally from solving an inequality that we need to complete the proof.

*Proof.* Assume instead that $|X| \geqslant s$. Since $X$ is elementary, $k \geqslant |\overline{\varphi}(X)| \geqslant |X|(k - \Delta) + 2 \geqslant s(k-\Delta)+2$. By adding $\Delta - k$ to the first and last expressions, and using the hypothesis $(s-1)k > s\Delta - 2$, we get $\Delta \geqslant (s-1)(k-\Delta)+2 = (s-1)k-(s-1)\Delta+2 > (s\Delta-2)-(s-1)\Delta+2 = \Delta$, which is a contradiction. □

We get Shannon's bound as a corollary.

**Corollary 3.35.** *Every multigraph $G$ satisfies $\chi'(G) \leqslant \frac{3}{2}\Delta$.*

*Proof.* Let $G$ be a $k$-critical graph and $\varphi$ be a $k$-edge-coloring of $G - e$, where $e = vw$. Choose $\alpha \in \overline{\varphi}(v)$ and $x \in N(w)$ such that $\varphi(wx) = \alpha$. Let $X := \{v, w, x\}$ and note that $X$ is elementary (we can prove this directly, but it is quicker to notice that $v, vw, w, wx, x$ is a Kierstead path, and apply Lemma 3.8). Suppose $\chi'(G) = k + 1$ with $k > \frac{3}{2}\Delta - 1$. Now applying Lemma 3.34 with $s = 3$ contradicts that $|X| = 3$. Thus, $k \leqslant \frac{3}{2}\Delta - 1$, which shows that $\chi'(G) \leqslant \frac{3}{2}\Delta$. □

To best take advantage of Lemma 3.34, we search for elementary sets that are as large as possible. For simple graphs, we saw this idea in our proof of Vizing's Theorem via Kierstead paths (and also Vizing fans). To get more power, we generalize this method to Tashkinov trees.

$k$ **Definition 3.36.** Let $G$ be a multigraph with $\chi'(G) = k+1$ for some integer $k \geqslant \Delta(G)+1$, and
$\varphi, v_i, e_i$ $\chi'(G-e) = k$ for every $e \in E(G)$. Choose $e_0 \in E(G)$, and let $\varphi$ be a $k$-edge-coloring of $G - e_0$.
Tashkinov tree A *Tashkinov tree* w.r.t. $\varphi$ is an ordered set of vertices and edges $(v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$ such that the vertices $v_i$ are distinct, $v_0v_1 = e_1$ and the following two conditions hold (Figure 3.15 shows an example):

(i) For each $v_i$ with $i \in [s]$, we have $e_i = v_iv_j$ for some $j \in \{0, \ldots, i-1\}$ and[9]

(ii) For each $i \in [s]$ we have $\varphi(e_i) \in \overline{\varphi}(v_\ell)$ for some $\ell \in \{0, \ldots, i-1\}$.

Condition (i) requires that for each $i$, the subgraph induced by $\{e_0, \ldots, e_i\}$ is a tree. Condition (ii) requires that each color used on an edge of the tree is unused at some vertex earlier in the tree. Note that every prefix of a Tashkinov tree is again a Tashkinov tree. For each $i \in [p]$,
$Tv_i$ we write $Tv_i$ for the Tashkinov tree $(v_0, e_0, v_1, \ldots, v_{i-1}, e_i, v_i)$.

[9]Since G is a multigraph, the vertex pair $(v_i, v_j)$ may be the endpoints for more than one edge. So, more precisely, we require that $e_i$ has endpoints $v_i$ and $v_j$. However, this technicality will not concern us.

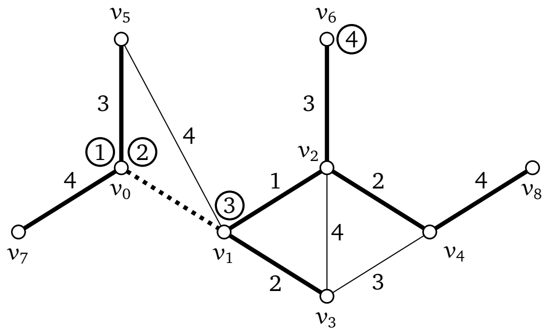



Figure 3.15: A Tashkinov tree as in Lemma 3.38. Tree edges are drawn in bold.

Note that Vizing fans are precisely those Tashkinov trees that are stars, and that Kierstead paths are precisely those that are paths with $e_0$ at one end.

Tashkinov proved that if $G-e$ has a k-edge-coloring $\varphi$ (for some edge $e$ and some $k \geqslant \Delta+1$), but $\chi'(G) > k$, then every Tashkinov tree w.r.t. $e$ and $\varphi$ must be elementary. This result is called Tashkinov's Lemma. Its proof is somewhat long, so we defer it to the next subsection (Theorem 3.43). First we assume the lemma and use it to show that the Goldberg–Seymour Conjecture is true asymptotically. We begin with a few simple results.

**Proposition 3.37.** *Consider a multigraph $G$, an edge $e \in E(G)$, an integer $k \geqslant \Delta + 1$, and a $k$-edge-coloring $\varphi$ of $G - e$. Let $T$ be a maximal Tashkinov tree w.r.t. $e$ and $\varphi$. If $T_0$ is some other Tashkinov tree w.r.t. $e$ and $\varphi$, then $V(T_0) \subseteq V(T)$. Thus, for any two maximal Tashkinov trees, $T$ and $T'$, w.r.t. $e$ and $\varphi$, we have $V(T) = V(T')$. Hence, every maximal Tashkinov tree w.r.t. $e$ and $\varphi$ is maximum.*

*Proof.* The proof is by induction on $\|T_0\|$. The base case, $\|T_0\| = 1$, is trivial, since it implies that $V(T_0)$ is simply the endpoints of $e$, which are contained in $V(T)$. Now suppose $\|T_0\| = s$, for some $s \geqslant 2$, and let $v_s$ and $e_s$ denote the final vertex of $T_0$, and its incident edge in $T_0$. By the induction hypothesis, $V(T_0) - v_s \subseteq V(T)$. Thus, $\varphi(e_s) \in \overline{\varphi}(T_0 - v_s) \subseteq \overline{\varphi}(V(T))$. So, if $v_s \notin V(T)$, then we can add $e_s$ to $T$ to get a larger Tashkinov tree, a contradiction. This proves the first statement. The second and third statements are immediate corollaries. $\square$

**Lemma 3.38.** *Given any multigraph $G$ with $\chi'(G) > k$, edge $e_0 \in E(G)$, and $k$-edge-coloring $\varphi$ of $G - e_0$, with $k \geqslant \Delta + 1$, there exists a maximum Tashkinov tree $T$ w.r.t. $e_0$ and $\varphi$ such that $T$ uses at most $(|V(T)| - 1)/2$ colors.*

*Proof.* Let $e_0 := v_0v_1$. Choose $\alpha \in \varphi(v_0)$, and let $e_1$ denote the edge incident to $v_1$ with $\varphi(e_1) = \alpha$; say $e_1 = v_1v_2$. Now $(v_0, e_0, v_1, e_1, v_2)$ is a Tashkinov tree $T_3$ on 3 vertices, that uses only $(3-1)/2 = 1$ color on its edges. We use induction on the number of vertices to repeatedly grow our Tashkinov tree, until it is maximum. (By Proposition 3.37, we cannot get stuck, since every maximal Tashkinov tree is maximum.)

By Tashkinov's Lemma, our Tashkinov tree $T_i$ is elementary, for each $i \geqslant 3$. If $T_i$ is not maximal, then there exists $\beta \in \overline{\varphi}(V(T_i))$ such that $\beta$ is used on some edge leaving $T_i$. Since $T_i$ is elementary, $\beta$ is unused at exactly one vertex of $T_i$. Since $|T_i|$ is odd, $\beta$ must be used on an even number of edges leaving $T_i$; so the number of such edges is at least 2. (Figure 3.15 gives an example.) Adding these edges to $T_i$ yields a new Tashkinov tree, $T_{i+2}$. Further, the number of colors used on $T_{i+2}$ is at most $1 + (|T_i| - 1)/2 \leqslant (|T_{i+2}| - 1)/2$, as desired. □

boundary

defective

closed

strongly closed

**Definition 3.39.** For a graph $G$ and $X \subseteq V(G)$, the *boundary* of $X$ is the set of edges with exactly one endpoint in $X$. Let $G$ be a $k$-critical graph, with $k \geqslant \Delta + 1$ and $\varphi$ be a $k$-coloring of $G - e$. A color is *defective* for $X$ if is used at least twice on the boundary of $X$. Suppose $X \subseteq V(G)$, $X$ contains the endpoints of $e$, and $X$ is elementary w.r.t. $\varphi$. Now $X$ is *closed* if there does not exist a color $\alpha$ such that $\alpha \in \overline{\varphi}(X)$ and $\alpha$ is used on the boundary of $X$. A closed set $X$ is *strongly closed* if no color is defective for $X$.

A closed set is clearly an obstruction to continuing to grow a Tashkinov tree. So a natural plan is to grow a Tashkinov tree until its vertex set is closed. By Lemma 3.37, we can't go wrong in this process, since every maximal Tashkinov tree (w.r.t. $\varphi$ and $e$) has the same vertex set. But what can we do once the vertex set of our Tashkinov tree becomes closed? If $G$ is elementary, then $G$ satisfies the Goldberg–Seymour Conjecture, so we have nothing to prove. Otherwise, the following lemma implies that no vertex subset $X$ is strongly closed. So, once we prove the lemma, we will focus on how we can make further progress when the vertex set of our tree is closed, but not strongly closed.

**Lemma 3.40.** *Let $G$ be a multigraph with $\chi'(G) = k + 1$ for some integer $k \geqslant \Delta$. If $G$ is critical, then the following two conditions are equivalent:*

*(a)* $G$ *is elementary; and*

*(b)* $G$ *has an edge $e$, a $k$-edge-coloring $\varphi$ of $G - e$, and $X \subseteq V(G)$ such that $X$ contains the endpoints of $e$, and $X$ is elementary and strongly closed w.r.t. $\varphi$.*

Assuming (a), we show that (b) holds with $X := V(G)$, and that for every $e \in E(G)$, there exists a $k$-edge-coloring of $G - e$. To prove (a) from (b), we show that $X$ induces $k(|X|-1)/2+1$ edges, so $\lceil \rho(G) \rceil = k + 1$.

*Proof.* Suppose $G$ is elementary. So there exists $H \subseteq G$ such that $\chi'(G) = k + 1 = \lceil \|H\| / \lfloor |H|/2 \rfloor \rceil$. Since $G$ is critical, $\chi'(G - e) = k$ for all $e \in E(G)$. So $H = G$, since otherwise there exists $e \in E(G) \setminus E(H)$ and $\chi'(G - e) = \chi'(G) = k + 1$. Thus $|G|$ is odd, since otherwise $\|G\| / \lfloor |G|/2 \rfloor \leqslant \frac{1}{2}\Delta |G| / \lfloor |G|/2 \rfloor = \Delta \leqslant k$, a contradiction. Since $\lceil \|G\| / \lfloor |G|/2 \rfloor \rceil = k + 1$ and $\lceil \|G - e\| / \lfloor |G|/2 \rfloor \rceil \leqslant k$, for some integer $t$, we have $|G| = 2t + 1$ and $\|G\| = kt + 1$. Since $G$ is $k$-critical, for every edge $e \in E(G)$, there exists a $k$-coloring $\varphi_e$ of $G - e$. Since $\|G - e\| = kt$, and each color class has size at most $t$, in fact every color class has size exactly $t$. Thus, $V(G)$ is elementary (and strongly closed) w.r.t $\varphi_e$. So condition (a) implies condition (b).

Now suppose (b) holds. Choose $\alpha \in \overline{\varphi}(X)$. Since $X$ is closed, $\alpha$ is not used on the boundary of $X$. Since $X$ is elementary, $\alpha$ is missed at only a single vertex of $X$. Say $\alpha \in \overline{\varphi}(v)$ for some $v \in X$. So in $G[X-v]$, the edges colored $\alpha$ induce a perfect matching. Thus $|X|$ is odd. Similarly, choose $\beta \in [k]\setminus\overline{\varphi}(X)$. Since $X$ is odd, $\beta$ is used on the boundary of $X$. Since $X$ is strongly closed, $\beta$ is used exactly once on the boundary of $X$. So, for each $\gamma \in [k]$, color $\gamma$ is used on exactly $(|X|-1)/2$ edges induced by $X$, whether or not $\gamma \in \overline{\varphi}(X)$. Thus, $\|G[X]\| = k(|X|-1)/2+1$, which implies that $\lceil \|G[X]\|/\lfloor |X|/2 \rfloor \rceil = k+1$. So, by definition, $G$ is elementary. That is, condition (b) implies condition (a). □

**Remark 3.41.** For many pictures in the rest of this section, we draw dotted ovals to depict trees or portions of trees. This choice emphasizes that the tree's shape is irrelevant to the proof.

Now we can prove the Goldberg–Seymour Conjecture asymptotically.

**Theorem 3.42.** *Every multigraph* $G$ *satisfies* $\chi'(G) \leqslant \max\{\lceil \rho(G) \rceil, \Delta + \sqrt{\frac{\Delta}{2}-1}\}$.

The main idea is to find a vertex $z \in V(T)$ such that each color in $\overline{\varphi}(z)$ is used on $T$. By Lemma 3.38, we have $(|T|-1)/2 \geqslant \overline{\varphi}(z) \geqslant k-\Delta$. Since $V(T)$ is elementary and $\overline{\varphi}(v) \geqslant k-\Delta$ for each $v \in V(T)$, we have $k \geqslant \overline{\varphi}(V(T)) \geqslant (k-\Delta)|T|+2$. To prove the theorem, we combine this inequality with the one in the previous sentence, and solve for $k$.

*Proof.* If $G$ is elementary, then the theorem holds, so suppose $G$ is not elementary. We assume that $G$ is critical. Let $k = \chi'(G)-1$. Choose an arbitrary edge $e$. Over all $k$-edge-colorings $\varphi$, we choose a maximum Tashkinov tree $T$. Subject to this, we choose $T$ to use as few colors as possible. By Tashkinov's Lemma, $V(T)$ must be elementary. That is, each color is missed at no more than one vertex. Since each vertex misses at least $k-\Delta$ colors, the total number of colors missing on $V(T)$ is at least $(k-\Delta)|T|+2$; thus, $k \geqslant (k-\Delta)|T|+2$.

We need one other idea: to show that for some vertex $z \in V(T)$, every color missing at $z$ is used on some edge of $T$. By Lemma 3.38, the edges of $T$ use at most $(|T|-1)/2$ colors. Hence $(|T|-1)/2 \geqslant \overline{\varphi}(z) \geqslant k-\Delta$, which gives

$$|T| \geqslant 2k-2\Delta+1. \tag{3.4}$$

Combining Inequality (3.4) with the previous inequality gives $k \geqslant (k-\Delta)|T|+2 \geqslant (k-\Delta)(2k-2\Delta+1)+2$. Solving the resulting quadratic in $k$ yields $k \leqslant \Delta+\sqrt{\frac{\Delta}{2}-1}$, as desired. Now we must simply prove (3.4), by showing that for some vertex $z \in V(T)$, every color $\delta \in \overline{\varphi}(v)$ is used on some edge of $T$.

By assumption, $G$ is not elementary. So, Lemma 3.40 implies that $V(T)$ is not strongly closed. By assumption, $T$ is maximum, so $V(T)$ is closed. Thus, some color is defective (for $\varphi$ and $V(T)$); call it $\beta$. This implies that $\beta$ is used on at least 2 edges on the boundary of $V(T)$. In fact, $\beta$ is used on least 3 such edges, since $|V(T)|$ is odd, and every edge of $V(T)$ is incident to an edge colored $\beta$.

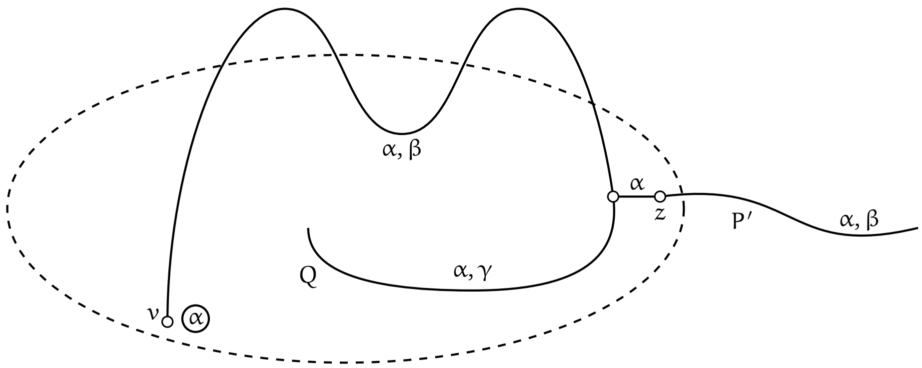



Figure 3.16: Claim 2 in the proof of Theorem 3.42.

**Claim 1.** *Choose colors* $\alpha, \beta$ *such that* $\alpha \in \overline{\varphi}(T)$ *but* $\alpha$ *is not used on* $T$*, and* $\beta$ *is defective. Let* $P := P_\nu(\alpha, \beta)$*, where* $\nu \in V(T)$ *such that* $\alpha \in \overline{\varphi}(\nu)$*. Now* $P$ *must contain every edge colored* $\beta$ *on the boundary of* $V(T)$*.*

Proof. Since $\beta$ is defective and $T$ is maximum, we must have $\beta \notin \overline{\varphi}(T)$; otherwise we could add some $\beta$-colored edge to $T$ to get a larger Tashkinov tree, a contradiction. Further, since $\beta \notin \overline{\varphi}(T)$, color $\beta$ is not used on $T$. Form a coloring $\varphi'$ from $\varphi$ by recoloring $P$. Since neither $\alpha$ nor $\beta$ is used on $T$ in $\varphi$, also neither color is used on $T$ in $\varphi'$. So $T$ is also a Tashkinov tree w.r.t. $\varphi'$. Suppose some $\beta$-colored edge $e$ in $\varphi$ is on the boundary of $V(T)$, but is not in $P$. Now also $\varphi'(e) = \beta$. However, since $\beta \in \overline{\varphi'}(\nu)$, we can add $e$ to $T$ to get a larger Tashkinov tree, a contradiction. ♢

Since $\beta \notin \overline{\varphi}(V(T))$ and $\alpha$ is missed only at $\nu$, path $P$ must end outside $V(T)$. And since $P$ contains at least 3 boundary edges of $V(T)$ (each of which is colored $\beta$), we know that $z \neq \nu$.

**Claim 2.** *Define* $\alpha$*,* $\beta$*, and* $P$ *as in Claim 1, and let* $z$ *be the final vertex on* $P$ *that lies in* $V(T)$*. See Figure 3.16. If* $\gamma \in \overline{\varphi}(z)$*, then* $\gamma$ *must be used on* $T$*.*

Proof. Suppose there exists $\gamma \in \overline{\varphi}(z)$ that is unused on $T$. Let $P'$ denote the subpath of $P$ from $z$ to the endpoint of $P$ other than $\nu$. Since $\alpha \in \overline{\varphi}(V(T))$, the edge of $P'$ incident to $z$ must be $\beta$-colored. Let $Q$ denote the $\alpha, \gamma$-chain starting at $z$. Form $\varphi''$ from $\varphi$ by recoloring $Q$. Since $\alpha, \gamma \in \overline{\varphi}(V(T))$, and $T$ is maximum, all vertices of $Q$ lie in $V(T)$; so recoloring $Q$ does not change any colors on $P'$. Since $\alpha \in \overline{\varphi''}(z)$, recoloring $P'$ in $\varphi''$ yields a proper coloring $\varphi'''$. Note that $T$ is again a Tashkinov tree for $\varphi'''$, since $\alpha$ and $\gamma$ are not used on $T$ in any of $\varphi$, $\varphi''$, and $\varphi'''$. However, $\beta \in \overline{\varphi'''}(z)$, so we can add some $\beta$-colored-edge to get a larger Tashkinov tree, which is a contradiction. ♢

Recall, by Lemma 3.38, that $T$ uses at most $(|T| - 1)/2$ colors. Clearly, $|\overline{\varphi}(z)| \geqslant k - \Delta$. So $(|T| - 1)/2 \geqslant |\overline{\varphi}(z)| \geqslant k - \Delta$. This proves (3.4), which finishes the proof of the theorem. □

### 3.4.2 Tashkinov Trees are Elementary

**Theorem 3.43 (Tashkinov's Lemma).** *Tashkinov trees are elementary.*

The proof builds heavily on Kierstead's Lemma, that all Kierstead paths are elementary. Given a non-elementary path, in that proof we "pushed" the color missed twice to be missed at the endpoints of the uncolored edge. Rather than trying to repeat that proof for every Tashkinov tree $T$, we instead reduce the general case to the case when $T$ is a path. Suppose we are given a $k$-edge-coloring $\varphi$ of $G - e_0$ and a non-elementary Tashkinov tree $T$. Our goal is to recolor $G - e_0$ to get a new coloring $\varphi'$ and non-elementary Tashkinov tree $T'$, such that $T'$ is more "path-like" than $T$. If $T$ eventually becomes a path, then it is a Kierstead path, so we are done by Kierstead's Lemma. To formalize this intuition, we need to introduce a number of definitions.

**Definition 3.44.** Let $T := (v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$. Let $\mathcal{C}^k_T$ be the set of $k$-edge-colorings of $G - e_0$ such that for each $\varphi \in \mathcal{C}^k_T$, tree $T$ is a Tashkinov tree w.r.t. $\varphi$ and $V(T)$ is non-elementary w.r.t. $\varphi$. When the context is clear, we write $\mathcal{C}$ for $\mathcal{C}^k_T$. When $\varphi \in \mathcal{C}$, we call $(T, \varphi)$ a *repeating pair*, since some color $\alpha$ is missed at two vertices of $T$. $T$ $\mathcal{C}$ repeating pair

The *tail* of a tree $T$ is the maximum subgraph $(v_j, e_{j+1}, \ldots, e_s, v_s)$ that is a path. The *body* of $T$ is the subgraph $Tv_j$. So $T$ is the edge-disjoint union of its body and its tail. The left side of Figure 3.18 shows an example, with the body in the dotted oval and the tail extending outside of it. Let $t(T)$ and $b(T)$ denote, respectively, the numbers of edges in the tail and the body of $T$. For short, we write $\overline{\varphi}(Tv_i)$ to mean $\overline{\varphi}(V(Tv_i))$. tail, body $t(T)$, $b(T)$ $\overline{\varphi}(Tv_i)$

We will show that given a repeating pair $(T, \varphi)$, we can recolor $G$ to reach a repeating pair $(T', \varphi')$ such that either $b(T') < b(T)$ or else $b(T') = b(T)$ and $t(T') < t(T)$. This implies that $G$ has no minimal repeating pair $(T, \varphi)$, which will prove the theorem.

*Proof of Theorem 3.43.* The proof can be viewed as a double induction, where the base case is that $b(T) = 0$, which means that $T$ is a Kierstead path w.r.t. $\varphi$. However, it is slightly cleaner when phrased in terms of minimality. We choose a repeating pair $(T, \varphi)$ such that

(a) the body has minimum size (that is, $b(T)$ is minimum) and, subject to that,

(b) the tail has minimum size (that is, $t(T)$ is minimum).

We call $(\varphi, T)$ a *minimal repeating pair*. When $\varphi$ is clear from context, we simply say that $T$ is *minimal*. Note that if $(\varphi, T)$ is a minimal repeating pair and $(\varphi', T')$ is smaller than $(\varphi, T)$, then $T'$ must be elementary w.r.t. $\varphi'$, by the minimality of $(\varphi, T)$. minimal repeating pair minimal

To break the proof into more manageable units, we prove five claims. The first three are essentially observations, which are easy to prove once stated. The bulk of the work is in proving the fourth and fifth claims, and in using these to prove the theorem.

During the proof, we often perform Kempe swaps to "move" a color $\alpha$ that is missing at one vertex $v_i$ of the tree $T$ to be missing at another vertex $v_j$. To ensure that $T$ is also a Tashkinov tree for this new coloring, we require that neither color $\alpha$ nor $\beta$ be used on $Tv_{\max\{i,j\}}$. This motivates our first two claims.

**Claim 1.** *If* $1 \leqslant i < s$*, then at least four colors in* $\overline{\varphi}(Tv_i)$ *are unused on* $Tv_i$*. Further, when* $i \in [s-2]$*, for each color* $\alpha$ *there exists* $\gamma \in \overline{\varphi}(Tv_i) \setminus \{\alpha\}$ *that is unused on* $Tv_{i+2}$.

Proof. The second statement follows from the first, since at most 3 colors are excluded by $\alpha$ and the colors used on $e_{i+1}$ and $e_{i+2}$. To prove the first, we simply count the colors in $\overline{\varphi}(Tv_i)$ and those used on $Tv_i$. Since $k \geqslant \Delta + 1$, each $v_j$ misses at least one color. Further, since $v_0v_1$ is uncolored, $v_0$ and $v_1$ each miss at least two colors. Since T is minimal, $V(Tv_i)$ is elementary. So $|\overline{\varphi}(Tv_i)| \geqslant 2(2) + 1(i-1) = i+3$. Since $\|Tv_i\| = i$ and $v_0v_1$ is uncolored, at most $i-1$ colors are used on $Tv_i$. Now we are done, since $(i+3)-(i-1) = 4$. ♢

The next claim allows us to move an unused color from one vertex of T to another.

**Claim 2.** *Suppose that* $\gamma \in \overline{\varphi}(v_i)$ *and* $\delta \in \overline{\varphi}(v_j)$ *and* $\gamma$ *is unused on* $Tv_j$ *for some* $\varphi \in \mathcal{C}$ *and* $0 \leqslant i < j < s$*. (See Figure 3.17.) Now* $\gamma \neq \delta$*, so let* $P := P_{v_j}(\gamma, \delta)$*. The other endvertex of* P *is* $v_i$*. Further, if* $\varphi'$ *is formed from* $\varphi$ *by recoloring* P*, then* $\varphi' \in \mathcal{C}$ *and*

$$\overline{\varphi'}(v) = \begin{cases} (\overline{\varphi}(v_i) \setminus \{\gamma\}) \cup \{\delta\} & \text{if } v = v_i; \\ (\overline{\varphi}(v_j) \setminus \{\delta\}) \cup \{\gamma\} & \text{if } v = v_j; \\ \overline{\varphi}(v) & \text{otherwise.} \end{cases}$$

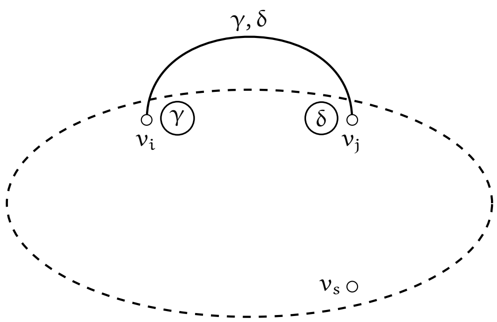



Figure 3.17: Path $P_{v_j}(\gamma, \delta)$ must end at $v_i$.

This is essentially an extension of the ideas used to prove the Fan Equation.

Proof. Since T is minimal, $V(Tv_j)$ is elementary. So $\delta \in \varphi(v_h)$ for each $h \in [j-1]$. Thus, $\delta$ is unused on $Tv_j$. By hypothesis, $\gamma$ is also unused on $Tv_j$. This proves that $Tv_j$ is a Tashkinov tree w.r.t. $\varphi'$. Now P must end at $v_i$, as shown[10] in Figure 3.17, since otherwise $\gamma \in \overline{\varphi'}(v_i) \cap \overline{\varphi'}(v_j)$, contradicting the minimality of T. This proves the above description of $\overline{\varphi'}$. As a result, $\overline{\varphi'}(Tv_j) = \overline{\varphi}(Tv_j) \supseteq \{\gamma, \delta\}$. So T is also a Tashkinov tree w.r.t. $\varphi'$. ♢

In the proof of the next claim, the body size of T decreases, but its tail size may increase arbitrarily. This is why the minimality of T is phrased primarily in terms of $b(T)$, and only secondarily in terms of $t(T)$, rather than in terms of $\|T\|$, since $\|T\| = b(T) + t(T)$.

[10]Figure 3.17 is perhaps misleading, since edges of $P_{v_i}(\gamma, \delta)$ might be incident to other vertices of $Tv_j$ (even having both endpoints in this set), but are forbidden only from being edges of $Tv_j$. The same is true of Figure 3.20.

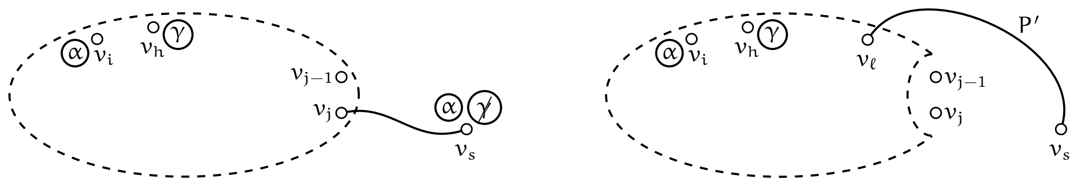



Figure 3.18: The proof of Claim 3.

**Claim 3.** *Let* $j := b(T)$. *Suppose there exist* $i, h \in [j-2]$ *and colors* $\alpha, \gamma$ *such that* $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$ *and* $\gamma \in \overline{\varphi}(v_h) \setminus \overline{\varphi}(v_s)$*; we allow the possibility that* $i = h$. *Let* $P := P_{v_s}(\alpha, \gamma)$*; see the right of Figure 3.18. We must have* $V(P) \cap V(Tv_{j-1}) = \emptyset$.

Proof. Suppose that $i, h, \alpha, \gamma$ satisfy the hypotheses, but $V(P) \cap V(Tv_{j-1}) \neq \emptyset$. Let $P'$ be the minimal subpath of $P$ containing $v_s$ such that $V(P') \cap V(Tv_{j-1}) \neq \emptyset$ and let $\{v_\ell\} := V(P') \cap V(Tv_{j-1})$. Now replace the tail of $T$ with $P'$, deleting $v_j$; further, if $\ell < j-1$, then also delete $v_{j-1}$. Formally, let $(w_0, f_1, w_1, \ldots, f_t, w_t)$ denote $P'$, where $w_0 = v_\ell$ and $w_t = v_s$. If $\ell = j-1$, then let

$$T' := (v_0, e_1, v_1, \ldots, e_{j-1}, v_{j-1}, f_1, w_1, \ldots, f_t, w_t).$$

Otherwise, let

$$T' := (v_0, e_1, v_1, \ldots, e_{j-2}, v_{j-2}, f_1, w_1, \ldots, f_t, w_t).$$

In each case $(T', \varphi)$ is a Tashkinov tree with $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(w_t)$, but with $b(T') \leqslant j-1 < b(T)$, which contradicts the minimality of $T$. ♢

In the next claim, we extract the main idea from the proof of Kierstead's Lemma. The hypothesis that $\alpha$ is unused on $Tv_j$ arises naturally when we rephrase the induction step from that proof to apply also in this more general setting. This is the most important claim for handling the case when $T$ has a non-trivial tail (that is, $t(T) > 0$).

**Claim 4.** *If* $T$ *has a non-trivial tail, then for all* $\varphi \in \mathcal{C}$ *there cannot exist* $i, j$*, and* $\alpha$ *with* $i \leqslant j < s$*, vertex* $v_j$ *a tail vertex, color* $\alpha$ *unused on* $Tv_j$*, and* $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$. *In particular, there cannot exist a tail vertex* $v_i$ *(distinct from* $v_s$*) and* $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$. *See Figure 3.19.*

Proof. We begin with the first statement. Assume that $i, j, \alpha$ satisfy the hypotheses; subject to this, choose $\varphi, i, j, \alpha$ so that $i$ is as large as possible, over all $\varphi \in \mathcal{C}$. If $i = s-1$, then let $\beta := \varphi(e_s)$, and recolor $e_s$ with $\alpha$. For the resulting coloring $\varphi'$, the pair $(T - v_s, \varphi')$ is a smaller repeating pair, since $\beta \in \overline{\varphi'}(Tv_{s-2}) = \overline{\varphi}(Tv_{s-2})$ and also $\beta \in \overline{\varphi'}(v_{s-1})$. So instead, we assume that $i < s-1$. If $v_i$ is a tail vertex, then we may assume $j = i+1$; otherwise, we may assume $j = b(T)$. Choose $\beta \in \overline{\varphi}(v_j)$. Clearly, $\beta \neq \alpha$. Let $P := P_{v_j}(\alpha, \beta)$ and form $\varphi'$ from $\varphi$ by recoloring $P$. By Claim 2, the pair $(T, \varphi')$ is again repeating.

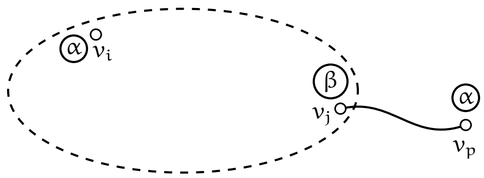



Figure 3.19: The proof of Claim 4.

Now observe that, since $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$, at most one of $v_i$ and $v_s$ appears in $V(P)$; this is the key insight needed. If $v_i \notin V(P)$, then $\alpha \in \overline{\varphi'}(v_i) \cap \overline{\varphi'}(v_j)$, which contradicts the minimality of T, specifically (b). If $v_s \notin V(P)$, then $\alpha \in \overline{\varphi'}(v_j) \cap \overline{\varphi'}(v_s)$, which contradicts the maximality of $i$. This proves the first statement.

Now we consider the second statement. The case when $i = s-1$ is the same as in the first paragraph, so we assume that $i < s-1$ and let $j := i+1$. To apply the first statement, we must show that $\alpha$ is unused on $Tv_j$. Since $\alpha \in \overline{\varphi}(v_i)$, and $Tv_{s-1}$ is elementary by the minimality of T, we know that $\alpha$ is unused on $Tv_i$. And since $e_{i+1} = v_i v_{i+1}$ and $\alpha \in \overline{\varphi}(v_i)$, we conclude that $\alpha$ is unused on $Tv_{i+1} = Tv_j$. Thus, the second statement follows from the first. ♢

The next claim is the natural analogue of Claim 4 for the case when T has a trivial tail.

**Claim 5.** *If* T *has a trivial tail, then for all* $\varphi \in \mathcal{C}$ *there cannot exist* $i$ *and* $\alpha$ *with* $i \leqslant s-2$ *such that color* $\alpha$ *is unused on* T *and* $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$*. See Figure 3.20.*

Proof. Our goal is to find $\varphi \in \mathcal{C}$ such that $\varphi(e_s) \notin \overline{\varphi}(v_{s-1})$ and also there exists $h \in [s-2]$ and color $\alpha$ such that $\alpha \in \overline{\varphi}(v_h) \cap \overline{\varphi}(v_s)$. Given such a $\varphi$, we let

$$T' := (v_0, e_1, \dots, e_{s-2}, v_{s-2}, e_s, v_s), \tag{3.5}$$

formed from T by deleting $v_{s-1}$. Now we are done, since $T'$ is also a Tashkinov tree with $\alpha \in \overline{\varphi}(v_h) \cap \overline{\varphi}(v_s)$, but $b(T') < b(T)$, which contradicts the minimality of T.

Suppose the claim is false and let $\varphi$, $\alpha$, and $i$ be a counterexample. We show that we can choose $\varphi$ and $\alpha$ so that $\varphi(e_s) \notin \overline{\varphi}(v_{s-1})$, although now possibly $i = s-1$, i.e., $\alpha \in$

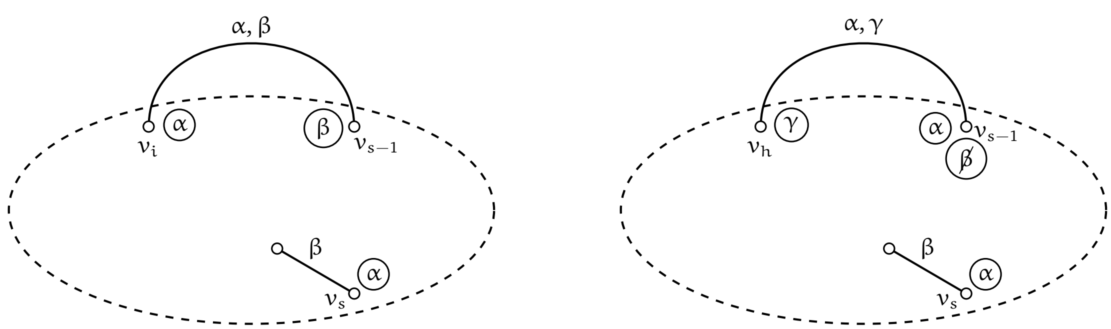



Figure 3.20: Two steps in the proof of Claim 5, ensuring that $\varphi(e_s) \in \overline{\varphi}(v_{s-1})$.

$\overline{\varphi}(v_{s-1}) \cap \overline{\varphi}(v_s)$. Let $\beta := \varphi(e_s)$. Assume to the contrary that $\beta \in \overline{\varphi}(v_{s-1})$; see the left of Figure 3.20. Let $P := P_{v_i}(\alpha, \beta)$ and form $\varphi'$ from $\varphi$ by recoloring $P$. By Claim 2, $P$ ends at $v_{s-1}$ and $\varphi' \in \mathcal{C}$. However, now $\varphi'(e_s) = \varphi(e_s) = \beta$ and $\beta \in \varphi'(v_{s-1})$, as desired.

As shown above in (3.5), to complete the proof it suffices to show that we can choose $\varphi \in \mathcal{C}$ such that $\varphi(e_s) \in \varphi(v_{s-1})$ and also there exists $h \in [s-2]$ such that $\overline{\varphi}(v_h) \cap \overline{\varphi}(v_s) \neq \emptyset$. So assume that $\overline{\varphi}(v_h) \cap \overline{\varphi}(v_s) = \emptyset$ for all $h \in [s-2]$. Since $T$ is non-elementary w.r.t. $\varphi$, we have $\alpha \in \overline{\varphi}(v_{s-1}) \cap \overline{\varphi}(v_s)$. By Claim 1, there exists $h \in [s-2]$ and $\gamma$ such that $\gamma \in \overline{\varphi}(v_h)$ and $\gamma$ is unused on $T$. By Claim 2, $P_{v_h}(\alpha, \gamma)$ must end at $v_{s-1}$; see the right of Figure 3.20. However, we now show that we can also swap the roles of $v_{s-1}$ and $v_s$, and so conclude that $P_{v_h}(\alpha, \gamma)$ must also end at $v_s$, a contradiction. Let

$$T' := (v_0, e_1, \ldots, e_{s-2}, v_{s-2}, e_s, v_s, e_{s-1}, v_{s-1}),$$

formed from $T$ by swapping the order of $v_{s-1}$ and $v_s$. Note that $b(T') = b(T)$ and $t(T') = t(T) = 0$. Further, $V(T'v_s)$ is elementary and $\gamma$ is unused on $T'v_s$. Thus, Claim 2 implies that $P_{v_h}(\alpha, \gamma)$ must end at $v_s$, the desired contradiction. $\diamondsuit$

Now we complete the proof of the theorem.

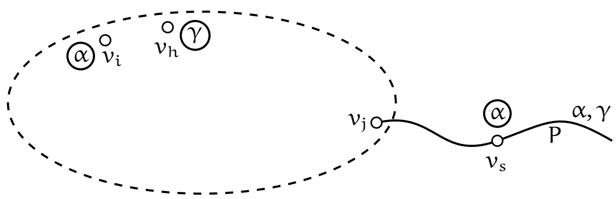



Figure 3.21: The end of the proof of Theorem 3.43.

First suppose that $T$ has a non-trivial tail; see Figure 3.21. Let $j := b(T)$. Since $(\varphi, T)$ is a minimal repeating pair, there exists $i < s$ and $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$. By the second statement of Claim 4, we must have $i \leqslant j-1$. We first ensure that $i \leqslant j-2$. Suppose, to the contrary, that $i = j-1$. By Claim 1, there exists $h \in [j-2]$ and $\gamma \in \overline{\varphi}(v_h) \setminus \{\alpha\}$ such that $\gamma$ is unused on $Tv_j$. Let $P := P_{v_h}(\alpha, \gamma)$. By Claim 2, we know that $P$ ends at $v_i$. So recoloring $P$ gives a coloring $\varphi'$ such that $(T, \varphi')$ is a repeating pair with $\alpha \in \overline{\varphi'}(v_h) \cap \overline{\varphi'}(v_s)$. Further, $\varphi'(e_s) \in \varphi'(v_{s-1})$, as desired. Thus, we can assume $i \leqslant j-2$.

Now we make a similar argument to finish the case. By Claim 1, there exist $h \in [j-2]$ and $\gamma \in \overline{\varphi}(v_h) \setminus \{\alpha\}$ such that $\gamma$ is unused on $Tv_j$. If $\gamma \in \overline{\varphi}(v_s)$, then $h$, $j$, and $\gamma$ violate Claim 4. So assume $\gamma \notin \overline{\varphi}(v_s)$. Let $P := P_{v_s}(\alpha, \gamma)$. By Claim 3, $V(P) \cap V(Tv_{j-1}) = \emptyset$. Form $\varphi'$ from $\varphi$ by recoloring $P$. Path $P$ might intersect $V(T) \setminus V(Tv_{j-1})$. But $\alpha, \gamma \in \overline{\varphi'}(Tv_{j-1}) = \overline{\varphi}(Tv_{j-1})$, so still $\varphi' \in \mathcal{C}$. Now $\gamma \in \overline{\varphi'}(v_s) \cap \overline{\varphi'}(v_h)$, so $h, j, \gamma$, and $\varphi'$ violate Claim 4; thus, we are done.

Suppose instead that $T$ has a trivial tail. By Claim 5, it suffices to find $\varphi \in \mathcal{C}$ and $i$ and $\alpha$ with $i < s-1$ such that color $\alpha$ is unused on $T$ and $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$. By assumption, we have $i \in [s-1]$ and $\alpha \in \overline{\varphi}(v_i) \cap \overline{\varphi}(v_s)$. We now show that we can also assume that both $i \in [s-2]$ and $\alpha$ is unused on $T$. First suppose that $i = s-1$. By Claim 1, there exist $j \in [s-2]$ and

$\beta \in \overline{\varphi}(v_j)$ such that $\beta$ is unused on $T$. By Claim 2, path $P_{v_j}(\alpha, \beta)$ ends at $v_{s-1}$ and recoloring $P_{v_j}(\alpha, \beta)$ yields a new coloring $\varphi'$ with $\varphi' \in \mathcal{C}$ and $\alpha \in \overline{\varphi'}(v_j) \cap \overline{\varphi'}(v_s)$. Thus, we can assume $i \leqslant s-2$. So assume $\alpha$ is used on $T$. Again, by Claim 1, there exists $j \in [s-2]$ and $\beta \in \overline{\varphi}(v_j)$ such that $\beta$ is unused on $T$. Let $P := P_{v_s}(\alpha, \beta)$. By Claim 3, we have $V(P) \cap V(Tv_{s-1}) = \emptyset$. Form $\varphi'$ from $\varphi$ by recoloring $P$. Now $\varphi' \in \mathcal{C}$ and $\beta \in \overline{\varphi'}(v_j) \cap \overline{\varphi'}(v_s)$ and $\beta$ is unused on $T$. Thus, we are done by Claim 5. □

## Notes

Kempe chains first appeared in 1879, in Kempe's false proof of the 4 Color Theorem [243]. Heawood noticed the error 11 years later, and salvaged the idea to prove the 5 Color Theorem [212]. For nearly a century, Heawood's was the only known proof of this result; finally, in the 1970s Kainen [235] discovered the proof we present in Section 4.1. König's Theorem [268] was proved in 1916; Schrijver [357, Section 16.7h] gives more history. An alternate proof uses Hall's Theorem, and induction on $\Delta$; see Exercise 4.

Vizing proved his eponymous theorem [398, 400] in 1964 and adjacency lemma [400] in 1965. In view of Vizing's Theorem, we call a simple graph $G$ *Class 1* if $\chi'(G) = \Delta$ and *Class 2* if $\chi'(G) = \Delta + 1$. Lemma 3.5 and the subsequent proof of Vizing's Theorem are due to Ehrenfeucht, Faber, and Kierstead [146]. Our first two proofs of Vizing's Theorem both easily extend to multigraphs; see Exercise 4 Our presentation of Lemma 3.10 follows [367].

Kierstead [244] strengthened Vizing's Theorem: If $G$ is a multigraph with $\chi'(G) \geqslant \Delta + t$, and $t > \frac{1}{2}(\mu(G) + 1)$, then $G$ contains a triangle $vwx$ with $\mu(vw) + \mu(vx) + \mu(wx) \geqslant 2t$. Using this result Kierstead and Schmerl [253] showed that if $G$ is a simple graph and $G$ does not induce $K_{1,3}$ or $K_5 - e$, then $\chi(G) \leqslant \omega(G) + 1$. This generalizes Vizing's Theorem for simple graphs, since neither $K_{1,3}$ nor $K_5 - e$ can appear as an induced subgraph in a line graph.

It is NP-hard to decide whether a simple $\Delta$-regular graph is Class 1 or Class 2. This was proved in 1981, by Holyer [220], when $\Delta = 3$. Two years later, it was extended by Leven and Galil [282] to all $\Delta \geqslant 3$. We include the proof in Section 2.7.2.

Most work on edge-coloring simple graphs provides sufficient conditions for a graph to be Class 1. This includes König's Theorem and Theorem 3.15. Tait showed that the 4 Color Theorem is equivalent to the following statement: every 2-connected, 3-regular planar graph is Class 1. Tutte conjectured [392] this could be extended to all 2-connected, 3-regular graphs with no subdivision of the Petersen graph. This conjecture was proved in a series of papers by Edwards, Robertson, Sanders, Seymour, and Thomas [145, 345, 346, 347, 350].

Vizing conjectured that every simple planar graph with $\Delta \geqslant 6$ is Class 1. We proved this for $\Delta \geqslant 8$ in Theorem 3.15. The case $\Delta = 7$ was proved by Zhao [352] and Zhang [429]. (Both proofs use the same general approach as when $\Delta \geqslant 8$, but the details are more technical, so we omit them.) The case $\Delta = 6$ remains open.

Jaeger posed the following intriguing conjecture. If true, this conjecture implies both the Berge–Fulkerson Conjecture and the Five Cycle Double Cover Conjecture.

**Conjecture 3.45.** *Let* $G$ *be* 3*-regular with no cut-edge. We can map the edges of* $G$ *to the edges of the Petersen graph,* $P$*, so that every* 3 *edges in* $G$ *incident to a common vertex are mapped to* 3 *edges of* $P$ *incident to a common vertex.*

We discuss the 4 Color Theorem in the Chapter 4 Notes. The material in Section 3.2 follows Steinberger [366].

We began Section 3.3 with a question of Vizing [399]: Starting from any proper edge-coloring of a graph $G$, can we reach an optimal proper edge-coloring by a sequence of Kempe swaps (suppressing empty color classes)? The proof of Vizing's Theorem gets us to an edge-coloring with at most $\Delta+1$ colors. But this stronger question remained open for many decades. The first significant progress was by Asratian, who proved it for bipartite graphs [30]. This was later extended by Bonamy, Defrain, Klimošová, Lagoutte, and Narboni to all triangle-free graphs [46]. Finally, the question was answered affirmatively by Narboni for all graphs [316].

The meta-question motivating Section 3.3 is this: For which graphs $G$ and which values $k$ are all $k$-colorings of $G$ Kempe equivalent? This area was first investigated by Las Vergnas and Meyniel [280], who proved Theorem 3.26, that this holds for $d$-degenerate graphs when $k > d$. Theorem 3.28, due to Meyniel [298], extends this to planar graphs when $k = 5$. Fisk [163] proved it for 3-colorable plane triangulations, and Mohar [303] extended this to all 3-colorable plane graphs. In contrast, Mohar [301] disproved it, when $k = 4$, for general plane triangulations. He found examples where the number of equivalence classes of 4-colorings is arbitrarily large.

Mohar conjectured that if $G$ is $d$-regular then all $d$-colorings are Kempe equivalent. Figure 3.10 gives a counterexample, when $d = 3$. But Feghali, Johnson, and Paulusma [159] proved the conjecture when $d = 3$, with that single exception. Bonamy, Bousquet, Feghali, and Johnson [44] proved the conjecture when $d \geqslant 4$. Cranston and Mahmoud [98] extended the notion of Kempe equivalence to list-coloring and proved the analogue of Mohar's conjecture in this more general context. (This also implies an alternate proof of the main result in [44].)

Early progress on the Goldberg–Seymour Conjecture proved it for graphs with $\Delta \leqslant k$, for increasing values of $k$. In 2000, Tashkinov [372] proved it for $\Delta \leqslant 11$ (a result proved earlier by Nishizeki and Kashiwagi [324]). However, his work introduced Tashkinov trees and proved Tashkinov's Lemma, which laid the foundation for all future work on the problem. In 2012, Stiebitz, Schiede, Toft, and Favrholdt [367] published the monograph *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*. McDonald [295] surveyed work until 2014. And in 2018, Chen, Jing, and Zang announced a proof of the full conjecture. In 2019 they uploaded a preprint [81] to arXiv. However, as of 2024, the author is not aware of this manuscript having appeared in a journal. In addition to being very long, the proof is complex enough that it does not yield a polynomial-time algorithm for constructing an optimal edge-coloring. In 2023, Jing [230] provided a "more natural" proof of the Goldberg–Seymour Conjecture, which is significantly shorter and does provide a polynomial-time coloring algorithm.

Kahn [233] was the first to prove that the Goldberg–Seymour Conjecture holds asymptotically, by using an iterative random coloring. (This result now has a much easier proof, which we present in Section 3.4.1; see Theorem 3.42.) Later [234] Khan extended this work to prove the same bound for list-coloring. Namely $\chi_\ell{}'(G) \leqslant (1+o(1))\max\{\Delta, \rho(G)\}$. For a more accessible presentation of these results, see [307, Chapters 22 and 23].

Although the Goldberg–Seymour Conjecture is now proved, many interesting questions on edge-coloring multigraphs remain open. Seymour conjectured [360, 362] that every planar graph satisfies $\chi'(G) = \max\{\Delta(G), \lceil\rho(G)\rceil\}$. Much work has studied the special case when G is r-regular and $\rho(G) = \Delta(G)$. The cases $r \in \{1, 2\}$ are trivial and the case $r = 3$ is equivalent to the 4 Color Theorem. The cases $r \in \{4, 5\}$ were proved by Guenin [326]. The cases $r \in \{6, 7, 8\}$ were handled in [131],[86],[87]. For each $r \geqslant 4$, the proof reduces to the case of smaller r. So all results for $r \geqslant 4$ assume the 4 Color Theorem. We remark more on some related problems in the Notes of Chapter 6.

## Exercises

3.1. Let G be an edge-critical graph. For each $i$ with $2 \leqslant i \leqslant \Delta$, show that G contains an edge-critical graph H with $\Delta(H) = i$. [400]

3.2. Determine all edge-critical graphs with $\chi' = \Delta$.

3.3. Prove König's Theorem using Hall's Theorem and induction on $\Delta$.

3.4. Extend our first two proofs of Vizing's Theorem (for simple graphs) to the case of multigraphs. Let G be a multigraph, and let $s := \mu(G)$. Use the second proof to show that if $\chi'(G) = \Delta(G) + \mu(G)$, then G contains vertices $v, w, x$ such that $\mu(xv) = s$, $\mu(xw) \geqslant s - 1$, and $\mu(vw) \geqslant 1$. [146, 244]

3.5. Prove the following stronger form of Vizing's Theorem. For a multigraph G, let $\mu$ denote the maximum edge multiplicity and let $k := \Delta + \mu$. For any maximal matching M, graph G has a k-edge-coloring in which one color class is M.

3.6. Show that if G is k-degenerate and $\Delta \geqslant 2k$, then $\chi'(G) = \Delta$. [399]

3.7. Use Vizing's Theorem to give a short proof of Theorem 3.16 in the special case of simple graphs.

3.8. Show that for every surface S, the set of 7-critical graphs embeddable in S is finite. [302]

3.9. Use Kempe chains to give an alternate proof (without vertex identification) that all planar graphs are 5-colorable.

3.10. Use Kempe chains to prove Brooks' Theorem. [297]

3.11. Prove that the two 3-colorings in Figure 3.10 are not 3-Kempe equivalent. [395]

3.12. Show that in the definition of $\rho(G)$ we can restrict the maximum to subgraphs H with $|H| \geqslant 3$ and $|H|$ odd (as long as $|G| \geqslant 3$). [356, §4.2]

# Chapter 4

# Vertex Identification: Coloring Planar Graphs

> When you first start off trying to solve a problem, the first solutions you come up with are very complex, and most people stop there. But if you keep going, and live with the problem and peel more layers of the onion off, you can often times arrive at some very elegant and simple solutions.
>
> —Steve Jobs

In this chapter we study vertex coloring problems for planar graphs. Our proofs typically follow the pattern familiar from Chapter 1: assume a minimal counterexample $G$, color a smaller graph $G'$ by minimality, and extend the coloring of $G'$ to $G$, which gives a contradiction. But rather than forming $G'$ by simply deleting vertices of $G$, we now contract edges. So to color $G'$ by minimality, we use more than just the observation that planar graphs form a hereditary class. In fact, planarity is preserved by edge-contraction. If our theorems also assume that $G$ is triangle-free (or, more generally, that $G$ has girth at least $g$), then we must be more careful about which edges we contract, to ensure that the resulting graph $G'$ has girth large enough to itself satisfy the hypotheses of the theorem.

We aim to use the coloring $\varphi'$ of $G'$ to induce a partial coloring $\varphi$ of $G$. To ensure that $\varphi$ is proper for $G$, we need to know that each pair of vertices identified in $G'$ is non-adjacent in $G$. But why is this approach better than simply deleting vertices? The key observation is that now each pair of vertices that are identified must use the same color in $\varphi$. So if two vertices are identified in $G'$ and they have a common neighbor $v$ in $G$, then they forbid only a single color from use on $v$, rather than the two colors they might forbid if we had used vertex deletion. (Our goal here is much the same as it was in the previous chapter: to ensure that the vertices we are about to color have colors that are repeated among their neighbors. But our means for achieving that goal are quite different.) This idea is illustrated well by our first theorem.

## 4.1 5-Coloring, 4-Coloring, and 3-Coloring

If G is planar, then $\chi(G) \leqslant \mathrm{col}(G) \leqslant 6$, as we saw in Corollary 1.7. As a warm-up, we strengthen this bound by 1. Following the suggestion above, we want to ensure that when we extend a 5-coloring of a planar graph to a previously deleted 5-vertex $v$ that the neighbors of $v$ use some color at least twice. Thus, we identify non-adjacent neighbors of $v$.

**Theorem 4.1 (5 Color Theorem).** *Every planar graph is* 5*-colorable.*

*Proof.* Assume the theorem is false, and let G be a counterexample minimizing $|G|$. Since G is planar, by Lemma 1.6 it has a $5^-$-vertex $v$. If $d(v) \leqslant 4$, then we 5-color $G - v$ by minimality, and greedily extend the coloring to $v$. So assume that $d(v) = 5$. Note that $K_6$ is non-planar, since $\|K_6\| = \binom{6}{2} > 3(6) - 6$. So $v$ has two neighbors, $w_1$ and $w_2$, that are non-adjacent. Form $G'$ from G by contracting edges $vw_1$ and $vw_2$, and suppressing any parallel edges this creates. Note that $G'$ is planar. By minimality, $G'$ has a 5-coloring $\varphi'$. Furthermore, $\varphi'$ gives a 5-coloring of $G - v$ that uses the same color on $w_1$ and $w_2$. Since $\varphi'$ uses at most 4 colors on neighbors of $v$, we can extend $\varphi'$ to G. □

The upper bound of 5 in Theorem 4.1 can be improved further to 4, and this is best possible, as shown by $K_4$. But $K_4$ is far from being the only obstruction to 3-coloring planar graphs. In fact, there are infinitely many 4-critical planar graphs. A simple family of examples are the "necklaces". Each necklace is formed from an odd cycle by expanding each vertex of some maximum independent set into an edge, with both endpoints of each new edge inheriting the two neighbors of the original vertex. Figure 12.1 shows the first 3 necklaces.

**Theorem 4.2 (4 Color Theorem).** *Every planar graph is* 4*-colorable.*

Proving the 5 Color Theorem is easy, as we just saw. In contrast, proving the 4 Color Theorem is quite hard. More precisely, all known proofs require extensive computer case-checking that is infeasible for a human. Not surprisingly, these proofs use reducibility and unavoidability, and the latter relies on discharging. But the former needs two new techniques: Kempe swaps (which we study in Chapter 3), and a more subtle use of minimality[1].

Entire books have been written on the 4 Color Theorem [23, 415, 167], so we will not address it at length. In Section 3.2 we prove a few properties of a minimal counterexample, and in the Notes we recommend places to read more. Instead, we now turn to Grötzsch's Theorem, that every triangle-free planar graph is 3-colorable.

### 4.1.1 3-Coloring Planar Graphs: Grötzsch's Theorem

separating cycle
triangle, minimal counterexample

**Definition 4.3.** A *separating cycle* in a plane graph G is a cycle C with vertices of G both inside and outside. A *triangle* is a 3-cycle. In this section, a *minimal counterexample* means a counterexample to Theorem 4.5 that minimizes $|G|$.

[1] The smaller graph that we color by induction is not just formed by contracting edges, but requires adding new vertices where old ones were deleted.

**Remark 4.4.** We often reuse the following trick. To prove a theorem, we consider some hypothetical minimal counterexample, and show that it has no short separating cycle $C$, as follows. Suppose such a cycle $C$ exists. By minimality, we get a coloring $\varphi_{\text{in}}$ of $C$ and the vertices inside, as well as a coloring $\varphi_{\text{out}}$ of $C$ and the vertices outside. When $\varphi_{\text{in}}$ and $\varphi_{\text{out}}$ agree on $C$, they combine to give a coloring of $G$, which is a contradiction. We may also need to modify one coloring first. For example, suppose a minimal counterexample $G$ to the 4 Color Theorem has a separating 3-cycle $C$. The vertices of $C$ receive distinct colors in both $\varphi_{\text{in}}$ and $\varphi_{\text{out}}$, so we can permute color classes of $\varphi_{\text{in}}$ to agree with $\varphi_{\text{out}}$ on $V(C)$. Thus, our minimal counterexample $G$ has no separating 3-cycle.

**Theorem 4.5 (Grötzsch's Theorem).** *Every triangle-free planar graph is 3-colorable.*

We assume a minimal counterexample $G$, and note that $\delta(G) \geqslant 3$, since $2^-$-vertices are reducible. Every such $G$ has a $5^-$-face, by face charging. So it would suffice to show that 4-faces and 5-faces are reducible. For a 4-face, we can always identify some non-adjacent pair of its vertices and 3-color the smaller graph by minimality, which gives a 3-coloring of $G$. We want to do something similar for a 5-face $f$, but now we cannot identify any vertices on $f$ (without creating a triangle or loop), so we instead identify some nearby vertices.

Identifying these nearby vertices is complicated, since we might create a triangle, if $G$ has a short separating cycle. So we design Lemma 4.8 to handle separating $6^-$-cycles. When $G$ has such a cycle, we restrict ourselves to working on a subgraph $H$ that lies inside an "innermost" such cycle. Now we perform the reduction within $H$, which sidesteps this pitfall. To formalize when a $k$-face is reducible, we need a new definition. When $k = 5$ the details are technical, but they arise naturally from our reducibility proof in Lemma 4.7.

**Definition 4.6.** A 4-face or 6-face $(v_1, \dots, v_s)$ is *safe* if every path of length at most 3 in $G$ from $v_1$ to $v_3$ is part of the cycle $v_1 \cdots v_s$. A 5-face $f$ is *safe* if $(v_1, \dots, v_5)$ satisfies the following four properties (see Figure 4.1): (i) $d(v_i) = 3$ for all $i \in [4]$, (ii) if $w_i$ denotes the neighbor of $v_i$ not on $f$, for each $i \in [4]$, then all vertices $w_i$ are distinct and non-adjacent, (iii) $G \setminus \{v_1, v_2, v_3, v_4\}$ has no path of length at most 3 joining $w_2$ and $v_5$, (iv) $G$ has at most one path of length at most 3 joining vertices $w_3$ and $w_4$ other than $w_3v_3v_4w_4$, and if such a path exists, then it has length 2; if $x$ is the common neighbor of $w_3$ and $w_4$, then $w_3xw_4v_4v_3$ is a 5-face. safe

The proof of Theorem 4.5 is now easy, assuming our three lemmas on reducibility and unavoidability, which we state and prove below.

*Proof of Theorem 4.5.* Suppose the theorem is false, and let $G$ be a minimal counterexample. Clearly $\delta(G) \geqslant 3$, since for any $2^-$-vertex $v$, we can 3-color $G - v$ by minimality, and greedily extend the coloring to $v$. Lemma 4.9 implies that $G$ contains a safe 4-face, 5-face, or 6-face, while Lemma 4.7 implies the opposite, since $G$ is a minimal counterexample. This contradiction completes the proof. □

**Lemma 4.7.** *Safe 4-faces, safe 5-faces, and safe 6-faces are all reducible for Theorem 4.5. That is, none of these appears in a minimal counterexample.*

*Proof.* Suppose that G is a minimal counterexample, and G contains a safe 4-face or safe 6-face $(v_1, \dots, v_s)$. Form $G'$ from G by identifying $v_1$ and $v_3$. By the definition of safe, G has no path of length at most 3 joining $v_1$ and $v_3$. So $G'$ is a triangle-free plane graph. Since G is minimal, $G'$ has a 3-coloring, which induces a 3-coloring of G, a contradiction.

Suppose instead that G contains a safe 5-face $(v_1, \dots, v_5)$, as shown in Figure 4.1. Form $G'$ from G by deleting $v_1, v_2, v_3, v_4$ and identifying $w_3$ with $w_4$ and also $w_2$ with $v_5$; call these new vertices $w_3 * w_4$ and $w_2 * v_5$, . Now (iii) and (iv), in the definition of safe 5-face, imply that $G'$ is a triangle-free plane graph. Since G is minimal, $G'$ has a 3-coloring, which induces a 3-coloring $\varphi'$ of $G \setminus \{v_1, v_2, v_3, v_4\}$. We now extend this 3-coloring $\varphi'$ to G.

$w_3 * w_4$
$w_2 * v_5$

Let $\alpha$, $\beta$, and $\gamma$ be the colors used, respectively, by $\varphi'$ on $w_1$, $w_2 * v_5$, and $w_3 * w_4$. If $\beta = \gamma$, then we can color greedily in the order $v_1, v_2, v_3, v_4$. Otherwise, we use color $\beta$ on $v_3$, and color greedily in the order $v_1, v_2, v_4$. This gives a 3-coloring of G, a contradiction. □

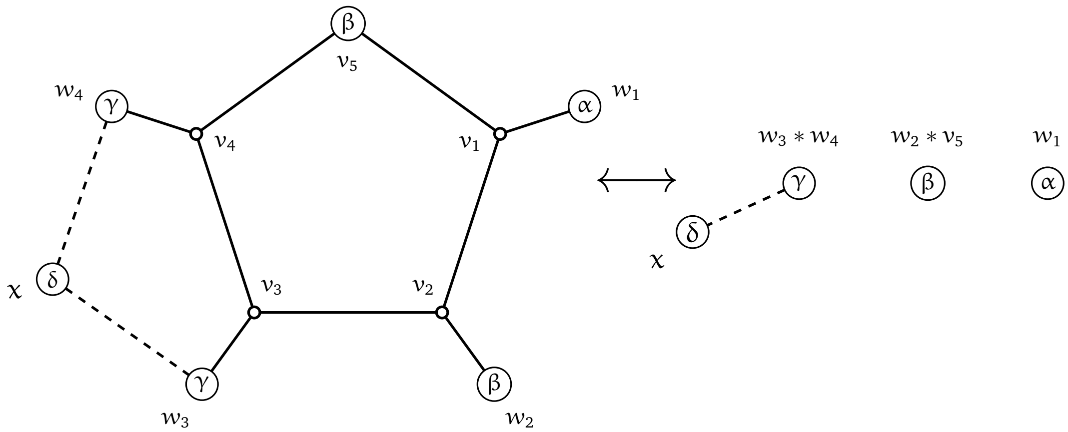



Figure 4.1: A safe 5-face in G and the corresponding subgraph in $G'$. The 3-coloring of $G'$ induces a partial 3-coloring of G, which we then extend to all of G.

Due to Lemma 4.7, our goal is to show that every triangle-free planar G with $\delta(G) \geqslant 3$ contains a safe 4-face, safe 5-face, or safe 6-face. But the definition of a safe 5-face is unwieldy. To improve clarity, we first show the unavoidability of certain configurations; only later do we show they are safe. To prove unavoidability we use discharging, much like in Lemma 1.43. The main difference is in our application, which is restricted to the subgraph inside an innermost separating $6^-$-cycle. Thus the outer face may actually be a separating $6^-$-cycle, rather than a real face; so we need a safe 4-face, safe 5-face, or safe 6-face that is *not* the outer face, $f_0$. To account for this, we give $f_0$ extra charge, to guarantee that it ends positive.

**Lemma 4.8.** *Let* G *be a connected triangle-free plane graph with* $\delta(G) \geqslant 2$*, with outer face* $f_0$ *of length at most 6, and with* $d(v) \geqslant 3$ *for each vertex* $v$ *not on* $f_0$*. Assume the boundary of* $f_0$ *is a cycle. If* $E(G) \neq E(f_0)$*, then* G *contains a face* f*, other than* $f_0$*, such that either (a)* f *is a* 4*-face or (b)* f *is a* 5*-face* $(v_1, \dots, v_5)$ *and, for each* $i \in [4]$*, both* $d(v_i) = 3$ *and* $v_i$ *is not on* $f_0$*.*

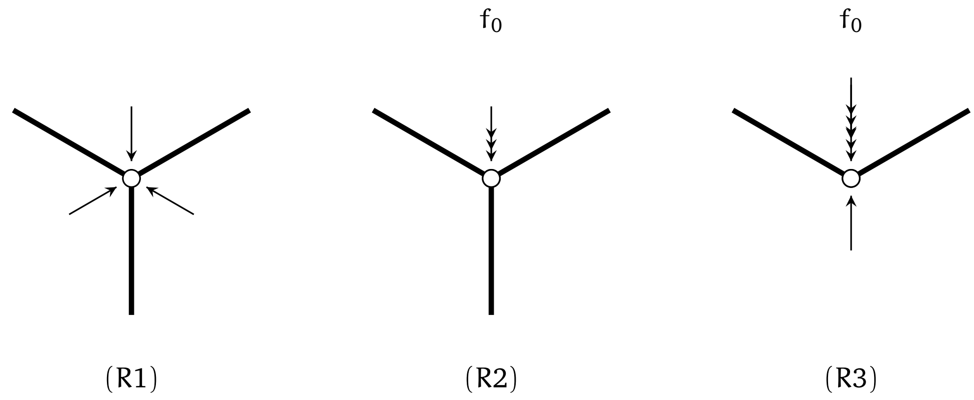



Figure 4.2: (R1)–(R3) give charge from faces to 3-vertices and 2-vertices. Here $\longrightarrow$, $\twoheadrightarrow$, and $\twoheadrightarrow$ denote sending 1/3, 3/3, and 5/3.

*Proof.* Assume the lemma is false, and let G be a counterexample. To reach a contradiction, we use discharging[2], giving $d(v)-4$ to each vertex $v$ and $\ell(f)-4$ to each face $f$ other than $f_0$, but giving $\ell(f_0)+4$ to $f_0$. By Euler's Formula, these charges sum to 0. We use three discharging rules, shown in Figure 4.2.

(R1) Each 3-vertex not on $f_0$ takes $\frac{1}{3}$ from each incident face.

(R2) Each 3-vertex on $f_0$ takes 1 from $f_0$.

(R3) Each 2-vertex on $f_0$ takes $\frac{5}{3}$ from $f_0$ and $\frac{1}{3}$ from its other incident face.

We show that each vertex and face ends happy, and $f_0$ ends positive. Since the initial charges sum to 0, this is a contradiction.

Each 3-vertex $v$ starts with $-1$ so must gain 1. If $v$ is on $f_0$, then $v$ gains 1 by (R2). If $v$ is not on $f_0$, then $v$ gains $3(\frac{1}{3})$, by (R1). Each 2-vertex $v$ starts with $-2$, so must gain 2. Note that $v$ must be on $f_0$, so $v$ gains $\frac{5}{3}+\frac{1}{3}$, by (R3). Thus, all vertices end happy.

Now we consider faces, starting with $f_0$. Let $s := \ell(f_0)$. By hypothesis, $s \leqslant 6$ and $f_0$ contains a $3^+$-vertex. Thus, $f_0$ ends positive, since $s+4-\frac{5}{3}(s-1)-1 = \frac{14-2s}{3} > 0$. Recall that G has no 3-face, since it is triangle-free, and G has no 4-face, since it is a counterexample to the lemma. So we only need to consider $5^+$-faces.

Let $f$ be a $5^+$-face other than $f_0$. By (R1) and (R3), $f$ gives at most $\frac{1}{3}$ to each incident vertex, so ends with at least $\ell(f)-4-\frac{1}{3}\ell(f) = \frac{2}{3}(\ell(f)-6)$. Thus $f$ ends happy when $\ell(f) \geqslant 6$. Further, if $\ell(f) = 5$ and $f$ ends unhappy, then $f$ gives charge to at least 4 vertices. So, to finish the proof, we show this is impossible.

[2] Balanced charging has initial charges summing to $-8$. Here we modify the initial charge of $f_0$ to exploit the "slack" between $-8$ and 0. We use a similar approach in the proof of Lemma 4.37.

Suppose that one of these vertices receiving charge from $f$, say $v$, is a 2-vertex on $f_0$. Consider the maximum path $P$ in $E(f) \cap E(f_0)$ containing $v$. The endvertices of $P$ must be distinct $3^+$-vertices that lie on both $f$ and $f_0$, since the boundary of $f_0$ is a cycle. So these end-vertices each receive no charge from $f$, and $f$ ends happy. Otherwise, $f$ has no incident 2-vertex, so $f$ is a 5-face with at least four incident 3-vertices not on $f_0$. But now $f$ satisfies (b) of the lemma, a contradiction. □

To complete the proof of Theorem 4.5, we show that (a) and (b) in the previous lemma both imply that $G$ has a safe 4-face, safe 5-face, or safe 6-face.

**Lemma 4.9.** *Every triangle-free plane graph $G$ with $\delta(G) \geqslant 3$ contains a safe 4-face, safe 5-face, or safe 6-face.*

*Proof.* We first show that every 4-face is safe. Let $f$ be a 4-face $(v_1, v_2, v_3, v_4)$ in a triangle-free plane graph. If $f$ is not safe, then $G$ contains a $v_1, v_3$-path $P_1$, of length 2 or 3 (edge-disjoint from $f$). By symmetry, $G$ also contains a $v_2, v_4$-path $P_2$ of length 2 or 3. By planarity, paths $P_1$ and $P_2$ have a common vertex $x$. But now $x$ induces a triangle with two vertices of $f$, a contradiction. Thus, $f$ is a safe 4-face; see Figure 12.5.[3] This is a very special case of the Folding Lemma, which we prove in Section 4.3.

Now we assume that $G$ has no 4-face. If $G$ has no separating $6^-$-cycle, then let $G' := G$ and choose an embedding of $G'$ with outer face of length at most 5 (by face charging, $G$ has such a face, since $\delta(G) \geqslant 3$). Otherwise, let $G'$ be an induced subgraph of $G$ bounded by a separating
$G'$ $6^-$-cycle, including all vertices inside the separating $6^-$-cycle; subject to this, choose $G'$ to be as small as possible. So $G'$ has no separating $6^-$-cycle.

Since $G$ has no 4-face, Lemma 4.8 implies that $G'$ contains a 5-face $(v_1, \ldots, v_5)$, other than $f_0$, such that both $d(v_i) = 3$ and $v_i$ is not on $f_0$, for each $i \in [4]$. We want to show that $v_1 \cdots v_5$ satisfies conditions (i)–(iv) for a safe 5-face, in Definition 4.6. For convenience, we repeat that
safe definition. A 5-face $f$ is *safe* if $(v_1, \ldots, v_5)$ satisfies the following four properties: (i) $d(v_i) = 3$ for all $i \in [4]$, (ii) if $w_i$ denotes the neighbor of $v_i$ not on $f$, for each $i \in [4]$, then all $w_i$ are distinct and non-adjacent, (iii) $G \setminus \{v_1, v_2, v_3, v_4\}$ has no path of length at most 3 joining $w_2$ and $v_5$, (iv) $G$ has at most one path of length at most 3 joining $w_3$ and $w_4$ other than $w_3v_3v_4w_4$, and if such a path exists, then it has length 2; if $x$ is the common neighbor of $w_3$ and $w_4$, then $w_3xw_4v_4v_3$ is a 5-face.

Clearly (i) holds. Note that (ii) also holds, for otherwise $G'$ has a 4-face or a separating $6^-$-cycle. Similarly, (iii) holds, for if $G \setminus \{v_1, v_2, v_3, v_4\}$ contains a path of length at most 3 joining $w_2$ and $v_5$, then with $w_2v_2v_1v_5$, this path forms a $6^-$-cycle in $G'$ that separates $w_1$ from $v_3$. Finally, suppose that (iv) fails and let $P$ be a path of length at most 3 joining $w_3$ and $w_4$, other than $w_3v_3v_4w_4$. Now $P$ together with $w_3v_3v_4w_4$ must form a face boundary, since otherwise $G'$ has a separating $6^-$-cycle; in particular, exactly one such path $P$ exists. Since (iv) fails and $G$ is triangle-free, $P$ must be a path $w_3x_1x_2w_4$, as in Figure 4.3. But now

[3] No, that figure number is not a typo.

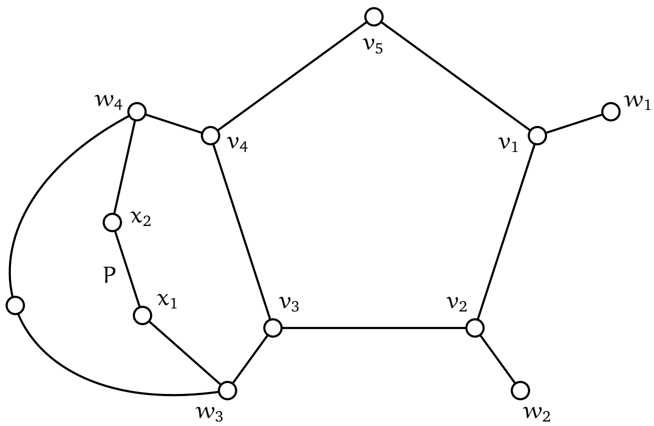



Figure 4.3: The 5-face in Lemma 4.8(b) must be safe.

$(v_3, v_4, w_4, x_2, x_1, w_3)$ must be a safe 6-face. If not, then $G'$ contains a path joining $w_3$ and $v_4$ of length at most 3; call it Q. (Note that Q contains $w_4$ or $v_5$ as an interior vertex, since $d(v_4) = 3$.) So $G'$ contains a separating 5-cycle $Qv_4v_3w_3$, which is a contradiction. □

## 4.2 $\frac{9}{2}$-Coloring

When a coloring conjecture seems difficult, we often consider its fractional relaxation. Now each vertex can be partially colored with various colors, say one half red, one third blue, and one sixth green. Here we prove a fractional coloring result for planar graphs, in the direction of the 4 Color Theorem.

**Definition 4.10.** To *fractionally color* a graph G, we give each independent set in G a nonnegative weight, such that each vertex appears in sets with weights summing to 1. A graph G is *fractionally* k*-colorable* if G has a weight assignment with weights summing to at most k. The *fractional chromatic number*, $\chi_f(G)$, is the minimum k such that G is fractionally k-colorable.

fractionally color

fractional chromatic number

By restricting each weight in a fractional coloring to be 0 or 1, we get the standard definition of vertex coloring. So always $\chi_f(G) \leqslant \chi(G)$. In 1997, Scheinerman and Ullman [356, p. 75] succinctly described the state of the art for fractionally coloring planar graphs:

> The fractional analogue of the four-color theorem is the assertion that the maximum value of $\chi_f(G)$ over all planar graphs G is 4. That this maximum is no more than 4 follows from the four-color theorem itself, while the example of $K_4$ shows that it is no less than 4. Given that the proof of the four-color theorem is so difficult, one might ask whether it is possible to prove an interesting upper bound for this

> maximum without appeal to the four-color theorem. Certainly $\chi_f(G) \leqslant 5$ for any planar G, because $\chi(G) \leqslant 5$, a result whose proof is elementary. But what about a simple proof of, say, $\chi_f(G) \leqslant \frac{9}{2}$ for all planar G? The only result in this direction is in a 1973 paper of Hilton, Rado, and Scott [217] that predates the proof of the four-color theorem; they prove $\chi_f(G) < 5$ for any planar graph G, although they are not able to find any constant $c < 5$ with $\chi_f(G) < c$ for all planar graphs G. This may be the first appearance in print of the invariant $\chi_f$.

Here we give exactly what Scheinerman and Ullman asked for—a simple proof that $\chi_f(G) \leqslant \frac{9}{2}$ for all planar G. This result follows from a stronger statement, which needs another definition.

2-fold 9-coloring

**Definition 4.11.** A *2-fold 9-coloring* of a graph G assigns to each vertex a 2-element subset of [9], such that adjacent vertices get disjoint sets. If G has a 2-fold 9-coloring, then $\chi_f(G) \leqslant \frac{9}{2}$: to the vertices of each color class, we assign weight $\frac{1}{2}$.

The main result of this section is the following theorem.

**Theorem 4.12.** *Every planar graph has a 2-fold 9-coloring.*

The proof is similar to our proof that planar graphs are 5-colorable, but here we use more reducible configurations. Suppose the theorem is false, and let G be a minimal counterexample. Adding edges never makes coloring easier, so we assume that G is a plane triangulation. By Remark 4.4, G has no separating 3-cycle.

Now we show that G cannot exist, since each planar graph contains a configuration H that is reducible. To prove unavoidability we use discharging. To show that each configuration H is reducible, we delete V(H) and identify certain sets of vertices in N(V(H)) to get a smaller graph G′. When a vertex $v$ of H has two of its neighbors identified in G′, they get the same colors in $\varphi'$, which saves colors for $v$ (as in the proof of the 5 Color Theorem). To color G′ by minimality, we must ensure that G′ is planar and has no loops. Such a loop would arise from a 3-cycle $vw_1w_2$ when $w_1$ and $w_2$ were identified. But this is typically impossible, since if $w_1$ and $w_2$ were adjacent, then $vw_1w_2$ would be a separating 3-cycle.

coloring

minimal

**Definition 4.13.** In this section, a *coloring* means a 2-fold 9-coloring. A *minimal counterexample* G to Theorem 4.12 is one that minimizes |G| and, subject to this, minimizes the number of non-triangular faces; for short, we say a *minimal* G.

Note that a minimal G must be a plane triangulation, since otherwise adding an edge contradicts the minimality of G. Recall, from Remark 4.4, that G has no separating 3-cycle. To prove Theorem 4.12, we formalize the outline above in a series of four lemmas.

**Lemma 4.14.** *Every minimal* G *has minimum degree* 5.

*Proof.* Since G is a plane triangulation, $\delta(G) \geqslant 3$. If G contains a 3-vertex, then its neighbors induce a separating 3-cycle, a contradiction. If G contains a 4-vertex $v$, then two neighbors, say $w_1$ and $w_2$, of $v$ are non-adjacent, since $K_5$ is non-planar. Form $G'$ from $G$ by deleting $v$ and identifying $w_1$ and $w_2$. By minimality, $G'$ has a coloring, which induces a coloring $\varphi$ of $G - v$ where $w_1$ and $w_2$ get the same colors. Thus, we can extend $\varphi$ to $G$. Hence $\delta(G) \geqslant 5$. Since $G$ is planar, it is 5-degenerate, so $\delta(G) = 5$. □

Given a coloring $\varphi$ of some subgraph of $G$, our next lemma helps us extend $\varphi$ to an uncolored induced $K_{1,3}$. In Lemma 4.16 we use it to forbid numerous configurations from appearing in our minimal counterexample $G$.

**Lemma 4.15.** *Let* $H := K_{1,3}$. *If each leaf has a list of size* 3 *and the center vertex has a list of size* 5*, then we can choose* 2 *colors for each vertex from its list such that adjacent vertices get disjoint sets of colors.*

*Proof.* Let $v$ denote the center vertex and $w_1, w_2, w_3$ the leaves. Since $2|L(v)| > |L(w_1)| + |L(w_2)| + |L(w_3)|$, some color $\alpha \in L(v)$ appears in $L(w_i)$ for at most one $w_i$. If such a $w_i$ exists, then say it is $w_1$, by symmetry; now color $v$ with $\alpha$ and some color not in $L(w_1)$. Otherwise color $v$ with $\alpha$ and an arbitrary color. Finally, color each $w_i$ arbitrarily from its at least 2 available colors. □

Now we reach our main reducibility lemma.

**Lemma 4.16.** *Every minimal* $G$ *has none of the following three configurations:*

*(a)* *a* 5*-vertex with a* 5*-neighbor and a non-adjacent* $6^-$*-neighbor,*

*(b)* *a* 6*-vertex with non-adjacent* $6^-$*-neighbors, or*

*(c)* *a* 7*-vertex with a* 5*-neighbor and two other* $6^-$*-neighbors such that all three are pairwise non-adjacent.*

*Proof.* Each configuration $H$ induces either $K_{1,2}$ or $K_{1,3}$. To prove $H$ is reducible, we (1) delete $V(H)$ and identify some vertices in $N(V(H))$ to get $G'$, (2) color $G'$ by minimality, and (3) use Lemma 4.15 to extend the coloring to $G$. The main question is how to identify vertices of $G - H$ so that the vertices of $H$ have lists large enough to apply Lemma 4.15.

In each diagram of Figures 4.4 and 4.5, the vertices of $H$ are $v$, $w_1$, $w_2$, and possibly $w_3$. Vertices to be identified in $G'$ are labeled with the same number. By assumption, $v$ and its neighbors are all distinct; however, pairs of vertices at distance two from $v$ that are drawn as distinct may not be. When this happens, certain prescribed vertex identifications will be impossible, since they create loops. Seeing all the cases is unenlightening, so we focus on the harder instances, those where prescribed $6^-$-vertices have degree 6, rather than 5. These are shown in Figures 4.4 and 4.5. The full details are in [103].

Let $v$ be a 6-vertex with non-adjacent 6-neighbors, $w_1$ and $w_2$. The 6-neighbors are either "across", as at the top of Figure 4.4, or "offset", as at the bottom of Figure 4.4 (on either the left or right). If we can form $G'$ as prescribed in any of these figures, by identifying each pair of vertices with the same label, then $v$ has 5 allowable colors, since it has only two neighbors in $G'$. Similarly, $w_1$ and $w_2$ each have at least 3 allowable colors, since they have only three neighbors in $G'$. By Lemma 4.15, we can thus extend any coloring of $G'$ to a coloring of $G$. So it suffices to show that we can identify vertices as prescribed in one of the three ways shown in Figure 4.4, without creating loops. Note that pairs of vertices drawn at distance 2 or 3 must always be distinct, since $G$ has no separating 3-cycle.

The only complication at the top of Figure 4.4 is that a vertex labeled 1 might be the same as a vertex labeled 4 that is drawn at distance four; suppose so, and call this vertex $x$. By symmetry, assume that $x$ is formed by identifying the vertices at the top left and bottom right. This is a problem only if also a vertex labeled 1 is adjacent to one labeled 4; so suppose this happens. Now the top right and bottom left vertices are non-adjacent, since they are on opposite sides of the cycle $xw_1vw_2$. Again by symmetry, we assume that $x$ is adjacent to the bottom left vertex labeled 1. But now $G$ has a separating 3-cycle (consisting of $x$, its neighbor labeled 1, and their common neighbor $w_1$); this contradicts Remark 4.4, which finishes the case.

The offset case, shown at the bottom of Figure 4.4 is similar. On the left, only the vertices labeled 1 and 3 that are drawn at distance four might be the same; if so, then call this vertex $x$. Now we switch to the identifications shown on the right, where the two vertices drawn in bold

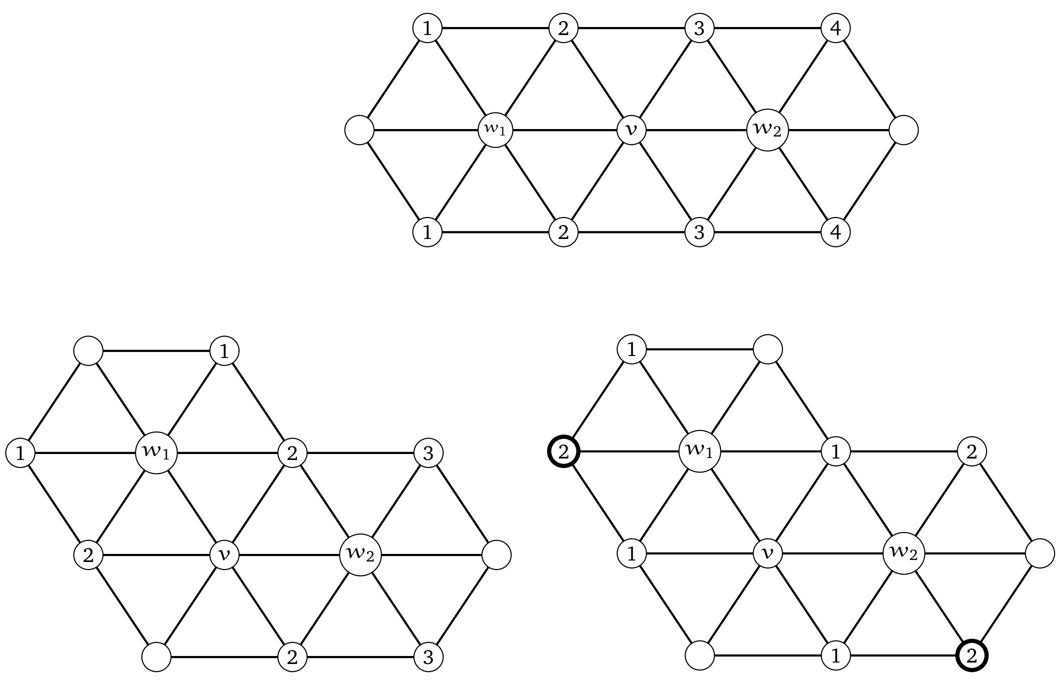



Figure 4.4: Two cases of Lemma 4.16(b). On top, the 6-neighbors of $v$, namely $w_1$ and $w_2$, are "across". On bottom, the 6-neighbors are "offset" (on the right the two vertices drawn in bold are identified).

are identified. All vertices with numeric labels are at pairwise distance at most three, due to the extra edges incident to $x$. Also, the two vertices labeled 1 that are drawn at distance three are non-adjacent, since they are separated by cycle $w_1vw_2x$. This finishes the case.

Finally, we consider the instance of case (c) on the left in Figure 4.5. By horizontal symmetry and planarity, we assume the vertices labeled 2 that are drawn at distance 3 are neither the same nor adjacent, by reflecting across edge $vw_2$ if necessary. (So the vertices labeled 1 and 2 drawn at distance 4 are distinct.) Hence, in forming $G'$ we can identify all vertices labeled 2; we can also identify all vertices labeled 3.

As in the previous cases, it is straightforward to check that no vertex labeled 2 or 3 is the same as any other labeled vertex. So we only need to consider the vertices labeled 1 and 4. The only possible problem is if some pair of vertices labeled 1 and 4 that are drawn at distance four are actually the same vertex $y$. Further, this only causes difficulty if another pair labeled 1 and 4 are adjacent. So, suppose this is the case.

If these identified and adjacent pairs are not disjoint, then two vertices with the same label (either 1 or 4) are adjacent. But now G has a separating 3-cycle, a contradiction. So assume the pairs are disjoint. Thus, the top vertices labeled 1 and 4 are the same vertex. Now $x_1$ is neither the same as, nor adjacent to, the top vertex labeled 2, since they are separated by a cycle through the pair labeled 1 and 4 that contains the top vertex labeled 1. If $x_1$ and $x_3$ are distinct, then we neglect the vertices labeled 1 and 4 altogether; instead we label $x_1$ as 2 and $x_3$ as 3. Due to the identified and adjacent pairs labeled 1 and 4, we can easily check that $G'$ is loopless, as above.

Assume instead that $x_1$ and $x_3$ are the same vertex, denoted by bold on the right in Figure 4.5. Now we switch the vertex identifications we use to form $G'$. Delete $v$, $w_1$, $w_2$, and $w_3$. Identify the two vertices labeled 4. Also identify the two neighbors of $w_1$ labeled 2, the top vertex that was labeled 3 (now 2), and $x_{1/3}$ (the bold vertex), as on the right in Figure 4.5. As in the previous cases, we can check that $G'$ is loopless. This finishes the case. □

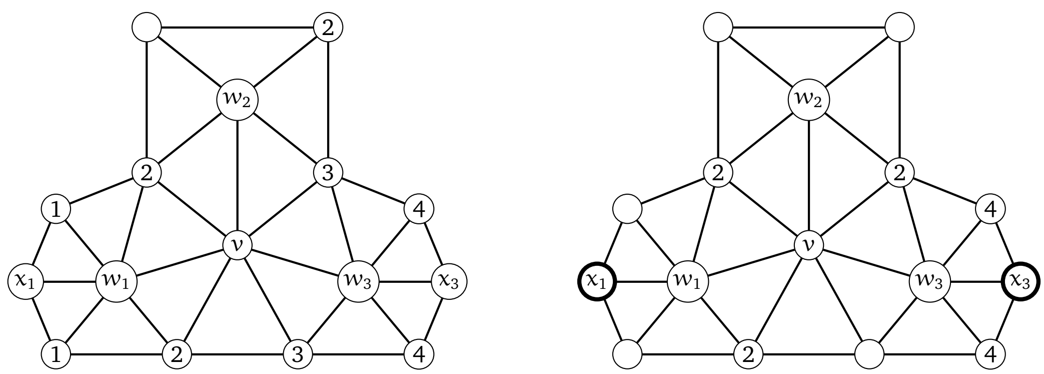


Figure 4.5: A harder case of Lemma 4.16(c): a 7-vertex with a 5-neighbor and two 6-neighbors that are pairwise nonadjacent (on the right $x_1$ and $x_3$ are identified).

Finally, we prove unavoidability, which completes the proof of Theorem 4.12.

**Lemma 4.17.** *No planar graph without separating 3-cycles satisfies the conclusions in both Lemmas 4.14, and 4.16. So no minimal counterexample exists, and Theorem 4.12 is true.*

*Proof.* Suppose that $G$ is a planar graph with no separating 3-cycles and that $G$ satisfies the conclusions of Lemmas 4.14 and 4.16. To reach a contradiction, we use vertex charging and rules (R1)–(R4), listed below.

Before providing the details, we offer some motivation. Recall that each 5-vertex must receive at least 1. Each 6-vertex must receive at least as much as it gives, and each $7^+$-vertex $v$ must give at most $d(v) - 4$. Intuitively, we would like to give $1/4$ from each $7^+$-vertex to each 5-neighbor. If we do this, $8^+$-vertices will not lose too much. So we must check that each 7-vertex does not lose too much, and that each 5-vertex receives enough (hopefully, because of Lemma 4.16). This admittedly optimistic approach does not quite work, but a modest refinement of it does.

$H_v$ We need a few definitions. For each vertex $v$, let $H_v$ denote the subgraph induced by the 5-neighbors and 6-neighbors of $v$. If some $w \in V(H_v)$ has $d_{H_v}(w) = 0$, then $w$ is an *isolated neighbor* of $v$; otherwise $w$ is a *non-isolated neighbor*. A non-isolated 5-neighbor of a vertex $v$ is *crowded* (with respect to $v$) if it has two 6-neighbors in $H_v$. We use crowded 5-neighbors to ensure that 7-vertices end happy, specifically for the configuration in Figure 4.6. We have the following 4 discharging rules.

(non)-isolated neighbor

crowded

(R1) Each $8^+$-vertex gives $\frac{1}{2}$ to each isolated 5-neighbor and $\frac{1}{4}$ to each non-isolated 5-neighbor.

(R2) Each 7-vertex gives $\frac{1}{2}$ to each isolated 5-neighbor, 0 to each crowded 5-neighbor, and $\frac{1}{4}$ to each remaining 5-neighbor.

(R3) Each $7^+$-vertex gives $\frac{1}{4}$ to each 6-neighbor.

(R4) Each 6-vertex gives $\frac{1}{2}$ to each 5-neighbor.

From Lemma 4.14, we know that $\delta(G) = 5$. And from the comment following Definition 4.13, we know that $G$ is a triangulation. Recall that every face (which is a triangle) starts and ends with charge 0; so we need not consider faces. To show that every vertex $v$ ends happy, we consider the possibilities for $d(v)$.

**Case 1: $d(v) \geqslant 8$.** Now $d(v) - 6 \geqslant \frac{d(v)}{4}$. Suppose that $v$ gives $\frac{1}{4}$ to each neighbor, rather than giving charge by (R1) and (R3). Now let each isolated 5-neighbor $w$ take also the $\frac{1}{4}$ that $v$ gave to its neighbor that (clockwise around $v$) follows $w$. Each neighbor of $v$ receives at least as much as by (R1) and (R3), and $v$ gives away $\frac{d(v)}{4}$. So when $v$ gives charge by (R1) and (R3), $v$ gives at most $\frac{d(v)}{4}$, and ends happy.

**Case 2: $d(v) = 7$.** Suppose that $v$ has an isolated 5-neighbor $w$. Let $x, y \in N(v)$ be the two $7^+$-vertices that are common neighbors of $v$ and $w$. We show that the total $v$ gives to $N(v) \setminus \{w, x, y\}$ is at most $\frac{1}{2}$. By Lemma 4.16(c), these four remaining vertices include at most two $6^-$-vertices. So, if $v$ gives them a total of more than $\frac{1}{2}$, then one of them is another isolated 5-neighbor. But now the final $6^-$-neighbor of $v$ must be at distance 2 from these first two 5-neighbors, violating Lemma 4.16(c).

So assume instead that $v$ has no isolated 5-neighbors. Thus, if $v$ loses more than 1, then $v$ loses charge to at least five $6^-$-neighbors, since they each take $\frac{1}{4}$. So assume that $|H_v| \geqslant 5$. Hence, $H_v$ consists of either (i) a 7-cycle or (ii) a single path or (iii) two paths. Recall from Lemma 4.16(b), that no 6-vertex has non-adjacent $6^-$-neighbors. So every vertex of degree 2 in $H_v$ is a 5-vertex; thus, every vertex on a cycle or in the interior of a path in $H_v$ is a 5-vertex.

In each of cases (i)–(iii), $H_v$ has an independent set of size 3 containing at least one 5-vertex; the only exception is when $H_v$ consists of a path on two vertices and a path on three vertices, and the only 5-vertex is the internal vertex on the longer path. But then the 5-vertex is a crowded neighbor of $v$, as in Figure 4.6, and receives no charge from $v$. So $v$ ends happy.

**Case 3: $\mathbf{d(v) = 6}$.** By Lemma 4.16(b), $v$ has at most two $6^-$-neighbors. So $v$ loses at most $2(\frac{1}{2})$ by (R4), and gains at least $4(\frac{1}{4})$ by (R3), and ends happy.

**Case 4: $\mathbf{d(v) = 5}$.** Since $v$ begins with $-1$, it must gain at least 1. If $v$ has at least two 6-neighbors, then it gains at least $2(\frac{1}{2})$, by (R4); so assume $v$ has at most one 6-neighbor. If $v$ has at least four $6^+$-neighbors, then it gains at least $4(\frac{1}{4})$, by (R1) and (R2) and (R4), and ends happy ($v$ has at most one 6-neighbor, so none of its 7-neighbors see $v$ as crowded). Instead assume $v$ has at least two 5-neighbors. By Lemma 4.16(a), these 5-neighbors must be adjacent and $v$ has no 6-neighbors. But now one of $v$'s three $7^+$-neighbors sees $v$ as an isolated 5-neighbor, so it sends $v$ $\frac{1}{2}$. Thus, $v$ gains at least $\frac{1}{2} + 2(\frac{1}{4})$, by (R2) and (R4), and ends happy. $\square$

It is typical, in discharging proofs, that vertices needing charge get at least as much from neighbors of higher degree as from those of lower degree. In contrast, here each 6-vertex gives each 5-neighbor $\frac{1}{2}$, while each 7-vertex may give a 5-neighbor $\frac{1}{4}$ or even 0. It is this observation that each 6-vertex can afford to give each 5-neighbor $\frac{1}{2}$ (because of Lemma 4.16(b)) that motivates (R2), and that ultimately makes the discharging portion of this proof so simple.

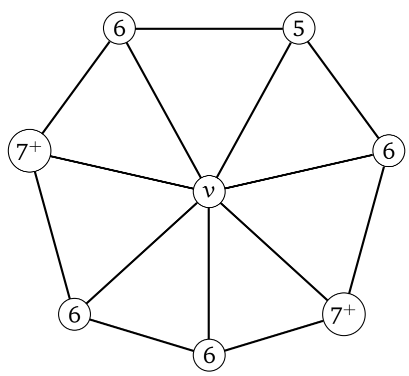



Figure 4.6: A 7-vertex $v$ does not give any charge to a crowded 5-neighbor.

## 4.3 The Folding Lemma: Graph Homomorphisms

By Grötzsch's Theorem, every triangle-free planar graph $G$ is 3-colorable. And if $G$ contains an odd cycle, then $\chi(G) = 3$. So what more can we say? Even cycles are bipartite. Intuitively, if $G$ has no short odd cycle, then $G$ is more "nearly bipartite" than $K_3$ or $C_5$. We formalize this idea with a few definitions.

odd-girth
graph homomorphism
maps into
k-thread

**Definition 4.18.** The *odd-girth* of a graph $G$ is the length of its shortest odd cycle. A *graph homomorphism* from $G$ to $H$ is a map $\varphi : V(G) \to V(H)$ such that if $vw \in E(G)$, then $\varphi(v)\varphi(w) \in E(H)$; that is, $\varphi$ preserves edges. If $G$ admits a homomorphism into $H$, then $G$ *maps into* $H$, and we write $G \to H$. If $G_1 \to G_2$ and $G_2 \to G_3$, then composing the maps shows that $G_1 \to G_3$. Figure 4.7 shows the case when $G_1$, $G_2$, $G_3$ are $C_7$, $C_5$, $C_3$. Recall that a $k$-*thread* in $G$ is a path with $k$ internal vertices, each with degree 2 in $G$.

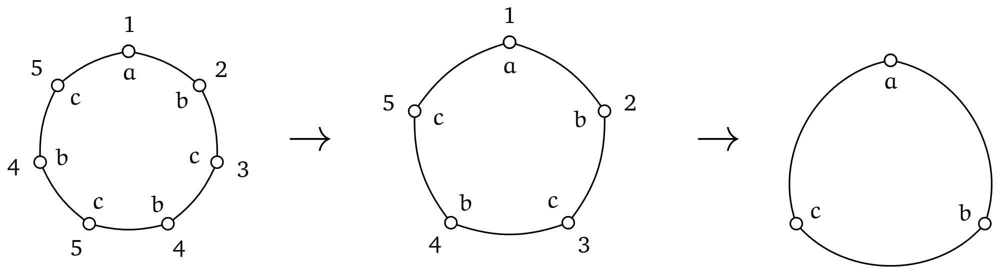



Figure 4.7: $C_7 \to C_5$ (with labels 1,2,3,4,5) and also $C_5 \to C_3$ (with labels a,b,c). Composing these maps shows that $C_7 \to C_3$ (with labels a,b,c).

Note that $G$ is $k$-colorable exactly when $G \to K_k$; further, the homomorphisms from $G$ to $K_k$ are in bijection with the $k$-colorings. For this reason, we often call a map $\varphi : G \to H$ an $H$-*coloring*. But there is no need to only consider maps into cliques. The Kneser Graph $K_{n:k}$ has as its vertex set the $k$-element subsets of $[n]$, and two vertices are adjacent when their subsets are disjoint.[4] So Theorem 4.12 says that every planar graph maps into $K_{9:2}$. But when does a graph map into $C_{2k+1}$ for some large $k$? Pavol Hell conjectured the following, and it was proved [262] shortly thereafter.

**Conjecture 4.19 (Proved).** *There exists a function* $f(k)$ *such that* $G \to C_{2k+1}$ *whenever* $G$ *is planar with odd-girth at least* $f(k)$.

When we replace 'odd-girth' by 'girth', this conjecture becomes easy to prove, by discharging. Every such $G$ contains a cut-vertex or a long thread, both of which are reducible. But how do we handle a short even face? As with 4-faces in the proof of Grötzsch's Theorem, we identify some pair of vertices at distance two along the face. In a sense, the Folding Lemma (which we will prove soon) just extracts that step and generalizes it.

[4]The most famous examples of Kneser graphs are the cliques $K_n$, which can be written as $K_{n:1}$. The next most famous example is the Petersen graph, which is $K_{5:2}$.

**Lemma 4.20 (Folding Lemma).** *Let* $G$ *be a connected plane graph with odd-girth* $g$. *If* $v_1 \cdots v_r$ *is a face boundary in* $G$ *and* $r \neq g$, *then there exists* $i \in [r]$ *such that identifying* $v_{i-1}$ *and* $v_{i+1}$ *(with subscripts modulo* $r$*) produces a graph* $G'$ *with odd-girth* $g$.

The Folding Lemma gives the following immediate corollary.

**Corollary 4.21.** *If* $G$ *is planar with odd-girth* $g$, *then* $G \to G''$ *for some* $G''$ *with odd-girth* $g$ *and with every face of length* $g$.

*Proof.* Assume not, and choose a counterexample $G$ minimizing $|G|$. If $G$ is disconnected, then we identify one vertex from each component, which gives a smaller counterexample. So $G$ is connected and has a face $f$ with $\ell(f) \neq g$. By the Folding Lemma, $G \to G'$ for some smaller $G'$ satisfying the corollary's hypothesis. By minimality $G' \to G''$ for some $G''$ satisfying the corollary's conclusion. But now $G \to G' \to G''$, so $G$ is not a counterexample. □

**Definition 4.22.** For walks $P$ and $Q$, with $Q$ starting at the end of $P$, form $PQ$ by appending $Q$ to $P$. For a directed cycle $D$ with $v_i, v_j \in V(D)$, let $v_iDv_j$ denote the directed subpath of $D$ from $v_i$ to $v_j$. When the orientation of $D$ is unspecified, we assume that its indices are increasing. Note that $v_iDv_j \neq v_jDv_i$; in fact, $(v_iDv_j)(v_jDv_i) = D$. For example, in Figure 4.8 path $v_iDv_j$ contains $v_{i+1}$ and $v_k$, but not $v_{i-1}$. In contrast, $v_jDv_i$ contains $v_{i-1}$, but neither $v_{i+1}$ nor $v_k$. For a graph $G$ with odd-girth $g$ and a specified face $f$ with boundary $v_1 \cdots v_r$, where $r \neq g$, let $C$ be the directed cycle $v_1 \cdots v_r$. Form $G_i$ from $G$ by identifying $v_{i-1}$ and $v_{i+1}$ (with subscripts modulo $r$) in the interior of $f$.[5]

$v_iDv_j$

$C$, $G_i$

A *critical cycle for* $v_i$ *(and* $f$*)* is a $g$-cycle that has as a subpath $v_{i-1}v_iv_{i+1}$. If $G_i$ has odd-girth less than $g$, then this is precisely because $G$ contains a critical cycle for $v_i$ and $f$. Given $C$, as above, and a critical cycle $D$ for $v_i$ and $f$, the *swath* of $D$ is its longest directed subpath $v_jCv_k$ such that $v_{i-1}v_iv_{i+1} \subseteq v_jCv_k$. Let $D$ be a directed cycle, $P$ a directed subpath of $D$, and $Q$ a walk with the same endpoints as $P$. Now $\text{splice}(D, P, Q)$ denotes the closed walk formed from $D$ by replacing the edges of $P$ with those of $Q$ (if $Q$ starts where $P$ ends, and vice versa, then we traverse $Q$ backwards). In Figure 4.8, $\text{splice}(D, v_jDv_k, v_kCv_j)$ is the directed cycle that follows $D$ along the top of the figure and follows $C$ (backwards) along the outside and bottom.

critical cycle for $v_i$

swath

For clarity, we split the proof of the Folding Lemma into two lemmas. Assuming a counterexample, the first says the following. When we take a critical cycle with longest swath, $P$, and a critical cycle for one endpoint of $P$, the endpoints of the two swaths must alternate along $C$. In other words, the case shown in Figure 4.9 is impossible. The second lemma says that when this happens, we can get a critical cycle with a longer swath, which is a contradiction.

**Lemma 4.23.** *Suppose* $G$ *is a counterexample to the Folding Lemma. Now* $G$ *must contain a face* $f$ *with* $\ell(f) > g$ *and distinct vertices* $v_i, v_j, v_{i+p}, v_{j+q}$ *(in that cyclic order) such that both (a)* $v_iCv_{i+p}$ *is a longest swath among all critical cycles for* $f$ *and also (b)* $v_jCv_{j+q}$ *is a swath of a critical cycle for* $v_{i+p}$.

[5] In $G_i$ vertex $v_i$ is not on the face arising from $f$, though it is on every other face that it is on in $G$.

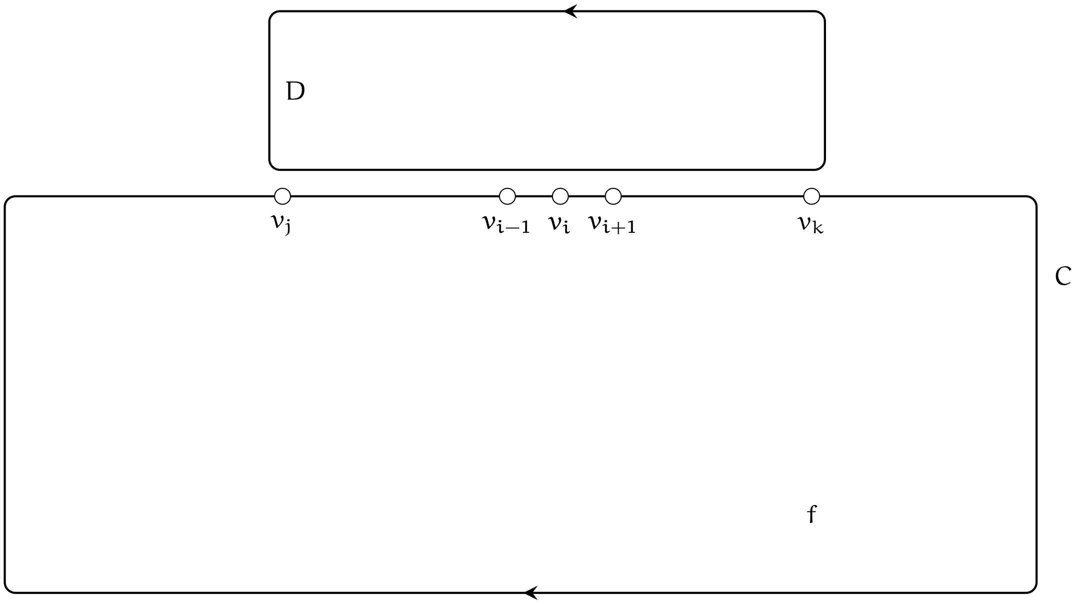



Figure 4.8: D is a critical cycle for $v_i$. The swath of D is $v_jCv_k$, which equals $v_jDv_k$. Now $\text{splice}(D, v_jDv_k, v_kCv_j)$ is the directed cycle formed by following D from $v_k$ to $v_j$ and following C (backwards) from $v_j$ to $v_k$.

The possible problem (when trying to prove Lemma 4.23) is that $v_j$ and $v_{j+q}$ may be “nested inside of” $v_i$ and $v_{i+p}$, so the order of the vertices is $v_i, v_{j+q}, v_j, v_{i+p}$, as in Figure 4.9. However, in this case $E(v_iCv_{i+p}) \cup E(v_jCv_{j+q}) = E(C)$; in particular $p > \frac{\ell(f)}{2}$. So we must prove this cannot happen; assume it does. When $\ell(f)$ is even, we go around f the other way, to get a shorter cycle, a contradiction. When $\ell(f)$ is odd, we do something similar, splicing from another critical cycle.

*Proof.* Suppose that G has a face f with $\ell(f) > g$ such that $G_i$ has odd girth $g-2$, for each $i \in [\ell(f)]$. (Notice that G must be 2-connected.[6]) Let $D_1$ be a critical cycle for C with longest swath; choose i and p so the swath is $v_iCv_{i+p}$. Since $G_{i+p}$ has odd girth $g-2$, there exists a critical cycle $D_2$ for $v_{i+p}$; choose j and q so that the swath of $D_2$ is $v_jCv_{j+q}$. If $v_{j+q}$ is an interior vertex of $v_{i+p}Cv_i$, then we are done, since the vertices are distinct and appear in cyclic order $v_i, v_j, v_{i+p}, v_{j+q}$.

So assume that $v_{j+q}$ is on $v_iCv_j$ (and possibly $v_{j+q} = v_i$) as is shown in Figure 4.9. This implies that $E(v_iCv_{i+p}) \cup E(v_jCv_{j+q}) = E(C)$; in particular $p+q > \ell(f)$, so $p > \frac{\ell(f)}{2}$. If $\ell(f)$ is even, then we go around f the other way to get a shorter cycle. Formally,

[6]If not, then consider vertices $v_{i-1}, v_i, v_{i+1}$ that are successive on the boundary of a face f, where $v_i$ is a cut-vertex and $v_{i-1}$ and $v_{i+1}$ are in distinct blocks. Now $G_i$ has no $(g-2)$-cycle, a contradiction.

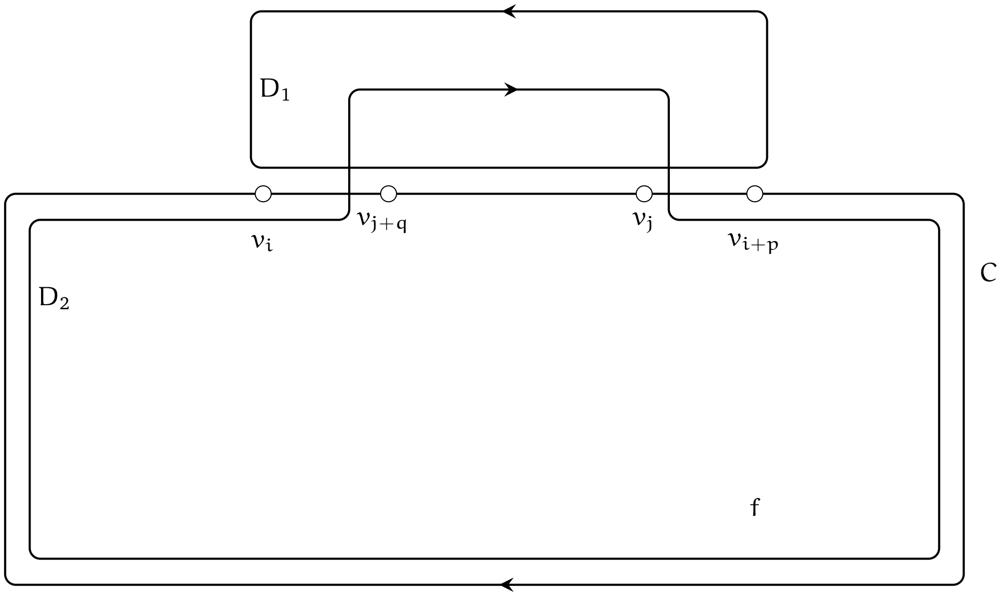



Figure 4.9: Vertices $v_i$, $v_{j+q}$, $v_j$, $v_{i+p}$ cannot appear around $f$ in the order shown.

$\text{splice}(D_1, v_iCv_{i+p}, v_{i+p}Cv_i)$ is an odd cycle with length less than $g$, a contradiction. So assume that $\ell(f)$ is odd.

Now we will show that $\text{splice}(D_1, v_{j+q}D_1v_j, v_{j+q}D_2v_j)$ is an odd closed walk, with length less than $g$, a contradiction. Note that $v_jD_2v_{j+q} = v_jCv_{j+q}$. Recall that $\ell(C)$ and $\ell(D_2)$ are both odd. Further, $\ell(D_2) = g$ and $\ell(C) > g$. Thus, $\ell(v_{j+q}Cv_j) - \ell(v_{j+q}D_2v_j)$ is positive and even. Since $v_{j+q}D_1v_j = v_{j+q}Cv_j$, we get that $\text{splice}(D_1, v_{j+q}D_1v_j, v_{j+q}D_2v_j)$ is an odd closed walk of length less than $g$. This gives the desired contradiction. □

**Lemma 4.24.** *If face* $f$ *and indices* $i, j, p, q$ *are as in Lemma 4.23, then* $f$ *has a critical cycle with a swath of length longer than* $p$*, a contradiction.*

Let $D_1$ and $D_2$ be the two critical cycles from the proof of Lemma 4.23. Since $G$ is planar, and $D_1$ and $D_2$ intersect on $f$, they also intersect outside of $f$, say at some vertex $w$. We direct and label the walks as in Figure 4.10, so $Q_1P_1P_2Q_3$ and $Q_2P_2P_3Q_4$ are $D_1$ and $D_2$. We show $Q_1P_1P_2P_3Q_4$ is also a critical cycle; but its swath, $P_1P_2P_3$, is longer than $P_1P_2$, a contradiction.

*Proof.* Since $G$ is planar, the two critical cycles intersect outside of $f$, say at a vertex $w$. Let $P_1 := v_iCv_j$, $P_2 := v_jCv_{i+p}$, and $P_3 := v_{i+p}Cv_{j+q}$. Let $D_1$ and $D_2$ denote, respectively, the critical cycles with swaths $v_iCv_{i+p}$ and $v_jCv_{j+q}$, and let $Q_1, Q_2, Q_3, Q_4$ denote paths ending at $w$ such that $D_1 = Q_1P_1P_2Q_3$ and $D_2 = Q_2P_2P_3Q_4$; see Figure 4.10. Note that every odd closed walk must contain an odd cycle. Since $G$ has odd-girth $g$, every odd closed walk must have length at least $g$.

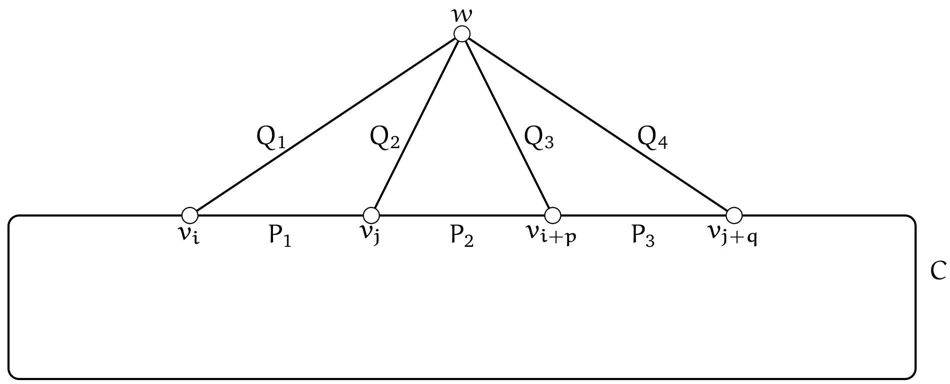



Figure 4.10: Critical cycles $Q_1P_1P_2Q_3$ and $Q_2P_2P_3Q_4$, and the vertex $w$ where they intersect outside cycle $C$.

**Case 1: $Q_2P_2Q_3$ is even.** Necessarily $\ell(Q_2) \geqslant \ell(P_2Q_3)$; otherwise splice$(D_1, P_2Q_3, Q_2)$ is an odd closed walk with length less than $g$, a contradiction. Similarly, we have $\ell(Q_3) \geqslant \ell(Q_2P_2)$. Adding these two inequalities, we get $\ell(Q_2)+\ell(Q_3) \geqslant \ell(P_2Q_3)+\ell(Q_2P_2) = \ell(Q_3)+\ell(Q_2)+2\ell(P_2)$, so $\ell(P_2) \leqslant 0$; this is a contradiction, since $v_j$ and $v_{i+p}$ are distinct.

**Case 2: $Q_2P_2Q_3$ is odd.** Let $W := Q_1P_1P_2P_3Q_4$. Now $\ell(W) = \ell(Q_1P_1P_2P_3Q_4) = \ell(D_1)+\ell(D_2)-\ell(Q_2P_2Q_3)$; since $D_1$ and $D_2$ are odd, so is $W$. Further $\ell(D_1) = \ell(D_2) = g$, and $\ell(Q_2P_2Q_3) \geqslant g$, since $Q_2P_2P_3$ is odd. But this implies that $\ell(W) \leqslant 2g - g = g$. Since $W$ is a closed odd walk with length $g$, it is a $g$-cycle; so it is a critical cycle for $v_j$. Further, it has swath longer than $v_iCv_{i+p}$, which gives the desired contradiction. □

Lemmas 4.23 and 4.24 prove the Folding Lemma. Now we can prove Conjecture 4.19.

**Theorem 4.25.** *If* $G$ *is planar with odd-girth at least* $10k-3$*, then* $G \to C_{2k+1}$.

*Proof.* Assume not, and let $G$ be a counterexample minimizing $|G|$. If $G$ is not 2-connected, then $B \to C_{2t+1}$, for each block $B$ of $G$. Since $C_{2t+1}$ is vertex transitive, we assume that the maps for all the blocks agree on the cut-vertices (this is analogous to permuting color classes of a coloring); together these maps give a map for $G$. So $G$ is 2-connected. Let $g$ be the odd-girth of $G$. By Corollary 4.21, we also assume that every face of $G$ has length $g$.

Suppose $G$ has an induced path $P$ of length $2k$ (a so-called $(2k-1)$-thread). Form $G'$ from $G$ by deleting the internal vertices of $P$. By minimality, $G'$ has a map $\varphi'$ to $C_{2k+1}$. Let $v$ and $w$ be the endpoints of $P$. To extend $\varphi'$ from $G'$ to $G$, we find a walk of length $2k$ from $\varphi'(v)$ to $\varphi'(w)$ in $C_{2k+1}$. Let $C$ denote a directed $(2k+1)$-cycle with vertices $v_1, \ldots, v_{2k+1}$. For distinct $i, j \in [2k+1]$, the lengths $\ell(v_iCv_j)$ and $\ell(v_jCv_i)$ sum to $2k+1$, and each is at most $2k$. So let $W$ be an even walk in $C$ between $v_i$ and $v_j$, with $\ell(W) \leqslant 2k$. If $\ell(W) < 2k$, then we go back and forth on the final edge until we reach length $2k$. Walk $W$ shows that we can extend $\varphi'$ to $G$, a contradiction. Thus, $G$ has no induced path of length at least $2k$.

Form $G_0$ from $G$ by replacing each maximal thread with a single edge (that is, suppressing each 2-vertex). Note that $\delta(G_0) \geqslant 3$, so $G_0$ has a $5^-$-face $f_0$, by face charging. Let $D$ be the boundary of the face in $G$ corresponding to $f_0$. By Corollary 4.21, $D$ has length $g \geqslant 10k-3$. Since $\ell(f_0) \leqslant 5$, cycle $D$ has an induced path of length $\lceil (10k-3)/5 \rceil = 2k$, a contradiction. $\square$

We should note that Theorem 4.25 fails if we require only that $G$ embeds in $\mathcal{S}$, for any surface $\mathcal{S}$ other than the plane. Youngs [425] and Klavzar and Mohar [261] constructed 4-chromatic graphs with odd-girth arbitrarily large, that embed in the projective plane and torus (respectively). One place where the proof breaks down is Lemma 4.24, since now our two critical cycles for face $f$ need not intersect outside of $f$.

## 4.4 Correspondence Coloring: 3-Choosability

The goal of this section is to prove the following theorem.

**Theorem 4.26.** *If* $G$ *is planar with no cycles of lengths* $4$ *to* $8$*, then* $G$ *is* $3$*-choosable.*

### 4.4.1 Overview and Discharging

Theorem 4.26 is similar to Theorem 1.44; the only difference is that now we allow 9-cycles. So our plan is simple: copy the proof of Theorem 1.44, see what goes wrong, and add technical finesses to overcome the difficulties. There we used face charging and three discharging rules: (R1) Each 3-face takes 1 from each incident vertex; (R2) Each 3-vertex $v$ incident to a 3-face takes $\frac{1}{2}$ from each other face incident to $v$; (R3) Each $10^+$-face $f$ takes $\frac{1}{2}$ from each incident $4^+$-vertex $v$ such that exactly one edge incident to $v$ and $f$ is on a 3-face. We use the same rules now, but substitute $9^+$-face for $10^+$-face in (R3). So what goes wrong?

All vertices end happy, as so do all faces, except for possibly 9-faces. But not all 9-faces are troublesome; only 9-faces with at least 7 incident 3-vertices that are each incident to a 3-face. This motivates the notion of a *tetrad,* 3-vertices $v_1, v_2, v_3, v_4$ that are consecutive along a face, and such that edges $v_1v_2$ and $v_3v_4$ are both in 3-faces; see the right of Figure 4.17. It is easy to check that every 9-face that finishes negative contains the vertices of a tetrad. So it suffices to show that tetrads are reducible for Theorem 4.26.

Fix a graph $G$ satisfying the hypothesis of Theorem 4.26, a 3-assignment $L$, and a tetrad $H$ in $G$. Since we cannot extend an arbitrary $L$-coloring $\varphi'$ of $G-H$ to all of $G$ (why not?), we must somehow constrain $\varphi'$, to make it easier to extend $\varphi'$ to $G$. The key idea is vertex identification. Rather than $L$-coloring $G-H$, we form a graph $G'$ from $G-H$ by identifying a pair of vertices. Since $\varphi'$ gives both vertices the same color, we can more easily extend $\varphi'$ to $G$, much like for Lemma 4.7 in our proof of Grötzsch's Theorem. This approach raises two obvious questions: (a) What list do we assign to this new vertex? (b) When we identify vertices, how do we avoid creating short cycles, so that we can $L$-color the smaller graph by minimality?

To answer (a), we need a truly innovative idea, which occupies much of the proof; we get to this soon. To answer (b), we reuse an idea from our proof of Grötzsch's Theorem. If $G$

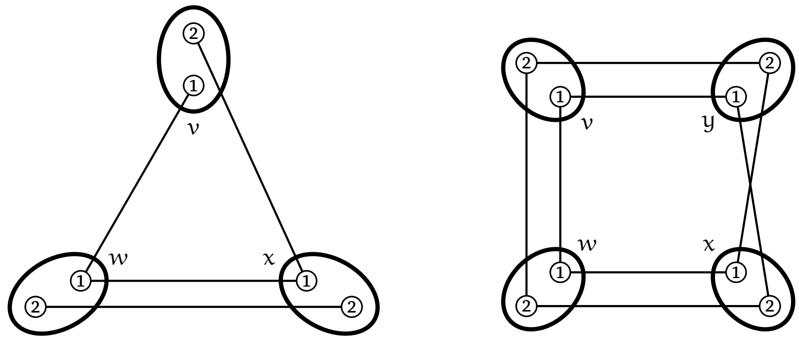



Figure 4.11: Correspondence assignments for $K_3$ and $C_4$. The assignment for $C_4$ illustrates that even cycles have $\chi_{\text{corr}} > 2$.

has a separating $12^-$-cycle D, then we restrict G to the subgraph $D_{\text{in}}$, induced by the vertices of D and its interior. By refining the discharging argument above, we find a tetrad H inside the "innermost" separating $12^-$-cycle. Now the vertices of H do not lie on any separating $12^-$-cycle so identifying vertices to form $G'$ cannot create any cycle of length 4 to 8. Thus, by the minimality of G, we can L-color $G'$.

What we need for (a) is a new type of coloring, *correspondence coloring*. We describe it informally now, and more precisely in Definition 4.27. Every vertex $v$ gets the same list $[k]$. Coloring a vertex $v$ with any color $\alpha \in [k]$ forbids at most one color on each neighbor $w$ of $v$, although the color forbidden by $\alpha$ *may vary from one neighbor to another*. Intuitively, correspondence coloring is "like list-coloring, but also allows vertex identification." Lemma 4.32 justifies this intuition.

correspondence assignment

**Definition 4.27.** A *correspondence assignment* for a graph G consists of a list assignment L and a function C that to every edge $vw \in E(G)$ assigns a partial matching $C_{vw}$ between $\{v\} \times L(v)$ and $\{w\} \times L(w)$. (We use the Cartesian product to distinguish between vertices of $C_{vw}$ when the same color appears in both $L(v)$ and $L(w)$.) See Figure 4.11. An $(L, C)$-*coloring* of G is a function $\varphi$ that assigns to each $v \in V(G)$ a color $\varphi(v) \in L(v)$ such that for every $vw \in E(G)$ the vertices $(v, \varphi(v))$ and $(w, \varphi(w))$ are non-adjacent in $C_{vw}$. Now G is $(L, C)$-*colorable* if such an $(L, C)$-coloring exists.

$(L, C)$-coloring

$(L, C)$-colorable

First, notice that correspondence coloring generalizes list coloring. For each edge $vw$ we simply let $C_{vw}$ match $(v, \alpha)$ and $(w, \alpha)$, for every $\alpha \in L(v) \cap L(w)$. Now an $(L, C)$-coloring is simply an L-coloring. When proving that a configuration is reducible, we want to identify vertices. For this identification to make sense, these vertices must have the same list. This insight motivates our next step.

The actual colors in a list $L(v)$ do not matter at all; we only care how they are matched to the colors in lists for neighbors of $v$. More precisely, suppose that $(L, C)$ is a correspondence assignment for graph G, with $\alpha \in L(v)$ and $\beta \notin L(v)$. Form $(L', C')$ from $(L, C)$ by letting $L'(v) := L(v) \setminus \{\alpha\} \cup \{\beta\}$ and letting $\beta$ replace $\alpha$ in every matching $C_{vw}$, where $w$ is a neighbor of $v$. We call this process *renaming* a color at $v$. Given any $(L', C')$-coloring $\varphi'$, we can get an $(L, C)$-coloring $\varphi$; simply let $\varphi := \varphi'$, except that if $\varphi'(v) := \beta$, then we let $\varphi(v) := \alpha$. Two correspondence assignments are *equivalent* if we can form one from the other by some sequence of renamings. Figure 4.12, center and right, shows an example. The following observation is easy to check by induction on the length of the renaming sequence. renaming equivalent

**Observation 4.28.** If correspondence assignments $(L, C)$ and $(L', C')$ are equivalent, then G is $(L, C)$-colorable if and only if G is $(L', C')$-colorable.

Since we can rename colors at one vertex independent of those at others, by induction on the order of the graph, we get the following.

**Observation 4.29.** If $(L, C)$ is a correspondence assignment for G and $|L(v)| = k$ for all $v \in V(G)$, then G has an equivalent assignment $(L', C')$ with $L'(v) = [k]$ for all $v \in V(G)$.

**Definition 4.30.** A $[k]$-*correspondence assignment* for G is a function C that assigns to each edge $vw \in E(G)$ a partial matching $C_{vw}$ between $\{v\} \times [k]$ and $\{w\} \times [k]$. In this case we write $C$-*coloring* and $C$-*colorable* rather than $(L, C)$-coloring and $(L, C)$-colorable. The *correspondence chromatic number*, denoted $\chi_{\text{corr}}(G)$, of G is the smallest integer k such that G is C-colorable for every $[k]$-correspondence assignment C. [k]-correspondence assignment C-coloring C-colorable correspondence chromatic number

Let $(L, C)$ be a correspondence assignment for a graph G, and let $v_1v_2 \cdots v_t$ with $v_t = v_1$ be a closed walk in G; call the walk $W$. The assignment $(L, C)$ is *inconsistent on* $W$ if there exist colors $\alpha_1, \cdots, \alpha_t$ such that $\alpha_i \in L(v_i)$ for all $i \in [t]$ and $(v_i, \alpha_i)(v_{i+1}, \alpha_{i+1})$ is an edge of $C_{v_iv_{i+1}}$ for all $i \in [t-1]$, but $\alpha_t \neq \alpha_1$. Otherwise $(L, C)$ is *consistent on* $W$. A correspondence assignment is *consistent* if it is consistent on every closed walk in G. On the left in Figure 4.11, the assignment $(L, C)$ is inconsistent on the walk $vwxv$, as shown by the colors 1, 1, 1, 2. But $(L, C)$ is consistent on the walk $wxvw$. inconsistent on $W$ consistent on $W$ consistent

The following easy observation will be useful.

**Observation 4.31.** Let $(L, C)$ and $(L', C')$ be equivalent correspondence assignments for a graph G. For every closed walk $W$ in G, assignment $(L, C)$ is consistent on $W$ if and only if assignment $(L', C')$ is consistent on $W$.

Theorem 4.26 follows from a more general result on correspondence coloring, Theorem 4.33 (with $P = \emptyset$). Its proof comprises Lemmas 4.35–4.38. To emphasize the high-level structure of the argument, we state the theorem and lemmas now, and give the proofs soon.

**Lemma 4.32.** *A graph* G *is* k*-choosable if and only if* G *is* C*-colorable for every consistent* $[k]$*-correspondence assignment* C*.*

**Theorem 4.33.** *Let* G *be a plane graph with no cycles of lengths 4 to 8. Fix* $P \subseteq V(G)$ *such that either (i)* $|P| \leqslant 1$ *or (ii)* P *consists of all vertices incident with some face of* G*. Let* C *be a 3-correspondence assignment for* G *that is consistent on every closed walk of length 3. If* $|P| \leqslant 12$*, then for any* C*-coloring* $\varphi_0$ *of* G[P]*, there is a* C*-coloring* $\varphi$ *of* G *such that* $\varphi$ *restricted to* P *is* $\varphi_0$*.* P $\varphi_0$

*Proof of Theorem 4.26.* Let $G$ satisfy the hypotheses of Theorem 4.26, and let $C$ be a consistent $[3]$-correspondence assignment. By Lemma 4.32 it suffices to show that $G$ is $C$-colorable. This follows from Theorem 4.33, with $P = \emptyset$. $\square$

The point of $P$, short for precolored, in Theorem 4.33 is to handle cut-vertices and short separating cycles. By symmetry, we assume $P$ is on the outer face. If $G$ contains a separating $12^-$-cycle $D$, then we get a coloring $\varphi_{\text{out}}$ of $D_{\text{out}}$, by minimality. Also by minimality, we get a coloring $\varphi_{\text{in}}$ of $D_{\text{in}}$, where now $P_{\text{in}} := V(D)$ and $\varphi_0$ is the restriction of $\varphi_{\text{out}}$ to $V(D)$. Together $\varphi_{\text{out}}$ and $\varphi_{\text{in}}$ give a $C$-coloring of $G$. (To prove that all planar graphs are 5-choosable, we use the same precoloring idea in the proof of Theorem 11.1, when the boundary of the outer face has a chord.) Handling cut-vertices is similar.

To prove Theorem 4.33, we naturally choose $G$ to minimize $|G|$; but we go further. Coloring $G$ is also easier when more of its edges are induced by $P$, since their constraints are already satisfied. Furthermore, we want to maximize $\sum_{vw \in E(G)} |\text{domain}(C_{vw})|$; this is akin to triangulating a plane graph, as in the proof of Theorem 4.12.

**Definition 4.34.** Let $G$, $P$, $C$, and $\varphi_0$ satisfy the hypotheses of Theorem 4.33, but not its conclusion, and let $B := (G, P, C, \varphi_0)$. Without loss of generality, when $P$ is non-empty we assume that its vertices are on the outer face, $f_0$. We choose $B$ to minimize $|G|$ and, subject to that, to minimize $\|G\| - \|G[P]\|$ and, subject to that, to maximize $\sum_{vw \in E(G)} |\text{domain}(C_{vw})|$. We call such a 4-tuple $B$ a *minimal counterexample*.

$B$

minimal counterexample

The next lemma states some properties of a minimal counterexample $B$ that we use to prove Lemmas 4.37 and 4.38. In reality (contrary to our order of presentation), we begin by trying to prove these lemmas and discovering which properties of $B$ are helpful to complete the proofs. But for ease of exposition, we present the properties first.

**Lemma 4.35.** *Every minimal counterexample* $(G, P, C, \varphi_0)$ *to Theorem 4.33 satisfies the following seven properties:*

*(a)* $V(G) \neq P$*,*

*(b)* $G$ *is* 2*-connected,*

*(c)* $G$ *has no separating* $12^-$*-cycle,*

*(d) if* $e_1$ *and* $e_2$ *are distinct chords of a* $12^-$*-cycle* $D$*, then* $e_1$ *and* $e_2$ *are not on a common* 3*-face,*

*(e) all* $2^-$*-vertices of* $G$ *are in* $P$*,*

*(f) the outer face* $f_0$ *is bounded by an induced cycle and* $P = V(f_0)$*, and*

*(g) if* $J$ *is a path in* $G$ *of length* 2 *or* 3 *with both ends in* $P$ *and no internal vertex in* $P$*, then no edge of* $J$ *is in a triangle that intersects* $P$ *in at most one vertex.*

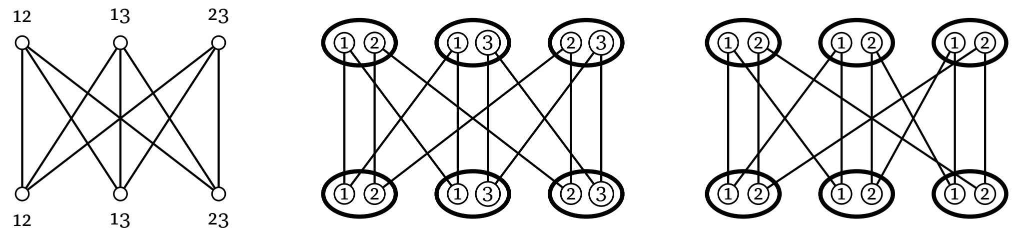



Figure 4.12: From left to right: A 2-list assignment L for $K_{3,3}$, the correspondence assignment that arises naturally from L, and an equivalent 2-correspondence assignment.

Now we define our primary reducible configuration.

**Definition 4.36.** In a minimal counterexample B, Lemma 4.35(c) implies every 3-cycle is a 3-face; we call this a *triangle*. A *tetrad* consists of 3-vertices $v_1, v_2, v_3, v_4$ that are consecutive along a face, with edges $v_1v_2$ and $v_3v_4$ both in triangles; see the right of Figure 4.17. triangle tetrad

**Lemma 4.37.** *If* $(G, P, C, \varphi_0)$ *satisfies (a)–(g) of Lemma 4.35, then* G *contains a tetrad with no vertex on the outer face.*

**Lemma 4.38.** *In a minimal counterexample to Theorem 4.33, every tetrad has a vertex on the outer face.*

Now we give the proofs. Proving Theorem 4.33 is easy, once we have the lemmas. Lemma 4.35 uses standard reducibility arguments. And Lemma 4.37 is also straightforward, though a bit tedious. For each tricky part of the proof, some property in Lemma 4.35 gives just what is needed. Most of our work goes into proving Lemma 4.38, that every tetrad disjoint from the outer face is reducible. For this we must better understand properties of correspondence coloring, which we study in the next section.

*Proof of Theorem 4.33.* Suppose the theorem is false, and $(G, P, C, \varphi_0)$ is a minimal counterexample. By Lemmas 4.35 and 4.37, G contains a tetrad with no vertex on the outer face, contradicting Lemma 4.38. Thus no counterexample exists, and Theorem 4.33 is true. □

*Proof of Lemma 4.32.* Suppose G is C-colorable for every consistent k-correspondence assignment C. (Figure 4.12 transforms a 2-list assignment for $K_{3,3}$ into a consistent [2]-correspondence assignment C. Since $K_{3,3}$ has no L-coloring, it also has no C-coloring.) Let L be a list assignment with $|L(v)| = k$ for all vertices $v$. For each edge $vw$ and each $\alpha \in L(v) \cap L(w)$, let $C_{vw}$ match $(v, \alpha)$ to $(w, \alpha)$. Clearly, C is consistent. By Observation 4.29, the correspondence assignment $(L, C)$ is equivalent to a [k]-correspondence assignment $C'$. By Observation 4.31, the assignment $C'$ is consistent. So, by hypothesis, G has a $C'$-coloring, $\varphi'$. Finally, by Observation 4.28, coloring $\varphi'$ implies that G has an $(L, C)$-coloring $\varphi$. Thus, G is k-choosable.

Now suppose that G is k-choosable, and let C be a consistent k-correspondence assignment. (Essentially we reverse the transformation in Figure 4.12.) Let H be the graph with vertex set

$V(G) \times [k]$ and edge set $\cup_{e \in E(G)} C_e$. Since C is consistent, for each $v \in V(G)$ every component of H intersects $\{v\} \times [k]$ in at most one vertex. Number the components of H arbitrarily, and let $L(v)$ the numbers of all components of H that intersect $\{v\} \times [k]$. Clearly $|L(v)| = k$ for all $v \in V(G)$. Since G is k-choosable, G has an L-coloring $\varphi$. To convert $\varphi$ to a C-coloring, we color $v$ with the unique $\alpha \in [k]$ such that $(v, \alpha)$ is in the component of H numbered $\varphi(v)$. □

*Proof of Lemma 4.35.* We prove each property in turn. Note that (a) is trivial, since if $V(G) = P$, then $\varphi_0$ is the desired C-coloring, a contradiction.

Consider (b). If G is disconnected, then by minimality each component $G_i$ of G has a C-coloring $\varphi_i$ extending the restriction of $\varphi_0$ to $G_i$; the union of these $\varphi_i$ is a C-coloring of G. So suppose that G has a cut-vertex $v$. Let H be one component of $G - v$, and let $G_1 := G - H$ and $G_2 := G[V(H) + v]$. First suppose that $v \in P$. For each $G_i$, by minimality we have a C-coloring $\varphi_i$ extending the restriction of $\varphi_0$ to $G_i$. Together these $\varphi_i$ give a C-coloring of G extending $\varphi_0$. Assume instead that $v \notin P$. By symmetry, assume that $P \cap V(G_2) = \emptyset$. By minimality, $G_1$ has a C-coloring $\varphi_1$. Also $G_2$ has a C-coloring $\varphi_2$ that agrees with $\varphi_1$ on $v$. Together $\varphi_1$ and $\varphi_2$ give a C-coloring for G extending $\varphi_0$, a contradiction. This proves (b).

Consider (c). Suppose that G has a $12^-$-cycle D with vertices both inside and outside of D. Denote by $D_{\text{out}}$ (resp. $D_{\text{in}}$) the subgraph induced by $V(D)$ and the vertices outside (resp. inside) of D. By minimality $D_{\text{out}}$ has a C-coloring $\varphi_{\text{out}}$ extending $\varphi_0$. Also $D_{\text{in}}$ has a C-coloring $\varphi_{\text{in}}$ that extends the restriction of $\varphi_{\text{out}}$ to $V(D)$, again by minimality. Since $D_{\text{in}}$ and $D_{\text{out}}$ agree on $V(D)$, together they give a C-coloring of G extending $\varphi_0$, a contradiction. This proves (c).

Consider (d). Suppose that D is a $12^-$-cycle $v_1 \cdots v_t$, and that $e_1$ and $e_2$ are chords of D in a triangle. By symmetry, assume that $e_1 = v_1v_i$ and $e_2 = v_1v_{i+1}$, for some $i$ such that $3 \leqslant i \leqslant 6$. To avoid a cycle $v_1v_2 \ldots v_i$ of length 4 to 8, we must have $i = 3$. But now $v_1v_2v_3v_4$ is a 4-cycle, using $e_2$ as its final edge, which is a contradiction. This proves (d).

Consider (e). Suppose that G has a $2^-$-vertex $v$ not in P. By minimality, $G - v$ has a C-coloring $\varphi$ extending $\varphi_0$. Since $d(v) \leqslant 2$, we can choose a color for $v$ that is not forbidden by any color used on its neighbors in $\varphi$, which is a contradiction. This proves (e).

Consider (f). We first show that $P = V(f_0)$. By assumption in Definition 4.36, $P \subseteq V(f_0)$, so assume that $|P| \leqslant 1$. If $P = \emptyset$, then we add to P an arbitrary vertex on $f_0$; so assume $|P| = 1$. First suppose that $v$ is contained in a $12^-$-cycle D. Now (c) implies that D is a face boundary.

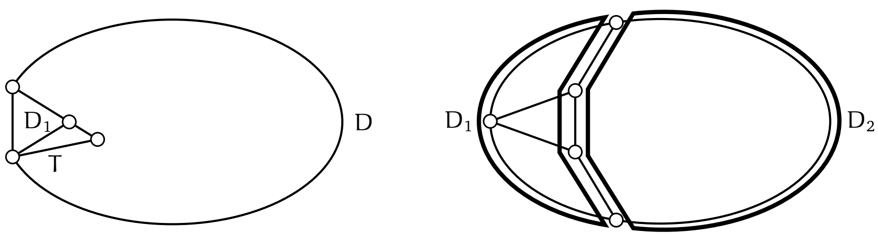



Figure 4.13: Two cases from the proof of (g) in Lemma 4.35.

So we redraw $G$ with $D$ as the outer face, let $P := V(D)$, and choose $\varphi_0$ to be an arbitrary $C$-coloring of $P$ ($\varphi_0$ exists by minimality when $V(D) \neq V(G)$; and if $V(D) = V(G)$, then we contradict (e), since $V(D)$ has at least two 2-vertices). Our ability to decrease $\|G\| - \|G[P]\|$ contradicts the minimality of $B$, so $v$ cannot be in a $12^-$-cycle. Now redraw $G$ with $v$ on the outer face $f_0$, and let $w$ and $x$ be its neighbors on $f_0$. Let $G' := G + wx$, drawn so that $vwx$ is the outer face. Let $P' := \{v, w, x\}$ and $\varphi_0'$ be an arbitrary $C$-coloring of $v, w, x$. By minimality $(G', C', P', \varphi_0')$ has a $C'$-coloring $\varphi'$, where $C'$ is formed from $C$ by letting $C_{vx}$ be null. But $\varphi'$ is a $C$-coloring of $(G, C, P, \varphi_0)$, which is a contradiction. This proves (f).

Consider (g). Let $J$ be a path in $G$ of length 2 or 3, with both ends in $P$ and no internal vertex in $P$. Suppose that an edge of $J$ is contained in a triangle $T$ that intersects $P$ in at most one vertex. Let $D$ be the boundary of $f_0$, and let $D_1$ and $D_2$ be the two cycles in $D \cup J$ distinct from $D$; see the left of Figure 4.13. If one of them, say $D_1$, is a 3-cycle, then the symmetric difference of $D_1$ and $E(T)$ is a 4-cycle, which is a contradiction. So neither $D_1$ nor $D_2$ is a 3-cycle. Since $G$ has no cycles of lengths 4 to 8, we have $|D_1| \geqslant 9$ and $|D_2| \geqslant 9$. Also, $|D_1| + |D_2| = |D| + 2\|J\| \leqslant 12 + 2(3)$. So $J$ has length 3 and $D_1$ and $D_2$ are both 9-cycles; see the right of Figure 4.13. But now one $D_i$ has an edge of $T \setminus E(J)$ as a chord, giving a cycle of length 4 to 8, a contradiction. This proves (g). □

To prove Lemma 4.37, we generally follow the outline sketched at the start of this section. Recall that a face ends negative only if it contains the vertices of a tetrad. The main difference from our sketch is in how we give charge to vertices on the outer face, $f_0$. In standard face charging, the initial charges sum to $-12$. Here we give $f_0$ an extra 12 to ensure that if a face ends negative, then no vertices in its tetrad lie on $f_0$.

Since the sum of the initial charges is 0, we assume that each tetrad in $G$ contains a vertex of $P$, and reach a contradiction by showing that $f_0$ ends positive and all other vertices and faces end happy. This proves the lemma. Most of this analysis is straightforward. The hardest case is a 9-face with an incident 2-vertex; this is the only place in the proof that we need Lemma 4.35(c,d).

*Proof of Lemma 4.37.* Recall that $P = V(f_0)$, by Lemma 4.35(f). Each vertex $v$ starts with $2d(v) - 6$, each face $f$ (other than $f_0$) with $\ell(f) - 6$, and $f_0$ with $\ell(f_0) + 6$. These charges sum to 0, by Euler's formula. We use these 5 discharging rules, shown in Figure 4.14:

(R1) Each 3-face, other than $f_0$, takes 1 from each incident vertex.

(R2) Each 3-vertex $v \in P$ takes 1 from $f_0$.

(R3) Each 3-vertex $v \notin P$ that is incident to a 3-face takes $\frac{1}{2}$ from each other incident face.

(R4) Each $9^+$-face $f$ takes $\frac{1}{2}$ from each incident $4^+$-vertex $v$ such that exactly one edge incident to $v$ and $f$ is on a 3-face.

(R5) Each 2-vertex takes $\frac{3}{2}$ from $f_0$ and takes $\frac{1}{2}$ from its other incident face.

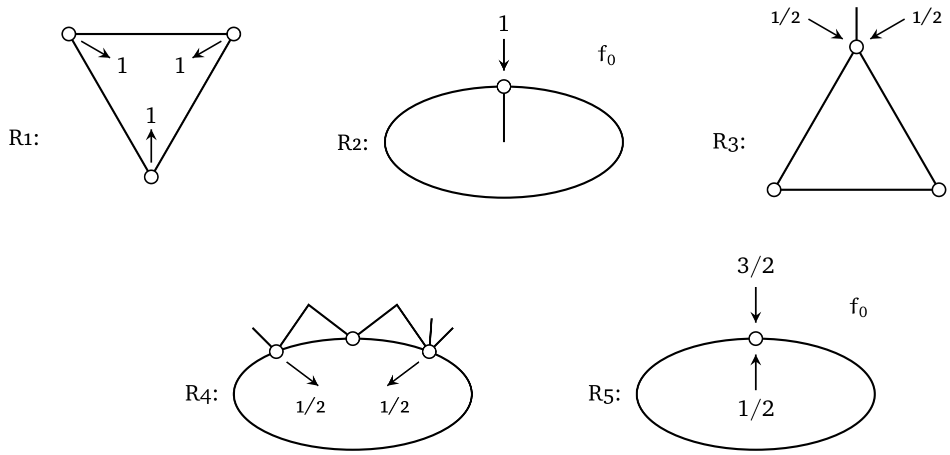



Figure 4.14: The 5 discharging rules in the proof of Lemma 4.37.

$s, t$ For an arbitrary vertex $v$, let $s$ and $t$ denote (respectively) its degree and number of incident triangles. Since $G$ has no 4-cycles, $t \leqslant \frac{1}{2}s$. Now $v$ gives at most $t$ by (R1) and at most $\frac{1}{2}(s-t)$ by (R4), for a total of $\frac{1}{2}(s+t)$, which is at most $\frac{3}{4}s$. Thus $v$ ends with at least $2s-6-\frac{3}{4}s = \frac{5}{4}s-6$, which is positive when $s \geqslant 5$. If $s \in \{3,4\}$ and $t = 0$, then $v$ ends with $2s-6$, and is happy. If $s = 4$ and $t = 1$, then $v$ ends with at least $2(4)-6-1-2(\frac{1}{2}) = 0$, by (R1) and (R4). If $s = 4$ and $t = 2$, then $v$ ends with $2(4)-6-2(1) = 0$, by (R1). Suppose $s = 3$ and $t = 1$. If $v \notin P$, then $v$ ends with $2(3)-6-1+2(\frac{1}{2}) = 0$, by (R1) and (R3). If $v \in P$, then $v$ ends with $2(3)-6-1+1 = 0$, by (R1) and (R2). Finally, suppose $s \leqslant 2$. By hypothesis, $\delta(G) \geqslant 2$ and every 2-vertex $v$ is on $f_0$. Since $G[P]$ is a chordless cycle, by Lemma 4.35(f), no 2-vertex lies on a 3-face. So $v$ ends with $2(2)-6+\frac{1}{2}+\frac{3}{2} = 0$, by (R5). Thus, every vertex ends happy.

Now we consider faces; we start with $f_0$. Lemma 4.35(a,f) gives $P = V(f_0)$ and $P \neq V(G)$. Since $G$ is 2-connected, at least two vertices on $f_0$ are $3^+$-vertices. Thus, $f_0$ ends with at least $|P|+6-\frac{3}{2}(|P|-2)-2(1) = 7-\frac{1}{2}|P|$; this is positive, since $|P| \leqslant 12$. So now we consider a face $f$ other than $f_0$; let $s := \ell(f)$. If $s = 3$, then $f$ ends with $3-6+3(1) = 0$, by (R1). Otherwise $f$ ends with at least $s-6-\frac{1}{2}s$, which is nonnegative when $s \geqslant 12$. So we must consider 9-faces,
$f$ 10-faces, and 11-faces. Let $f$ be such a face.

$V_3'$ Let $V_3'$ denote the set of 3-vertices on $f$ that are not in $P$ and that are incident to 3-faces. Let
$n_2, n_3'$ $n_2$ denote the number of 2-vertices on $f$, and let $n_3' = |V_3'|$. Note that $n_2 + n_3' \leqslant |f|$. Further, if $n_2 > 0$, then $n_2 + n_3' \leqslant |f|-2$, since $G$ is 2-connected, so the two $3^+$-vertices nearby a 2-vertex on $f$ must both be in $P$. Thus, if $n_2 > 0$, then $f$ ends with at least $s-6-\frac{1}{2}(s-2) = \frac{1}{2}s-5$, which is enough when $s \geqslant 10$. Suppose that $n_2 = 0$. If $s = 11$, then $n_3' \leqslant 10$ (by parity), so $f$ ends happy. If $s = 10$ and $n_3' \leqslant 8$, then again $f$ ends happy. If instead $s = 10$ and $n_3' \geqslant 9$, then $f$ contains a tetrad with no vertex on $f_0$. The analysis of 9-faces is more detailed.

Suppose that $s = 9$ and $n_2 = 0$. If $n_3' \leqslant 6$, then $f$ ends happy, so assume $n_3' \geqslant 7$.
$v_1, \ldots, v_9$ Let $v_1, \ldots, v_9$ denote the vertices of $f$, in order. See the left of Figure 4.15. By assump-

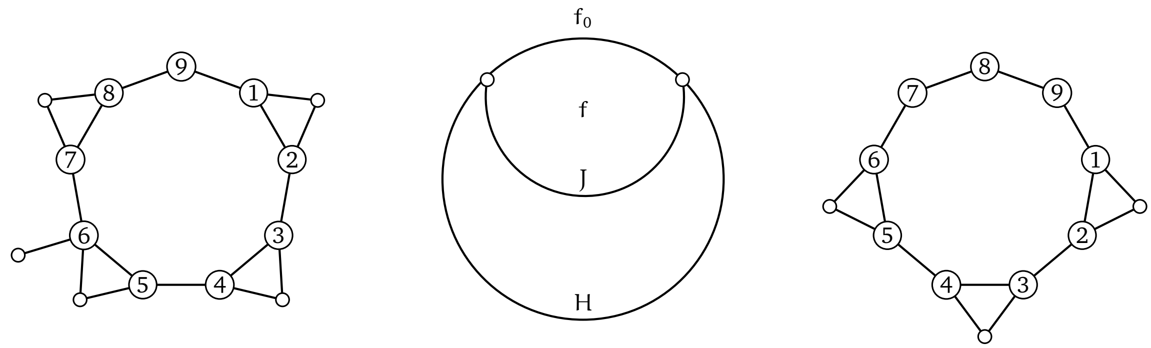



Figure 4.15: Instances from the final three paragraphs of the proof of Lemma 4.32.

tion, no tetrad has all its vertices in $V_3'$. So by symmetry we assume that $v_1, v_6 \notin V_3'$ and $v_2, v_3, v_4, v_5, v_7, v_8, v_9 \in V_3'$. Further, we must have each of edges $v_1v_2$, $v_3v_4$, $v_5v_6$ in a triangle. By symmetry between $v_7$ and $v_9$, we also assume that $v_7v_8$ is in a triangle. Since $v_5, v_7 \notin P$, if $d(v_6) = 3$ then $v_6 \notin P$, which implies that $v_6 \in V_3'$, a contradiction. So $d(v_6) \geqslant 4$. Since $d(v_7) = 3$ and $v_7v_8$ is in a triangle, $v_6v_7$ is not in a triangle. Thus, f takes $\frac{1}{2}$ from $v_6$, by (R4). Hence f ends with at least $9 - 6 - 7(\frac{1}{2}) + \frac{1}{2} = 0$.

Finally, suppose that $s = 9$ and $n_2 > 0$. If $n_2 + n_3' \leqslant 6$, then f ends happy, so assume $n_2 + n_3' \geqslant 7$. Since G is 2-connected, at least two $3^+$-vertices of f are not in $V_3'$. So $n_2 + n_3' = 7$. Further, the 2-vertices of f form a subpath of its boundary. See the center of Figure 4.15. If $n_2 = 7$, then $V(f) \subseteq V(f_0)$. Either $f_0$ has a chord or $V(f) = V(f_0) = P$, so $V(G) = P$; this contradicts Lemma 4.35(a,f). Thus, $n_2 < 7$ and $V_3' \neq \emptyset$. Let J be the path formed from f by deleting all its 2-vertices; let H be the cycle in $f \cup f_0$ other than f and $f_0$. Now all internal H vertices of J are in $V_3'$, since $n_2 + n_3' = 7 = |f| - 2$.

Suppose $n_3' \leqslant 3$. Now $|H| = |f_0| - n_2 + n_3' < |f_0|$. By Lemma 4.35(f), the outer face $f_0$ is a $12^-$-cycle, so H is also. By Lemma 4.35(c), G has no separating $12^-$-cycle, so $V(f_0) \cup V_3' = V(G)$. Since $|f_0| \leqslant 12$, and the vertices of $V_3'$ are 3-vertices, G contains a cycle of length between 4 and 8, a contradiction (we can verify this by a short case analysis based on the value of $n_3'$; it is helpful to note that $|f_0| \geqslant 4$, which implies $|f_0| \geqslant 9$). So instead assume $n_3' \geqslant 4$. Conversely, since G has no tetrad with all its vertices in $V_3'$, and $V_3'$ induces a path, $n_3' \leqslant 4$. Hence $n_3' = 4$. See the right of Figure 4.15.

Let $v_1, \ldots, v_9$ denote the vertices of f, in order, with $V_3' = \{v_2, v_3, v_4, v_5\}$. Since no tetrad has all its vertices in $V_3'$, edges $v_1v_2$, $v_3v_4$, and $v_5v_6$ must each be in a triangle; denote the first and last of these triangles by $v_1v_2w_1$ and $v_5v_6w_6$. If $v_1$ or $v_6$ is a $4^+$-vertex, then it sends $\frac{1}{2}$ to f by (R4), so f ends happy. Thus, $v_1$ and $v_6$ are 3-vertices, which implies that $w_1, w_6 \in P$. Recall that G has no separating $12^-$-cycle. By applying this fact to the cycles in $E(f_0) \cup E(w_1v_2v_3v_4v_5w_6)$, we conclude that $V(f_0) \cup V_3' = V(G)$. But now the edges of the triangle containing $v_3$ and $v_4$ (other than $v_3v_4$) are chords of a $12^-$-cycle, which contradicts Lemma 4.35(d). This completes the proof. □

### 4.4.2 Reducing Tetrads: Properties of Correspondence Coloring

To prove Lemma 4.38, we need more lemmas to shrink the number of cases we must consider.

equivalent **Definition 4.39.** Two $[k]$-correspondence assignments $C$ and $C'$ are *equivalent* if there exists a permutation $\pi_v : [k] \to [k]$ for each $v \in V(G)$ such that for every edge $vw$ and each $\alpha \in [k]$, we have $\pi_w(C_{vw}(\alpha)) = C'_{vw}(\pi_v(\alpha)))$. In other words, we can follow the matching $C_{vw}$ and afterwards permute the result, or we can first permute the colors at $v$ and afterwards follow the matching $C'_{vw}$; both choices give the same outcome.

fixed straight straighten full A vertex $v$ is *fixed* in this equivalence if $\pi_v$ is the identity. An edge $vw$ is *straight* in a k-correspondence assignment $C$ if $C_{vw}(\alpha) = \alpha$ for every $\alpha \in \text{domain}(C_{vw})$. Given a $[k]$-correspondence assignment $C$, a vertex $v$, and edge $vw$, to *straighten* edge $vw$, we form an equivalent correspondence assignment $C'$ by taking $\pi_x = \text{id}$ (the identity map) for all $x \in V - w$ and taking $\pi_w$ such that $\pi_w(C_{vw}) = \text{id}$. An edge $vw$ is *full* if $\text{domain}(C_{vw}) = [k]$.

**Lemma 4.40 (Equivalence Lemma).** *Let $C$ and $C'$ be equivalent $k$-correspondence assignments for a graph $G$. If $\varphi$ is a $C$-coloring of $G$, then there exists a $C'$-coloring $\varphi'$ of $G$ such that $\varphi'(v) = \varphi(v)$ for every fixed vertex $v$. In particular, $G$ is $C'$-colorable if and only if $G$ is $C$-colorable; here the second equality uses the definition of equivalent.*

*Proof.* Let $\pi_v$, for all $v \in V(G)$, be the permuations showing that $C$ and $C'$ are equivalent. So $\pi_w(C_{vw}(\alpha)) = C'_{vw}(\pi_v(\alpha)))$ for every $vw \in E(G)$ and $\alpha \in [k]$. For a $C$-coloring $\varphi$ of $G$, the function $\varphi'$ given by letting $\varphi'(v) := \pi_v(\varphi(v))$ is a $C'$-coloring of $G$ that matches $\varphi$ on every fixed vertex $v$. Since $\varphi(w) \neq C_{vw}(\varphi(v))$, we get $C'_{vw}(\varphi'(v)) = C'_{vw}(\pi_v(\varphi(v)) = \pi_w(C_{vw}(\varphi(v))) \neq \pi_w(\varphi_w)) = \varphi'(w)$. □

Fix a graph $G$ and a $k$-correspondence assignment $C$. The following lemma allows us to "straighten" edges of a subgraph $H$, as long as $C$ is consistent on $H$, and all edges of every cycle in $H$ are full. If we drop the hypothesis that the edges of every cycle in $H$ are full, then the lemma becomes false. We leave the details to Exercise 4.

**Lemma 4.41 (Straightening Lemma).** *Let $G$ be a graph with a $[k]$-correspondence assignment $C$. Let $H$ be a subgraph of $G$ such that for every cycle $D$ in $H$ the assignment $C$ is consistent on $D$, and all edges of $D$ are full. Now there exists a $k$-correspondence assignment $C'$ equivalent to $C$ with all edges of $H$ straight in $C'$, and all vertices not in $H$ fixed.*

In each component of $H$ we straighten edges one by one, so that the subgraph induced by straightened edges is always connected.

*Proof.* For each component of $H$ we only need to straighten the edges of a spanning tree $T$ of $H$. We order the edges of $T$ so that for each $i \in [\|T\|]$ the subgraph induced by the first $i$ edges is connected. On step $i$, we straighten edge $i$, say it is $vw$, where $v$ is incident to an already straightened edge and $w$ is not. (For edge 1, we pick the fixed endpoint, $v$, arbitrarily.) We keep each vertex fixed except for $w$. Given $\pi_v$, we choose $\pi_w$ so that $vw$ becomes straight. If,

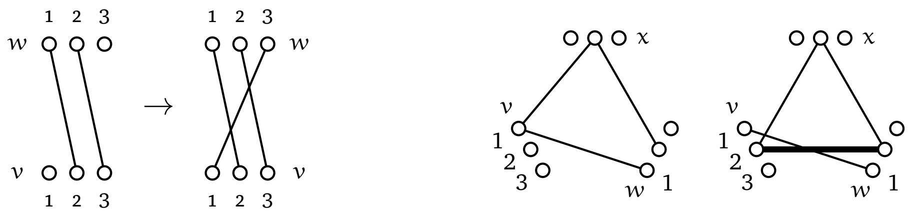



Figure 4.16: The proof of Lemma 4.42. Left: Adding an edge to the matching $C_{vw}$, when $vw$ is not on a triangle. Near Right: Finding an inconsistent walk of length 3, when $2 \notin \text{domain}(C_{vx})$. Far Right: Adding an edge (shown in bold) to the matching $C_{vw}$ without creating an inconsistent walk of length 3, when $2 \in \text{domain}(C_{vx})$.

after straightening all edge of T, some edge $vw$ in H remains unstraightened, then we combine $vw$ with a $v, w$-path in T to get a closed walk for which C is inconsistent. Such an inconsistent walk contradicts the hypothesis, which proves the lemma. □

**Lemma 4.42.** *Let* $(G, P, C, \varphi_0)$ *be a minimal counterexample. If* $vw$ *is an edge of* G *that does not join two vertices of* P*, then* $|domain(C_{vw})| \geqslant 2$*. If* $vw$ *is also not in a triangle, then* $vw$ *is full, i.e.,* $|domain(C_{vw})| = 3$.

*Proof.* Suppose instead that $vw$ is not full and not in any triangle. Choose $\alpha, \beta \in \{1, 2, 3\}$ such that $\alpha \notin \text{domain}(C_{vw})$ and $\beta \notin \text{domain}(C_{wv})$. Form $C'$ from C by adding the correspondence $C_{vw}(\alpha) = \beta$; since $vw$ is not in any triangle, $C'$ is consistent on all closed walks of length 3. Thus $C'$ contradicts the minimality of $(G, P, C, \varphi_0)$. This proves the second statement.

Now we consider the first statement. Suppose that $vw$ is in a triangle $vwx$ (note that $x$ is unique, since G has no 4-cycles). If $|\text{domain}(C_{vw})| = 0$, then we delete $vw$, which contradicts minimality. So instead assume that $|\text{domain}(C_{vw})| = 1$. By the Straightening and Equivalence Lemmas, we assume that $C_{vw}(1) = 1$. We will show that we can add a correspondence on edge $vw$, and thus contradict minimality. For all $\alpha, \beta \in \{2, 3\}$, form $C^{\alpha,\beta}$ from C by adding the $C^{\alpha,\beta}$
correspondence $C_{vw}(\alpha) = \beta$. Clearly any $C^{\alpha,\beta}$-coloring is a C-coloring; so no $C^{\alpha,\beta}$-coloring exists. Thus, by minimality, each correspondence assignment $C^{\alpha,\beta}$ must be inconsistent on some closed walk around the 3-cycle $vwx$.

For a walk $W$ longer than a single edge, let $e$ denote its first edge and let $W'$ denote the rest. Now let $C_W() := C_{W'}(C_e())$. This is the iteratively composed correspondence along the walk, from one endpoint to the other.

Suppose $2 \notin \text{domain}(C_{vx})$. Now for each $\beta \in \{2, 3\}$, assignment $C^{2,\beta}$ must be inconsistent on $vwxv$; so $2 \in \text{domain}(C^{2,\beta}_{vwxv})$ and $\beta \in \text{domain}(C_{wxv})$. Furthermore, $C_{wxv}(2) \neq 2$ and $C_{wxv}(3) \neq 2$, so either $C_{wxv}(2) = 1$ or $C_{wxv}(3) = 1$, which implies $C_{vxw}(1) \in \{2, 3\}$. Recall that $C_{vw}(1) = 1$. Now $C_{wvxw}(1) = C_{vxw}(1) \in \{2, 3\}$; thus C is inconsistent, a contradiction.

So assume $2 \in \text{domain}(C_{vx})$; by symmetry between 2 and 3 (and $v$ and $w$) we know $\{2, 3\} \subseteq \text{domain}(C_{vx}) \cap \text{domain}(C_{wx})$. By Pigeonhole, there is $\beta \in \{2, 3\}$ with $\beta \in \text{domain}(C_{vxw})$ and $C_{vxw}(\beta) \in \{2, 3\}$. So $C^{\beta, C_{vxw}(\beta)}$ is consistent on triangle $vwx$, a contradiction. □

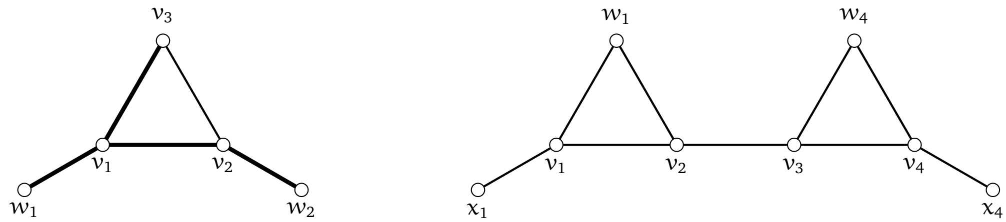



Figure 4.17: Left: A triangle with two 3-vertices not in P, in the proof of Lemma 4.43. Right: A tetrad with none of its vertices in P, in the proof of Lemma 4.38.

To prove that tetrads are reducible, we focus on triangles with two incident 3-vertices not in $P$. For these we can further strengthen Lemma 4.42.

**Lemma 4.43.** *Let $(G, P, C, \varphi_0)$ be a minimal counterexample. If $T$ is a triangle $v_1v_2v_3$ with at least two $v_i$ that are 3-vertices not in $P$, then all edges of $T$ are full in $C$.*

*Proof.* Assume the lemma is false. Let $v_1$ and $v_2$ be 3-vertices not in $P$, and let $w_1$ and $w_2$ be their neighbors outside of $T$, as on the left in Figure 4.17. By the Straightening and Equivalence Lemmas, we assume that all five edges induced by $\{v_1, v_2, v_3, w_1, w_2\}$ are straight (though perhaps not full), except for possibly $v_2v_3$. Lemma 4.42 implies that there exists $\alpha \in \text{domain}(C_{v_1v_2}) \cap \text{domain}(C_{v_1v_3})$; by symmetry, we assume that $\alpha = 1$. Since $v_1v_2$ and $v_1v_3$ are straight, $C_{v_1v_2}(1) = C_{v_1v_3}(1) = 1$. Since $C$ is consistent on $T$, either $C_{v_2v_3}(1) = 1$ or else $1 \notin \text{domain}(C_{v_2v_3})$ and $1 \notin \text{domain}(C_{v_3v_2})$. In the latter case, we form $C'$ from $C$ by adding the correspondence $C'_{v_2v_3}(1) = 1$; this contradicts the minimality of $(G, P, C, \varphi_0)$. So assume we are in the former case.

Form $C'$ from $C$ by changing the correspondence on $E(T)$ to be straight and full. By minimality, there exists a $C'$-coloring $\varphi'$ of $G$; but $\varphi'$ must not be a $C$-coloring. Since all edges of $T$ other than $v_2v_3$ are straight in $C$, by symmetry between colors 2 and 3 we assume that $\varphi'(v_2) = 2$, $\varphi'(v_3) = 3$, and $C_{v_2v_3}(2) = 3$. If $3 \notin \text{domain}(C_{v_3v_1})$, then we modify $\varphi'$ by uncoloring $v_1$ and $v_2$, and greedily coloring $v_2$ followed by $v_1$. So we assume that $C_{v_3v_1}(3) = 3$. By Lemma 4.42, $|\text{domain}(C_{v_1v_2})| \geqslant 2$; since $v_1v_2$ is straight, either $C_{v_1v_2}(2) = 2$ or $C_{v_1v_2}(3) = 3$. But now $C_{v_1v_2v_3v_1}(2) = 3$ or $C_{v_2v_3v_1v_2}(2) = 3$; in each case $C$ is inconsistent, which is a contradiction. □

Now we can prove Lemma 4.38, which completes the proof of Theorem 4.33.

**Lemma 4.38.** *In a minimal counterexample to Theorem 4.33, every tetrad has a vertex on the outer face.*

*Proof.* Let $(G, P, C, \varphi_0)$ be a minimal counterexample. Assume instead that $v_1v_2v_3v_4$ is a tetrad disjoint from the outer face $f_0$; recall from Lemma 4.35(f) that $P = V(f_0)$. Let $v_1v_2w_1$ and $v_3v_4w_4$ be the 3-cycles of the tetrad, and let $x_1$ and $x_4$ be the other neighbors of $v_1$ and $v_4$; see Figure 4.17 (right). We will (1) form $G'$ from $G - \{v_1, v_2, v_3, v_4\}$ by identifying $w_1$ and $x_4$, (2) color $G'$ by minimality, and (3) extend the coloring to $G$. To color $G'$ by minimality, we must show that it contains no cycles of lengths 4 to 8; we must also ensure that the restriction of $\varphi_0$ to $G'$ is a proper coloring of the vertices of its outer face. The first criteria is satisfied since $G$ has no short separating cycles (as we will show) and the tetrad is disjoint from $P$. For the second, it suffices that either (i) $w_1, x_4 \notin P$ or (ii) $w_1 \notin P$ and also $w_1$ has no neighbors in $P$. We now show that either (i) or (ii) holds.

Suppose that $w_1, w_4 \notin P$. By symmetry, if $x_1 \notin P$, then we are done; so assume $x_1 \in P$. For every neighbor $y$ of $w_1$, applying Lemma 4.35(g) to path $x_1 v_1 w_1 y$ shows that $y \notin P$; so (ii) holds, and we are done. Thus, $w_1 \in P$ or $w_4 \in P$ (or both). But applying Lemma 4.35(g) to $w_1 v_2 v_3 w_4$ shows that either $w_1 \notin P$ or $w_4 \notin P$; by symmetry, assume that $w_1 \notin P$ and $w_4 \in P$. Applying Lemma 4.35(g) to $w_4 v_4 x_4$ shows that $x_4 \notin P$. Now (i) holds, so we are done.

Since either (i) or (ii) holds, when we form $G'$ from $G$, we do not create any new edges between vertices of $P$. We must also check that we do not create short cycles. Now $G - \{v_1, v_2, v_3, v_4\}$ cannot contain any path of length from 4 to 8 between $w_1$ and $x_4$. If it did, then, together with $w_1 v_2 v_3 v_4 x_4$, such a path would form a cycle $D$ of length at most 12 in $G$, where $D$ encloses two edges of either triangle $v_1 v_2 w_1$ or triangle $v_3 v_4 w_4$. If neither $v_1$ nor $w_4$ lies on $D$, then $D$ is separating, which contradicts Lemma 4.35(c); otherwise we contradict Lemma 4.35(d). Thus, $(G', P, C, \varphi_0)$ is smaller than our minimal counterexample, so it admits a coloring $\varphi'$, which induces a $C$-coloring of $G - \{v_1, v_2, v_3, v_4\}$. Lemmas 4.42 and 4.43 imply that all edges in $G$ incident to $\{v_1, v_2, v_3, v_4\}$ are full; and the Straightening and Equivalence Lemmas allow us to assume that under $C$ these edges are also straight. To extend $\varphi'$ to $G$, we first greedily color $v_1$ and $v_2$. Since $v_2$ and $x_4$ use distinct colors, we can further extend the coloring to $v_3$ and $v_4$, which is a contradiction. □

## Notes

The 5 Color Theorem was proved in 1890 by Heawood [212], using Kempe swaps, which we present in Chapter 3. Our proof here follows Kainen [235].

In 1852 Francis Guthrie, a mapmaker, asked his brother Frederick, a mathematician, whether every map was 4-colorable. This problem appeared in the *Athenæum* in 1854 [296], and Cayley announced it to the London Mathematical Society in 1878. Proofs were published by Kempe [243], in 1879, and Tait [370], in 1880, but both were later shown to be erroneous. The first correct proof was due to Appel and Haken (working with Koch) [21, 24], in 1977. Their proof was revolutionary in its use of computers for extensive case-checking. But as a result, many doubted its validity [22, 23].

In 1993 Robertson, Sanders, Seymour, and Thomas [344] reproved the 4 Color Theorem, following the same paradigm of reducibility and unavoidability. Their proof was simpler and written to encourage external verification, and it is now widely accepted. Thomas [373] provides an accessible introduction to the problem. In 2005 Gonthier [180] encoded the 1993 proof to be checked by a formal proof checker. A history of the problem is discussed in

Wilson [415]. For more on the technical aspects, we recommend the treatments of Thomas [373] and Steinberger [366].

Grötzsch's Theorem was first proved in 1959 [185]. It was generalized by Aksenov [6] and Grünbaum [186], as follows: Every planar graph with at most three triangles is 3-colorable. This is best possible, due to $K_4$. In fact, there are infinitely many planar 4-critical graphs with exactly 4 triangles. These were characterized by Borodin, Dvořák, Kostochka, Lidický, and Yancey [55]. In Chapter 12, we discuss this result of Aksenov and Grünbaum, as well as other extensions of Grötzsch's Theorem.

Our proof of Grötzsch's Theorem follows Dvořák, Král', and Thomas [133], who gave the first algorithm to color triangle-free planar graphs in linear time. Their proof was based on a result of Thomassen [377]: Every planar graph with girth at least 5 is 3-choosable (we prove this result in Section 11.4). As in the proof of Theorem 4.26, he proves a stronger result that allows precoloring the vertices of a short outer face (this helps to handle short separating cycles). Thomassen also extended this result to other surfaces, even proving an exponential lower bound on the number of 3-list colorings; see [382] and its references.

This precoloring method pioneered by Thomassen has also been used to prove many other results. We study it in depth in Chapter 11, but here we just mention one that is closely related to material we studied in the present chapter. Borodin, Glebov, Raspaud, and Salavatipour [59] proved the 3-colorability of planar graphs with no cycles of lengths 4 to 7. (Later, it was shown [418, 58] that forbidding cycles of lengths 4, 5, and 7 is enough.) This is similar to Theorem 1.44, which also forbids 8-cycles and 9-cycles, but the proof is harder. In addition to tetrads, they also need two other reducible configurations. For more examples of the precoloring method, we recommend the habilitation of Dvořák [127].

In Section 4.2 we defined the fractional chromatic number, $\chi_f$. In short, we phrased the coloring problem as an integer program and considered its linear relaxation. This "rationalization" process can be applied to most graph parameters [356] and this approach is attractive for numerous reasons. (1) The resulting min/max theorems are often more elegant and have simpler proofs. (2) Determining the values of these fractional parameters, such as $\chi_f$, is often easier in practice. (3) We always have $\chi_f(G) \leqslant \chi(G)$, which frequently provides helpful insight into a particular problem of interest.

It is noteworthy that, despite the advantages mentioned above, computing $\chi_f$ is still NP-hard, as shown by Grötzschel, Lovász, and Schrijver [184]. However, computing $\chi'_f$, the fractional chromatic index, can be done in polynomial time. This is in large part because the matching polytope (see Section A.12) is so well-behaved.

The $\frac{9}{2}$ Color Theorem is due to Cranston and Rabern [103]. We form a graph $W$ from an 8-cycle by adding the four "diagonals". Wagner [408] showed that every $K_5$-minor-free graph can be formed from planar graphs and copies of $W$ by repeatedly pasting along cliques of size at most 3, and possibly deleting some edges. Since $W$ has a 2-fold 9-coloring, the $\frac{9}{2}$ Color Theorem extends to $K_5$-minor-free graphs. By the 4 Color Theorem, every planar graph maps into $K_4$. We may also seek a graph $H$ and a proof, independent of 4CT, such that every planar $G$ maps into $H$. Clearly, the smallest such $H$ is $K_5$. If we require that $\omega(H) = 4$, then the smallest known $H$ has 180 vertices; it is the "categorical product" $K_5 \times K_{9:2}$ (see [103] for more details).

We asked when $G \to C_{2k+1}$ (for an input planar graph $G$ and fixed $k$). To attack this question, we relied heavily on the Folding Lemma, which was proved by Klostermeyer and Zhang [262]. The question also fits into a larger context. A *circular* $(p,q)$*-coloring* of $G$ is a map $\varphi : V(G) \to [p]$ such that $q \leqslant |f(u) - f(v)| \leqslant p - q$ whenever $uv \in E(G)$. We represent $[p]$ as points equally spaced around a circle, and we must map the endpoints of each edge to points on the circle at least $q$ apart. The *circular chromatic number*, $\chi_c(G)$, of $G$ is the minimum $\frac{p}{q}$ such that $G$ has a circular $(p,q)$-coloring. Note that a circular $(k,1)$-coloring is just a $k$-coloring, so $\chi_c(G) \leqslant \chi(G)$. In fact, always $\chi(G) = \lceil \chi_c(G) \rceil$. Zhu [432] gives an extensive survey of circular coloring; see also Hell and Nešetřil [214, Chapter 6].

circular $(p,q)$-coloring

circular chromatic number

It is easy to check that $G \to C_{2k+1}$ precisely when $G$ has a circular $(2k+1,k)$-coloring. This condition is equivalent to $\chi_c(G) \leqslant 2 + \frac{1}{k}$. Jaeger [226] suggested the following.

**Conjecture 4.44.** *Every planar graph with girth at least* $4k$ *has* $\chi_c(G) \leqslant 2 + \frac{1}{k}$.

This was generalized by Zhang [428].

**Conjecture 4.45.** *Every planar graph with odd-girth at least* $4k+1$ *has* $\chi_c(G) \leqslant 2 + \frac{1}{k}$.

Both conjectures were actually posed more generally, in terms of flows, which we study in Chapter 6. Devos constructed planar graphs $G_k$ with girth $4k-1$ and $\chi_c(G_k) > 2 + \frac{1}{k}$ (see Exercise 1). Thus, the girth and odd-girth hypotheses in these conjectures cannot be weakened. For $k = 1$, both conjectures are equivalent to Grötzsch's Theorem. For $k = 2$, Conjecture 4.45 says that every planar graph with odd-girth at least 9 maps into $C_5$. Using the Folding Lemma, Dvořák, Škrekovski, and Valla [141] showed that such graphs map into the Petersen graph. Zhu [433] proved that $G \to C_{2k+1}$ when $G$ is planar with odd-girth at least $8k-3$. Borodin et al. [63] showed that $G \to C_{2k+1}$ when $G$ has girth at least $6k-2$ and $\mathrm{mad}(G) < 2 + \frac{6}{10k-4}$. This proves Conjecture 4.44 for graphs with girth at least $\frac{20k-2}{3}$. When $k = 2$, Borodin et al. [60] strengthened this to triangle-free graphs with $\mathrm{mad}(G) < \frac{12}{5}$, which includes planar graphs of girth 12 (we present a proof in Section 12.2). The strongest result towards Conjecture 4.44 is by Lovász, Thomassen, Wu, and Zhang [291], who proved the conjecture for planar graphs with girth at least $6k$. They proved this result, which we study in Section 6.4, in the more general context of nowhere-zero flows.

Correspondence coloring has been studied broadly. Thomassen showed that each planar graph is 5-choosable, and when its girth is at least 5 it is 3-choosable. Dvořák and Postle [139] noted that analogous results hold for correspondence coloring, essentially by mimicking Thomassen's list-coloring proofs. Bernshteyn, Kostochka, and Pron [39] studied the analogue of degree-choosability for correspondence coloring. Now each vertex $v$ can be colored from $[d(v)]$, and still each color for a vertex $v$ forbids at most one color from use on each neighbor. For a degree correspondence assignment $C$, a connected graph may fail to be $C$-colorable only when each block of $G$ is a clique or a cycle (not necessarily odd). Recall that even cycles have correspondence chromatic number 3. This fact is important because it shows that no analogue of the Alon–Tarsi Theorem holds for correspondence coloring. Likewise, the Kernel Lemma, an important list-coloring tool that we see in Chapter 5, has no analogue for correspondence coloring. Dirac proved a lower bound on the number of edges in an $n$-vertex $k$-list-critical graph. Using correspondence coloring, Bernshteyn and Kostochka [38] characterized the graphs where Dirac's bound holds with equality.

## Exercises

4.1. For each positive integer $k$, construct a planar graph $G_k$ with girth $4k-1$ that has no homomorphism into $C_{2k+1}$.

4.2. Prove a version of Brooks' Theorem for correspondence coloring.

4.3. (a) Show that every $k$-degenerate graph is correspondence $(k+1)$-colorable. (b) Mimic the proof of Theorem 11.1 to show that every planar graph is correspondence 5-colorable.

4.4. Show that the Straightening Lemma becomes false if we drop the hypothesis that all edges are full.

4.5. A *degree-correspondence assignment* assigns to each vertex $v$ a list $L(v)$ with $|L(v)| = d(v)$ and to each edge $vw$ a matching between elements of $L(v)$ and $L(w)$. A graph $G$ is *degree-correspondence colorable* if every degree-correspondence assignment $(L, C)$ admits an $(L, C)$-coloring. For a simple graph $H$, let $H^k$ denote the multigraph with every edge of multiplicity $k$ and with $H$ as its underlying simple graph. (a) Show that neither $K_n^k$ nor $C_n^k$ is degree correspondence colorable. Let $G_1$ and $G_2$ be arbitrary graphs, with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. (b) Form $G$ from $G_1$ and $G_2$ by identifying $v_1$ and $v_2$. Show that $G$ fails to be degree correspondence colorable if and only if both $G_1$ and $G_2$ do. So, by induction, a graph fails to be degree-correspondence colorable if each block is $C_n^t$ or $K_n^t$ (where $t$ and $n$ can vary between blocks). In fact, these are the only graphs that do.

# Chapter 5

# The Kernel Method

> To many, mathematics is a collection of theorems. For me, mathematics is a collection of examples; a theorem is a statement about a collection of examples and the purpose of proving theorems is to classify and explain the examples..."
>
> —John B. Conway

Our coloring methods so far have been mainly *local*. Our reducible configurations have usually had bounded size. And Kempe swaps, although they can recolor long paths, only change the colors available at the two endpoints. Here we explore a more global approach.

## 5.1 Planar Bipartite Graphs are 3-Choosable

**Definition 5.1.** A *kernel* of a digraph $D$ is an independent set $S$ such that every vertex not in $S$ has an outneighbor in $S$. A digraph $D$ is *kernel-perfect* if every induced subgraph of $D$ has a kernel. A digraph is *strongly connected* if each vertex has a directed path to each other vertex. Let $N^+(x)$ (resp. $N^-(x)$) denote the set of outneighbors (resp. inneighbors) of $x$, and $N(x) := N^+(x) \cup N^-(x)$.

kernel
kernel-perfect
strongly connected

We study kernel-perfect orientations because of the following lemma. (See Figure 5.1.)

**Lemma 5.2 (Kernel Lemma).** *Let* $D$ *be a digraph with* $G$ *as its underlying simple graph. Let* $L$ *be a list assignment such that* $|L(v)| > d^+_D(v)$ *for all* $v$*. If* $D$ *is kernel-perfect, then* $G$ *is* $L$*-colorable.*

*Proof.* We assume $G$ is connected; otherwise we consider each component separately. We use induction on $|G|$, with base case $|G| = 1$. For the induction step, choose $\alpha \in \cup_{v\in V(G)} L(v)$. Let $D_\alpha$ be the subgraph of $D$ induced by all vertices $v$ with $\alpha \in L(v)$. Since $D$ is kernel-perfect, $D_\alpha$ has a kernel, $U$. For all $v \in V(G) \setminus U$, let $L'(v) := L(v) \setminus \{\alpha\}$. Each $v \in D_\alpha \setminus U$ has an outneighbor in $U$, so $|L'(v)| > d^+_{D\setminus U}(v)$ for all $v \in V(G) \setminus U$. Thus, by hypothesis, $G - U$ has an $L'$-coloring $\varphi$. To get an $L$-coloring of $G$, start with $\varphi$ and use $\alpha$ on each vertex of $U$. $\square$

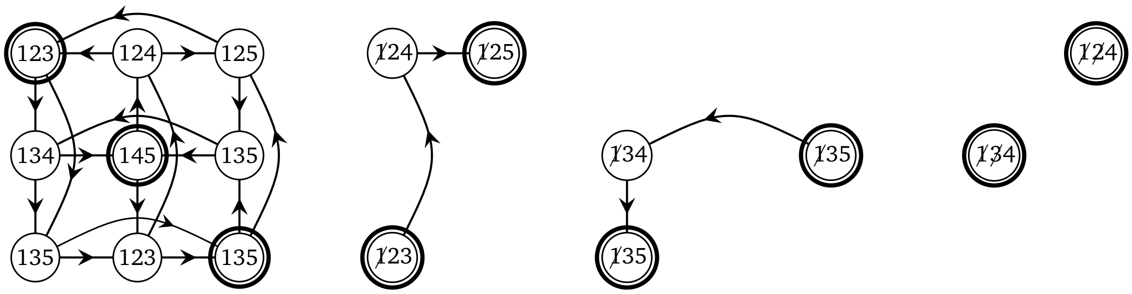



Figure 5.1: An example of the proof of the Kernel Lemma.

**Lemma 5.3 (Richardson's Theorem).** *If a digraph* $D$ *has no directed odd cycle, then* $D$ *is kernel-perfect.*

The key idea is to find $V_1 \subseteq V(D)$ and $S_1 \subseteq V_1$ such that all outneighbors of $S_1$ are in $V_1$ and also $S_1$ is a kernel for the subgraph induced by $V_1$. By induction, $D - V_1 - N(S_1)$ has a kernel $T$. Now $S_1 \cup T$ is a kernel for $D$.

*Proof.* Let $\mathcal{D}$ be the class of digraphs with no directed odd cycle. Suppose the lemma is false and choose a counterexample $D \in \mathcal{D}$ with fewest vertices. By the minimality of $D$, each of its proper induced subgraphs has a kernel, since $\mathcal{D}$ is hereditary. So to reach a contradiction we only need to show that also $D$ has a kernel.

If two strongly connected subgraphs have a common vertex, then their union is also strongly connected. So let $V_1, \ldots, V_k$ be a vertex partition such that each $V_i$ induces a maximal strongly
$V_1$, $D_1$ connected digraph $D_i$. By symmetry, assume each $v \in V_1$ has all its outneighbors in $V_1$. (If each $V_i$ has outneighbors outside $V_i$, then merging some $V_i$'s gives a larger strongly connected subgraph, contradicting our assumption about the partition $V_1, \ldots, V_k$.)

Given $v, w \in V_1$, every directed $v,w$-walk must have the same parity as every directed $w,v$-walk. If not, then directed $v,w$- and $w,v$-walks of opposite parity combine to give an odd closed walk, which contains a directed odd cycle, contradicting the hypothesis. Similarly, every directed $v,w$-walk must have the same parity. For if two have opposite parities, then one will combine with a directed $w,v$-walk (which exists, since $D_1$ is strongly connected) to give an odd closed walk, again contradicting the hypothesis.

The argument above implies that $D_1$ is bipartite. That is, $V_1$ has a partition into $S_0$ and $S_1$, with each edge of $D_1$ between $S_0$ and $S_1$, as follows. Pick an arbitrary vertex $v \in D_1$. For each $w \in V_1$, let $f(w)$ denote the parity of all directed $v,w$-walks. Since $D_1$ is strongly connected,
$S_i$ the argument above shows that $f$ is well-defined. Let $S_i := \{w : f(w) = i\}$ for each $i \in \{0, 1\}$. Now $D_1$ is bipartite with parts $S_0$ and $S_1$, as we now show. Suppose, to the contrary, that there exist $w, x \in S_0$ with $\overrightarrow{wx} \in E(D_1)$. Let $P$ be a directed $v,w$-walk, of even length. Now $P + \overrightarrow{wx}$ is an odd directed $v,x$-walk, contradicting that $x \in S_0$. So $S_0$ is independent; the same is true for $S_1$. Thus, $D_1$ is bipartite, as claimed.

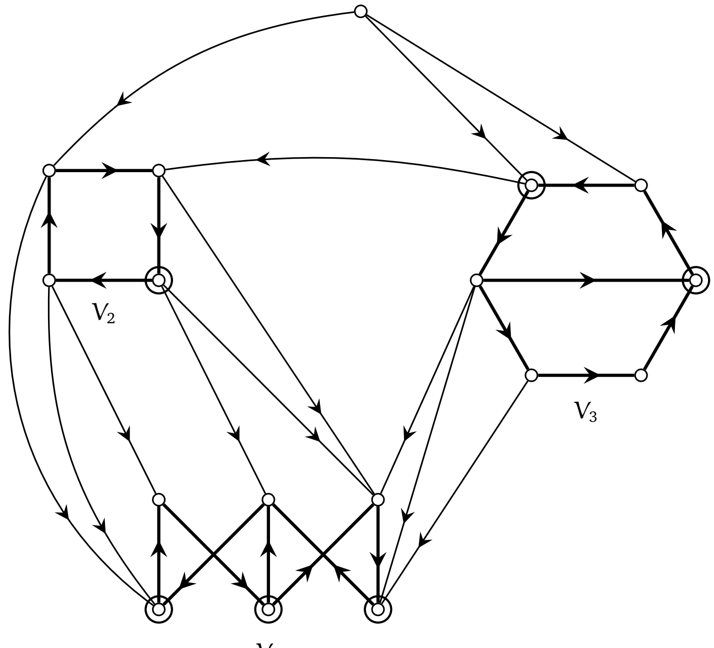



Figure 5.2: An example of the proof of Richardson's Theorem.

Since $D_1$ is strongly connected, each vertex of $S_0$ has an outneighbor in $S_1$, so $S_1$ is a kernel for $D_1$. Recall that $N^+(S_1) \subseteq V_1$, so $N(S_1) \setminus V_1 \subseteq N^-(S_1)$. By the minimality of D, subgraph $D - V_1 - N(S_1)$ has a kernel T. So $T \cup S_1$ is a kernel of D. $\square$

We typically list-color from lists with equal sizes. So, to apply the Kernel Lemma, among kernel-perfect orientations, we seek one with the smallest maximum outdegree, $\Delta^+$.

**Lemma 5.4.** *If mad*$(G) \leqslant 2k$*, then* G *has an orientation* D *with* $\Delta^+(D) \leqslant k$.

*Proof.* Let G be a graph with $\text{mad}(G) \leqslant 2k$. Let D be an orientation that minimizes the maximum outdegree and, subject to that, minimizes the number of vertices with maximum outdegree. We call a vertex $v$ *excessive* if $d^+_D(v) > k$. If D has no excessive vertex, then we are done. So suppose D has an excessive vertex, and let $v$ be one of maximum outdegree. D excessive $v$

Let $W$ be the set of all vertices that are reachable by some directed path from $v$. If $W$ contains a vertex $w$ with $d^+_D(w) \leqslant d^+_D(v) - 2$, then reversing a directed path from $v$ to $w$ reduces the number of vertices with maximum outdegree, contradicting our choice of D. So assume instead that every vertex $w \in W$ has $d^+_D(w) \geqslant d^+_D(v) - 1 \geqslant k$. Consider the subgraph H induced by $W$ (including $v$). Since $d^+_D(w) \geqslant k$ for all $w \in W$, we have $\|H\| \geqslant k|H| + 1$. Thus $\overline{d}(H) > 2k$, contradicting our assumption that $\text{mad}(H) \leqslant 2k$. $\square$

This lemma holds in a more general form. See Exercise 1.

**Theorem 5.5.** *If* $G$ *is bipartite with* $mad(G) \leqslant 2(k-1)$*, then* $G$ *is* $k$*-choosable.*

*Proof.* Let $G$ satisfy the hypotheses. By Lemma 5.4, $G$ has an orientation $D$ with maximum outdegree at most $k-1$. Since $G$ is bipartite, $D$ has no directed odd cycle. So Richardson's Theorem (Lemma 5.3) implies that $D$ is kernel perfect. Now the Kernel Lemma (Lemma 5.2) shows that $G$ is $k$-choosable. □

Theorem 5.5 is strikingly sharp. In Section 2.8.1 we construct bipartite graphs $G$ such that $mad(G-e) \leqslant 2(k-1)$, for every edge $e \in E(G)$, but $\chi_\ell(G) > k$.

**Corollary 5.6.** *If* $G$ *is planar and bipartite, then* $G$ *is* $3$*-choosable.*

*Proof.* If $G$ is planar with girth at least 4, then Lemma 1.6 implies that $mad(G) < 4$. □

## 5.2 Bipartite Graphs are Δ-Edge-Choosable

Below is the most famous conjecture on edge list-coloring.

**Conjecture 5.7 (List Coloring Conjecture).** *Every graph* $G$ *satisfies* $\chi'_\ell(G) = \chi'(G)$.

In this section we use the kernel method to prove Conjecture 5.7 for all bipartite graphs. Our proof uses the following definitions and theorem.

**Definition 5.8.** Consider a set of $n$ men and a set of $n$ women, in which each man and woman has ranked all members of the opposite sex (with distinct ranks from 1 to $n$) in order of who they most prefer to marry. These rankings are *preference lists*. We aim to pair men and women into $n$ married couples but avoid having any man and woman who are not married to each other but each prefer the other over their current partner. Such a pairing is a *stable matching*.

preference lists

stable matching

**Theorem 5.9 (Proposal Algorithm).** *Given* $n$ *men and* $n$ *women, for every set of preference lists there exists a stable matching.*

*Proof.* We use the following Proposal Algorithm, which proceeds in rounds. On each round, each unengaged man proposes to the woman he most prefers, among those who have not yet rejected him (including women who are tentatively engaged). After all the men propose, each woman says "maybe" to the proposal she most prefers and "no" to all others, including possibly the man to whom she was tentatively engaged from the previous round.

Each woman who said "maybe" is then tentatively engaged to that man to whom she said it, and he is tentatively engaged to her. All other men and women remain unengaged. This process repeats until a round when no man is rejected. At that point, each tentatively engaged couple becomes married, and the algorithm ends.

To show this algorithm yields a stable matching, our proof consists of three claims.

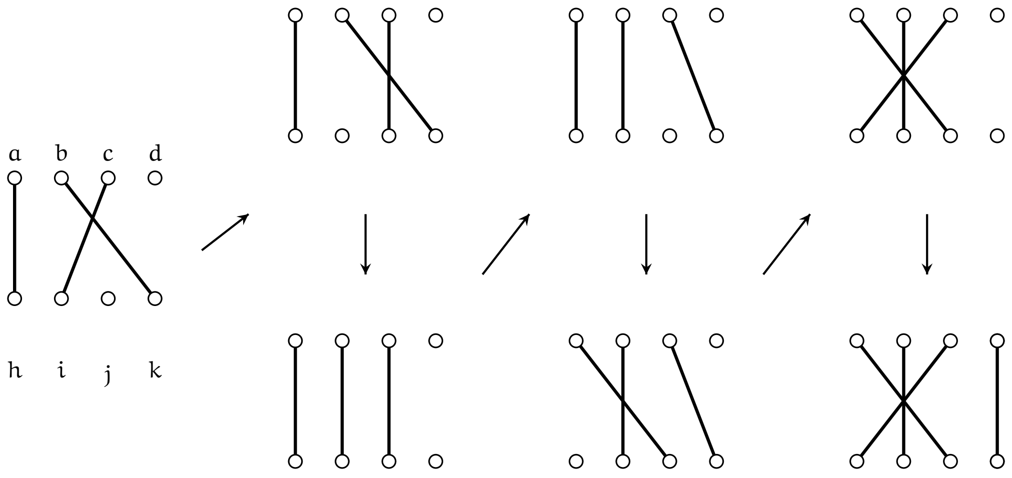



Figure 5.3: 7 rounds of the Proposal Algorithm with the preference lists for the women as a:ijhk, b:ikjh, c:hkji, d:hkij and those for the men as h:abcd, i:cbad, j:bcad, k:bacd. We only depict rounds during which one or more tentative engagements change.

**Claim 1.** *The algorithm terminates.*

Proof. On each round except the last, some man is rejected and so the total number of rejections increases. Each man is rejected at most once by each woman, and so the algorithm terminates after at most $n^2 + 1$ rounds (in fact fewer). ♢

**Claim 2.** *Everyone ends up married.*

Proof. Suppose a man and woman are both unmarried. At some point, the man must have proposed to the woman. And she rejected him only if she had a proposal she preferred more. In that case, she is now married. Thus, every woman ends married. Since the numbers of men and women are equal, so does every man. ♢

**Claim 3.** *The algorithm produces a stable marriage.*

Proof. Suppose that a man $m$ and a woman $w$ each prefer each other over their current partner. As some point, $m$ proposed to $w$ (before proposing to his current wife). And if $w$ rejected $m$ (either immediately or later on), then she had a proposal she preferred more, and is thus married to someone she prefers more than $m$. ♢

Claim 3 proves the theorem. □

Below we will need a generalization of Theorem 5.9. Now each man and woman ranks all members of the opposite sex, but we allow unequal numbers of men and women and also allow each person to designate some of these members as "unranked"; these are the ones that he or she refuses to marry. We call these lists *generalized preference lists*.

**Lemma 5.10 (Generalized Proposal Algorithm).** *For all sets of men and women (possibly of unequal sizes) and all generalized preference lists, there exists a stable matching.*

*Proof.* We simply apply Theorem 5.9, but first we modify the original generalized preference lists as follows. For each person, the candidates they left as unranked appear at the end of their preference list, in arbitrary order. Further, if woman $w$ has man $m$ as unranked, then we modify the preference list of $m$ to also have $w$ as unranked, and vice versa. Consider a matching $M$ resulting from the proof of Theorem 5.9. Now if both people in a marriage of $M$ had the other as unranked, then we view both as being unmarried. We can easily check that $M$ remains stable when we account for this possibility of being unmarried.

If we have more women than men, then we add "fake" men so that the numbers of men and women become equal. When we extend the preference lists, each woman lists each fake man as unranked. (The case with more men than women is analogous.) □

**Theorem 5.11.** *Every bipartite graph* $G$ *satisfies* $\chi'_\ell(G) = \chi'(G)$.

*Proof.* We apply the Kernel Lemma to the line graph $L(G)$ of $G$. So we need an orientation $D$ of $L(G)$ where $D$ is kernel-perfect and $d^+(x) \leqslant \Delta(G) - 1$ for each $x \in V(L(G))$. Let $U$ and $W$ denote the parts in the bipartition of $G$. We refer to edges of $G$ equivalently as vertices of $L(G)$.

$U, W$

We form $D$ as follows. Let $\varphi$ be a $\Delta(G)$-edge-coloring of $G$ with colors $\{1, \dots, \Delta(G)\}$; such a coloring exists by König's Theorem. Choose arbitrary $x, y \in V(L(G))$. If $x$ and $y$ are adjacent, then they correspond to edges in $G$ with a common endpoint either in $U$ or in $W$. Suppose $\varphi(x) > \varphi(y)$. If $x$ and $y$ have a common endpoint in $U$, then orient the edge[1] in $L(G)$ as $\overrightarrow{xy}$, and if in $W$, then as $\overrightarrow{yx}$. Let $\alpha := \varphi(x)$. Now $x$ has at most $\alpha - 1$ out-edges in $L(G)$ toward edges in $G$ with a common endpoint in $U$ and at most $\Delta(G) - \alpha$ toward edges with a common endpoint in $W$. So $x$ has outdegree in $D$ at most $\Delta(G) - 1$. Since $x$ was arbitrary, $d^+_D(x) \leqslant \Delta(G) - 1$ for every $x \in V(L(G))$.

(The left of Figure 5.1 shows the line graph of $K_{3,3}$ oriented as described in the previous paragraph. Edges with a common endpoint in $U$ appear in the same column and those with a common endpoint in $W$ appear in the same row. So the three vertices in the kernel at the first step are edges colored 3 in the edge-coloring of $K_{3,3}$. Now Figure 5.1 in its entirety shows how to find an edge-$L$-coloring for the edge list-assignment shown there.)

Now we must show that orientation $D$ is kernel-perfect. For this we use Lemma 5.10. Since bipartite graphs form a hereditary class, it suffices to show that $D$ has a kernel. Let $x_1 := u_1w_1$, $x_2 := u_2w_1$, $y_1 := u_3w_2$, and $y_2 := u_3w_3$, where $u_1, u_2, u_3 \in U$, $w_1, w_2, w_3 \in W$, and $x_1, x_2, y_1, y_2 \in L(G)$. We view $\overrightarrow{x_2x_1} \in D$ as meaning that $w_1$ prefers $u_1$ over $u_2$. Similarly, $\overrightarrow{y_1y_2}$ means that $u_3$ prefers $w_3$ over $w_2$. If $uw \notin E(G)$, then $u$ is unranked for $w$ and vice versa. In this way, every orientation of a line graph of a bipartite graph gives rise to a set of generalized preference lists. A stable matching for these preferences corresponds to a kernel $S$ in $D$, since a vertex $x$ of $L(G)$ is excluded from $S$ whenever one of its outneighbors is included. □

[1] If $x$ and $y$ are parallel edges in $G$, then in $L(G)$ they are joined by two edges; we orient them as $\overrightarrow{xy}$ and as $\overrightarrow{yx}$.

To close this section, we briefly mention a strengthening of list-coloring.

**Definition 5.12.** Given a graph $G$ and a list assignment $L$ for $G$, a *list-coloring packing* is a set of functions $\varphi_1, \dots, \varphi_p$ such that  list-coloring packing

(a) $\varphi_i(v) \in L(v)$ for all $v \in V(G)$ and all $i \in [p]$, and

(b) $\varphi_i(v) \neq \varphi_i(w)$ for all $v, w$ such that $vw \in E(G)$, and

(c) $\varphi_i(v) \neq \varphi_j(v)$ for all $v \in V(G)$ and all distinct $i, j \in [p]$.

The *list-coloring packing number* $\chi_\ell^*(G)$ is the minimum $k$ such that if $|L(v)| = k$ for all $v \in V(G)$, then $G$ has a list-coloring packing $\varphi_1, \dots, \varphi_k$.  list-coloring packing number

It is not immediately clear that $\chi_\ell^*$ is well-defined for all graphs, so we include a short proof. (Note that $\chi_\ell^*(G) \leqslant \chi_\ell^*(K_{|G|})$ for all $G$, since a list-coloring of $K_{|G|}$ gives a list-coloring of $G$.)

**Theorem 5.13.** $\chi_\ell^*(K_n) = n$ *for every positive integer* $n$.

*Proof.* The lower bound is easy: $\chi_\ell^*(K_n) \geqslant \chi_\ell(K_n) \geqslant \chi(K_n) = n$. Now we prove the upper.

Fix positive integers $m$ and $n$ with $m \geqslant n$, and an $m$-assignment $L$ for $K_n$. We denote the vertices of $K_n$ by $w_1, \dots, w_n$. Let $H := K_m \square K_n$ (this is the Cartesian product), where vertices of $H$ are denoted $(v_i, w_j)$ with $i \in [m]$ and $j \in [n]$ and $E(H) := \{(v_1, w_1)(v_2, w_2)$ if either (a) $v_1 = v_2$ and $w_1 \neq w_2$ or (b) $v_1 \neq v_2$ and $w_1 = w_2\}$. (The right of Figure 5.4 shows $K_3 \square K_3$.) Let $\tilde{L}$ be an $m$-assignment for $H$ given by $\tilde{L}(v_i, w_j) := L(w_j)$.  $m, n$  $H$  $\tilde{L}$

It suffices to show that $H$ is $\tilde{L}$-colorable, as follows. Given an $\tilde{L}$-coloring $\varphi$ of $H$, we let $\varphi_i(w_j) := \varphi(v_i, w_j)$. That is, each $\varphi_i$ is a restriction of $\varphi$ to a (disjoint) copy of $K_n$. It is easy to check that $\varphi_1, \dots, \varphi_m$ is indeed a list-coloring packing of $K_n$. To get $\varphi$, note that $H$ is the line graph of the complete bipartite graph $K_{m,n}$. Thus, by Theorem 5.11 we have $\chi_\ell(H) = \chi_\ell'(K_{m,n}) = \chi'(K_{m,n}) = \max\{m, n\} = m = |\tilde{L}(v_i, w_j)|$ for all $i, j$. Hence, $H$ has an $\tilde{L}$-coloring $\varphi$, as desired. □

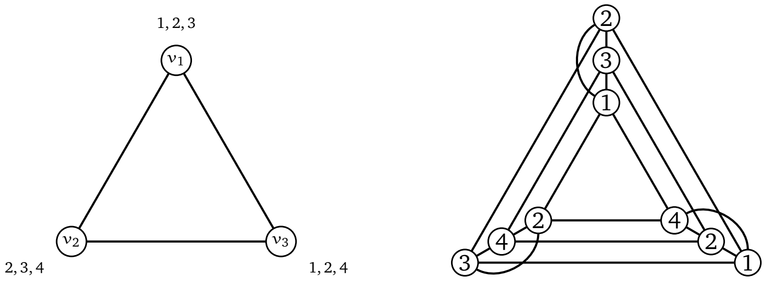



Figure 5.4: Left: A 3-assignment $L$ to $K_3$. Right an $\tilde{L}$-coloring of $K_3 \square K_3$, which corresponds to an $L$-coloring packing of $K_3$.

## 5.3 An Easy Strengthening and an Application

In this section we sketch a proof of the following result.

**Theorem 5.14.** *The List Coloring Conjecture is true for all line-perfect multigraphs.*

line-perfect

A multigraph $G$ is *line-perfect* if its line graph $J_G$ is perfect; that is, $\chi(H) = \omega(H)$ for each induced subgraph $H$ of $J_G$. Theorem 5.14 generalizes the main result from the previous section, since bipartite graphs form a proper subclass of line-perfect graphs. The proof relies heavily on the following theorem (which we restate and prove in the appendix, as Theorem A.10).

**Theorem 5.15.** *For a multigraph $G$, the following properties are equivalent.*

*(a)* $G$ *is line-perfect.*

*(b)* $G$ *does not contain any odd cycle with length at least* 5.

*(c) Every block of* $G$ *has as its underlying simple graph either (i) a bipartite graph, (ii)* $K_4$*, or (iii) the complete tripartite graph* $K_{1,1,t}$*, for some integer* $t \geqslant 1$.

The idea of the proof of Theorem 5.14 is to color the edges one block at a time, and to order the blocks to make this as easy as possible. We pick an arbitrary block to be the root in the block tree and color the blocks in order of increasing distance from the root. So, when we color each block $B$, at most one cut-vertex of $B$ is incident to edges already colored. We formalize this approach after our next lemma, which ensures that we can color each additional block.

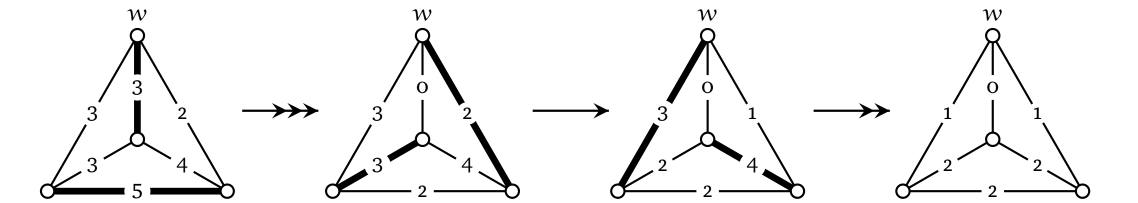
**Lemma 5.16.** *Let $G$ be a 2-connected multigraph with underlying simple graph either (i) bipartite, (ii) $K_4$, or (iii) $K_{1,1,t}$, for some integer $t \geqslant 1$. Fix a vertex $w$ and an edge list-assignment $L$ such that $|L(e)| = d(w)$ if $e$ is incident to $w$ and $|L(e)| = \chi'(G)$ otherwise. Now $G$ has an edge-$L$-coloring.*

*Proof sketch.* We give the full proof in cases (i) and (ii). The proof for case (iii) uses ideas similar to those for (ii), but is much more complicated, so we omit the details (see [329, 330]).

The proof for (i) is nearly the same as the proof of Theorem 5.11. In that proof, we showed that every proper edge-coloring of a bipartite graph gives rise to a kernel-perfect orientation of its line graph. But we didn't really use the flexibility we have in choosing our edge-coloring. Here we choose a $\Delta(G)$-edge-coloring $\varphi$ such that the colors used incident to $w$ are precisely $1, \ldots, d(w)$. Further, we assume that $w$ is in part $W$. Suppose $e = uw$ and $\varphi(e) = \alpha$. So the outneighbors of $e$ in $D$ that share endpoint $u$ have colors in $1, \ldots, \alpha - 1$ and the outneighbors of $e$ in $D$ that share endpoint $w$ have colors $\alpha + 1, \ldots, d(w)$. Thus, $d^+_D(e) \leqslant (\alpha - 1) + (d(w) - \alpha) = d(w) - 1$. The remainder of the proof is exactly the same.

t

Now consider (ii). We let $t(\{v_1v_2v_3\}) := \mu(v_1v_2) + \mu(v_1v_3) + \mu(v_2v_3)$ and also let $t(G) := \max t(\{v_1v_2v_3\})$, where the maximum is over all distinct $v_1, v_2, v_3 \in V(G)$. By Theorem 5.15, $G$ is line-perfect. For every line graph $J_H$ of a graph $H$ we have $\omega(J_H) = \max\{\Delta(H), t(H)\}$. Thus,




$d(w) = 8$, $\Delta(G) = 11$, and $t(G) = \max\{9, 10, 9, 12\} = 12$ | $d(w) = 5$, $\Delta(G) = 8$, and $t(G) = \max\{6, 7, 6, 9\} = 9$ | $d(w) = 4$, $\Delta(G) = 7$, and $t(G) = \max\{5, 6, 5, 8\} = 8$ | $d(w) = 2$, $\Delta(G) = 5$, and $t(G) = \max\{3, 4, 3, 6\} = 6$

Figure 5.5: An example of 6 successive applications of the inductive step in the proof of Lemma 5.16(ii). Bold edges indicate the good $2K_2$; numbers indicate edge multiplicities.

$\chi'(G) = \max\{\Delta(G), t(G)\}$. Fix an edge list-assignment $L$ as in the statement of the lemma (for $K_4$). A *good* $2K_2$, w.r.t. $L$ and $G$, is a matching $e_1, e_2$ such that there exists some color $\alpha \in L(e_1) \cap L(e_2)$. If some $e_1, e_2$ form a good $2K_2$, then we use $\alpha$ on $e_1$ and $e_2$ and proceed by induction, since $\chi'(G) = 1 + \chi'(G - e_1 - e_2)$. Our base case is when no good $2K_2$ exists. Now we use Hall's Theorem to give every edge its own color. Consider a set $S$ of edges. Let $L(S)$ denote $\cup_{e \in S} L(e)$.

good $2K_2$

$L(S)$

Suppose $S$ has no $2K_2$. So either all edges of $S$ have a common endpoint or they all lie on a triangle (with endpoints among the same three vertices). If all edges are incident to $w$, then for an arbitrary $e' \in S$, we have $|L(S)| \geqslant |L(e')| = d(w) \geqslant |S|$, as needed. Otherwise, some $e' \in S$ is not incident to $w$, so $|L(S)| \geqslant |L(e')| = \chi'(G) = \max\{\Delta(G), t(G)\} \geqslant |S|$, as needed.

So assume instead that $S$ has a $2K_2$; call it $e_1, e_2$, and assume that $e_1$ is incident with $w$. Denote the other vertices of the $K_4$ by $x, y, z$. Since $e_1, e_2$ is not a *good* $2K_2$, we know that $L(e_1) \cap L(e_2) = \emptyset$. So $|L(S)| \geqslant |L(e_1)| + |L(e_2)| \geqslant d(w) + t(\{xyz\}) = |E(G)| \geqslant |S|$, as needed.

Since always $|L(S)| \geqslant |S|$, by Hall's Theorem, we can give each edge its own color. □

Now we combine Theorem 5.15 and Lemma 5.16 to prove Theorem 5.14.

*Proof of Theorem 5.14.* Fix a line-perfect multigraph $G$ and an edge list-assignment $L$ with $|L(e)| = \chi'(G)$ for all $e \in E(G)$. By Theorem 5.15, each block of $G$ satisfies (i), (ii), or (iii) in that theorem. We use induction on the number of blocks in $G$. The base case follows immediately from Lemma 5.16.

For the induction step, consider a leaf block $B$ in the block tree of $G$, and let $v$ be the unique cut-vertex in $B$. By hypothesis, $G - (B - v)$ has an $L$-edge-coloring $\varphi$. For each $e \in E(B)$ that is incident to $v$, form $L'(e)$ from $L(e)$ by removing all colors used by $\varphi$ on edges incident to $v$. So $|L'(e)| \geqslant |L(e)| - (d_G(v) - d_B(v)) \geqslant d_B(v)$. For each $e \in E(B)$ not incident to $v$, let $L'(e) := L(e)$. So $|L'(e)| = |L(e)| = \chi'(G) \geqslant \chi'(B)$. Thus, by Lemma 5.16, $B$ has an $L'$-edge-coloring $\varphi'$. Finally, $\varphi \cup \varphi'$ is an $L$-edge-coloring of $G$. □

## 5.4 A Harder Strengthening and 3 Applications

### 5.4.1 The Strengthening

We will soon present 3 applications. For these the following strengthening of Theorem 5.11 is more useful for proving that various bipartite subgraphs are reducible.

**Theorem 5.17.** *Let $G$ be a bipartite graph, with parts $U$ and $W$. For each edge $e = uw$, let $f(e) := \max\{d(u), d(w)\}$. Now $G$ is $f$-edge-choosable. (Figure 5.7 shows an example.)*

The proof follows the same outline as that of Theorem 5.11. The key difference is that we now find an edge-coloring $\varphi$ of $G$ such that in the orientation $D$ arising from $\varphi$, we have $d^+_D(e) < f(e)$ for each edge $e \in E(G)$, rather than only $d^+_D(e) < \Delta(G)$. To do this, we use Lemmas 5.18 and 5.20 below.

**Lemma 5.18.** *Let $G$ be a bipartite graph with parts $U$ and $W$, and no isolated vertices. If $|U| \leqslant |W|$, then $G$ has a non-empty matching $M$ such that for each edge $uw \in E(G)$, with $u \in U$ and $w \in W$, if $w \in V(M)$ then $u \in V(M)$.*

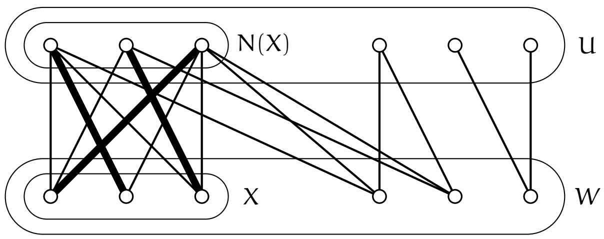



Figure 5.6: An example of Lemma 5.18.

One way to prove König's Theorem (Theorem 3.3) is by induction on $\Delta$: we find a matching $M$ saturating all vertices of maximum degree, color $M$ with a single color, and repeat this process on $G - M$. To prove Theorem 5.17, we want our matching $M$ to have slightly different properties. The present lemma guarantees that this $M$ exists.

*Proof.* Let $X$ be a minimal subset of $W$ such that $|N(X)| \leqslant |X|$. This set $X$ exists, since any $Y \subseteq W$ with $|Y| = |U|$ satisfies $|N(Y)| \leqslant |Y|$. We construct the matching $M$ by letting $V(M) \cap W = X$ and $V(M) \cap U = N(X)$; see Figure 5.6. If $|X| = 1$, then $|N(X)| = 1$, since $G$ has no isolated vertices; so $M$ exists. Assume instead that $|X| \geqslant 2$. By the minimality of $X$, every non-empty $Y \subsetneq X$ satisfies $|N(Y)| > |Y|$. So the desired matching $M$ exists by Hall's Theorem. □

defers to

$m_\varphi(e)$

**Definition 5.19.** Let $G$ be a bipartite graph with parts $U$ and $W$. For an edge-coloring $\varphi$ of $G$, edge $e_1$ *defers to* edge $e_2$ if either (i) $\varphi(e_1) < \varphi(e_2)$ and $e_1$ and $e_2$ share an endpoint in $U$ or (ii) $\varphi(e_1) > \varphi(e_2)$ and $e_1$ and $e_2$ share an endpoint in $W$. Let $m_\varphi(e)$ denote the total number of edges that $e$ defers to with respect to $\varphi$. Given a function $f : E(G) \to \mathbb{Z}^+$, an edge-coloring

$\varphi$ of $G$ *respects* $f$ if $m_\varphi(e) < f(e)$ for every edge $e \in E(G)$. (In the line graph, a vertex has as its outneighbor exactly the edges that it defers to w.r.t. $\varphi$.) respects

Now we can construct the desired edge-coloring of $G$.

**Lemma 5.20.** *For a graph* $G$*, let* $f(uw) := \max\{d(u), d(w)\}$ *for each edge* $uw$*. If* $G$ *is bipartite, then* $G$ *has an edge-coloring* $\varphi$ *that respects* $f$.

Before proving Lemma 5.20, we use it to prove Theorem 5.17, by slightly modifying our proof of Theorem 5.11.

*Proof of Theorem 5.17.* The proof is nearly the same as that of Theorem 5.11, but we replace the arbitrary $\Delta(G)$-edge-coloring there (guaranteed by König's Theorem) with an edge-coloring $\varphi$ that respects $f$, from Lemma 5.20. Note that $\varphi$ may use arbitrarily many colors. But since $\varphi$ respects $f$, graph $G$ is $L$-edge-colorable for every $f$-assignment $L$. $\square$

*Proof of Lemma 5.20.* Our proof is by induction on $\|G\|$, and Figure 5.7 shows an example. Suppose that $|U| \leqslant |W|$. Let $M$ be the matching guaranteed by Lemma 5.18. Let $X :=$ $M, X$

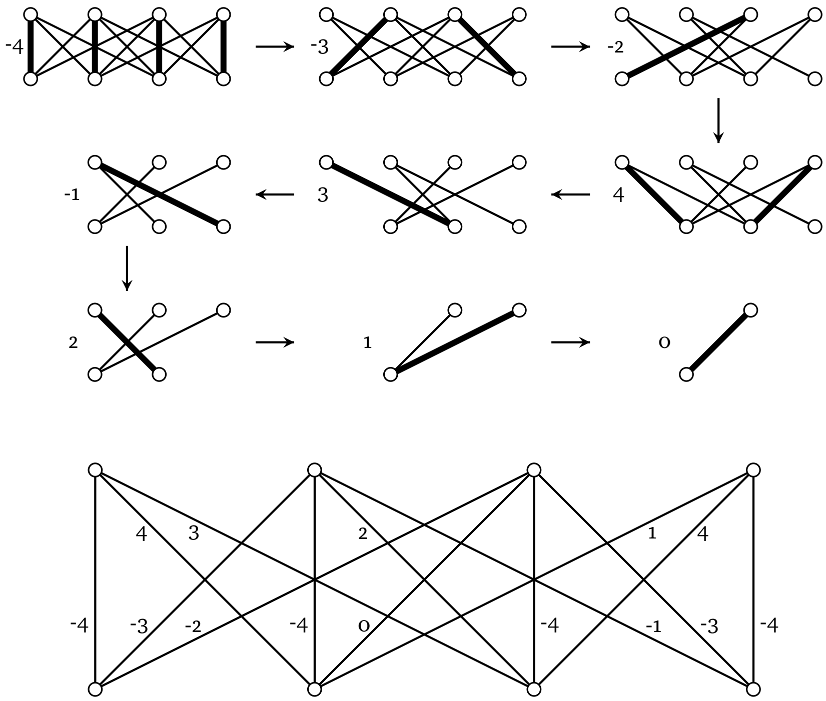



Figure 5.7: The proof of Theorem 5.20, run on $K_{4,4} - 2K_2$. In the final edge-coloring, each edge defers to at most three adjacent edges, and the left and right vertical edges each defer to only two adjacent edges.

$G', f'$ $\varphi'$ $\alpha$

$V(M) \cap W$, and recall that $N(X) = V(M) \cap U$. Let $G' := G - M$. For each edge $uw \in E(G) - M$, let $f'(uw) := \max\{d_{G'}(u), d_{G'}(w)\}$. By induction, $G'$ has an edge-coloring $\varphi'$ that respects $f'$. Let $\alpha$ be a color smaller than all colors used by $\varphi'$. To extend $\varphi'$ to $G$, color $M$ with $\alpha$; call this coloring $\varphi$. Now each edge $uw \in M$ defers to no edges at $w$ and defers to $d(u) - 1$ edges at $u$. Thus, $m_\varphi(uw) < d(u) \leqslant f(uw)$.

Instead consider an arbitrary edge $uw \in E(G - M)$. If $w \in X$, then $u \in N(X)$. So $d_{G'}(u) = d_G(u) - 1$ and $d_{G'}(w) = d(w) - 1$, which implies $f'(uw) = f(uw) - 1$. Now $uw$ defers to no new edges at $u$ and at most one new edge at $w$. So, $m_\varphi(uw) \leqslant m_{\varphi'}(uw) + 1 < f(uw)$, as desired. Assume instead that $w \notin X$. We may have $u \in N(X)$, in which case $uw$ shares an endpoint in $U$ with an edge $e$ of $M$. However, $uw$ does not defer to $e$, since $\varphi(e) < \varphi(uw)$. So $uw$ defers to at most $f'(uw)$ edges. Thus, $\varphi$ respects $f$.

Assume instead that $|W| < |U|$. The proof is nearly the same as above. Now we find $X \subseteq U$ and matching $M$ such that $V(M) \cap U = X$ and $V(M) \cap W = N(X)$. Given a coloring $\varphi'$ of $G - M$ that respects $f'$, we color the edges of $M$ with a color *larger* than all colors used by $\varphi'$. Again, the resulting coloring $\varphi$ respects $f$. □

### 5.4.2 Planar Graphs with $\Delta \geqslant 12$ are $\Delta$-Edge-Choosable

Now we will prove that $\chi'_\ell = \Delta$ for every planar graph with $\Delta \geqslant 12$.

2-alternating cycle

A 2-*alternating cycle* is an even cycle on which every second vertex has degree 2. It is easy to check that such cycles are reducible for $\Delta$-edge-choosability. Lemma 5.17 allows us to significantly generalize this reducibility argument.

weight
$i$-alternating subgraph
3-alternator

**Definition 5.21.** The *weight* of an edge $uw$ is $d(u) + d(w)$. An $i$-*alternating subgraph* $H$ of a graph $G$ is a bipartite subgraph with parts $U$ and $W$ such that $d_H(u) = d_G(u) \leqslant i$ and $i \leqslant d_H(w) \leqslant d_G(w)$ for all $u \in U$ and $w \in W$. The left of Figure 5.8 shows a 4-alternating subgraph. A 3-*alternator* is a bipartite subgraph $H$ of a graph $G$ such that (i) $2 \leqslant d_H(u) = d_G(u) \leqslant 3$ for all $u \in U$ and (ii) for each $w \in W$ either (a) $w$ has at least three $H$-neighbors in $U$ or (b) $w$ has exactly two $H$-neighbors in $U$, both with degree exactly $14 - d_G(w)$. (Condition (ii.b) is only possible when $d_G(w) \in \{11, 12\}$.) The right of Figure 5.8 shows a 3-alternator.

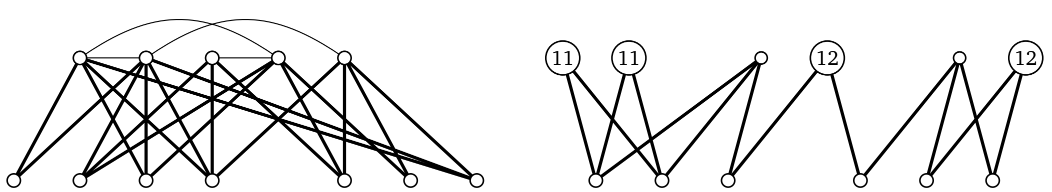



Figure 5.8: Left: A 4-alternating subgraph, with $U$ on bottom and $W$ on top. Thin edges are excluded from the subgraph. Vertices in $W$ may have more incident edges. Right: A 3-alternator, where labels prescribe degrees of vertices in the whole graph.

**Lemma 5.22.** *If* $G$ *is a minimal graph such that* $\chi'_\ell > \Delta$*, then* $G$ *contains neither (a) an edge of weight at most* $\Delta + 1$ *nor (b) an* $i$*-alternating subgraph. Further, if* $\Delta \geqslant 12$*, then* $G$ *also contains no* 3*-alternator.*

*Proof.* Let $G$ be a minimal graph (under subgraph inclusion) such that $\chi'_\ell > \Delta$, and let $L$ be an edge $\Delta$-assignment such that $G$ has no edge $L$-coloring.

(a) If $G$ contains an edge $e$ of weight at most $\Delta + 1$, then $G - e$ has an edge $L$-coloring $\varphi$, by minimality. Since edges incident to $e$ forbid at most $\Delta + 1 - 2$ colors, we can extend $\varphi$ to $e$, a contradiction. This proves (a).

(b) Instead suppose $G$ has an $i$-alternating subgraph $H$. By minimality, $G - E(H)$ has an edge $L$-coloring $\varphi$. We can easily check that each $uw \in E(H)$ has a list of remaining available colors of size at least $\max\{d(u), d(w)\}$ (this is a simpler version of our next case, so we omit the details). So Theorem 5.17 allows us to extend the coloring $\varphi$ to $E(H)$. This proves (b).

Finally, suppose $G$ contains a 3-alternator $H$. By minimality, $G - E(H)$ has an edge $L$-coloring $\varphi$. To extend $\varphi$ to $E(H)$, we again use Theorem 5.17. So we must check that each edge has enough available colors. Consider an edge $uw \in E(H)$. If $w$ satisfies (ii.a) in Definition 5.21, then edge $uw$ loses no colors to edges incident to $u$ and at most $d_G(w) - d_H(w)$ colors to edges incident to $w$, so $|L(uw)| \geqslant d_H(w) \geqslant 3 \geqslant d_H(u)$. Now assume instead that $w$ satisfies (ii.b). If $d_G(w) = 12$, then the same argument shows that $|L(uw)| \geqslant d_H(w) \geqslant 2 \geqslant d_H(u)$. Suppose instead that $d_G(w) = 11$. Now $|L(uw)| \geqslant \Delta - (d_G(w) - d_H(w)) \geqslant 12 - (11 - 2) = 3 = d_H(u) > d_H(w)$. So, again $|L(uw)|$ is big enough to apply Theorem 5.17. □

In view of Lemma 5.22, the following lemma will imply that $\chi'_\ell = \Delta$ for every simple planar graph with $\Delta \geqslant 12$. (For completeness, we give the details after proving the lemma.) We phrase the theorem to include multigraphs, since this simplifies the proof.

**Lemma 5.23.** *Let* $G$ *be a plane multigraph embedded such that every face has length at least* 3 *and no* 2*-vertex separates two* 3*-faces. If* $\delta(G) \geqslant 2$*, then* $G$ *contains (i) a* 2*-alternating cycle, (ii) a* 3*-alternator, or (iii) an edge* $uw$ *such that* $d(u) + d(w) \leqslant 13$*.*

We assume the theorem is false, and let $G$ be a counterexample that minimizes $|G|$ and, subject to that, maximizes $\|G\|$. To motivate the details of the proof, we first sketch the discharging argument. We use vertex charging.

Each $5^-$-vertex $v$ needs charge. Since $G$ has no edge of weight at most 13, we let $v$ take charge $\frac{1}{2}$ from each neighbor. This takes care of $4^+$-vertices, but 2-vertices and 3-vertices need more charge. For each 3-vertex $v$, we assign one $11^+$-vertex to "sponsor" $v$, by sending it 2 (rather than $\frac{1}{2}$). Similarly, for each 2-vertex $v$, we assign one $12^+$-vertex $v$ and one $4^+$-face to sponsor $v$, each of which send 2. The key is to assign these sponsors so that no $11^+$-vertex sponsors too many $3^-$-vertices. This is where the absence of 2-alternating subgraphs and 3-alternators will help. More generally, we need to show that $11^+$-vertices don't lose too much charge. The argument is simpler when the graph is a triangulation, or close to it, which is why we allow $G$ to be a multigraph and why we chose $G$ to maximize $\|G\|$.

*Proof.* We assume the theorem is false, and let $G$ be a counterexample that minimizes $|G|$ and, subject to that, maximizes $\|G\|$. For every 2-vertex $v$, its neighbors $w$ and $x$ must be adjacent, since otherwise we could add edge $wx$ near path $wvx$, contradicting the maximality of $\|G\|$.
H Form $H$ from $G$ by deleting all its 2-vertices. Now $H$ is a triangulation, as follows. Suppose to the contrary that $H$ has some $4^+$-face $f$. Choose $v, w, x$ consecutive along the boundary of $f$ so as to minimize $d_G(w)$. If $d_G(w) \leqslant 6$, then $d_G(v) \geqslant 8$ and $d_G(x) \geqslant 8$, so we can add the edge $vx$ while still satisfying the hypotheses of the theorem. This contradicts the maximality of $\|G\|$. Similarly, if $d_G(w) \geqslant 7$, then also $d_G(v) \geqslant 7$ and $d_G(x) \geqslant 7$, so we can again add edge $vx$. Thus, $H$ is indeed a triangulation.

Similarly, in $G$ each 2-vertex $v$ lies on a 3-face and on a 4-face or 5-face. This follows from the fact that $H$ is a triangulation, and that $G$ has no 2-alternating cycle. Specifically, each 3-face $f$ of $H$ contains at most two 2-vertices in $G$. And if $f$ contains exactly two 2-vertices, then they have distinct neighborhoods.

Now we assign sponsors to each 2-vertex and 3-vertex. Consider the subgraph $J$ induced by all edges incident to $3^-$-vertices. Note that $J$ is bipartite; call its parts $U$ and $W$, where all $3^-$-vertices are in $U$. Since $J$ is not a 3-alternating subgraph or a 3-alternator, by definition, some $w \in W$ has at most two neighbors in $U$, at most one of which has degree $14 - d_G(w)$. (An 11-vertex $w$ has no 2-neighbor by (iii). Thus, each 11-vertex $w$ has at most one neighbor $x$ in $J$; if $x$ exists, then $d_G(x) = 3$.) We assign $w$ to sponsor its neighbors in $U$. Form $J'$ from $J$ by deleting $w \cup N_U(w)$. Now $J'$ is also not a 3-alternating subgraph, so we can again find some vertex in $W$ and assign it to sponsor its at most two neighbors in $W \cap V(J')$. Repeating this process, we eventually assign each $u \in U$ a sponsor.

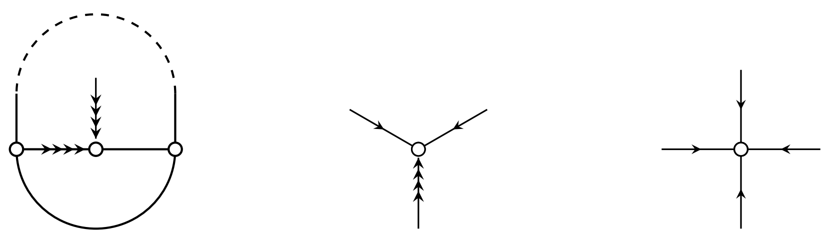
We use vertex charging and the following three discharging rules, shown in Figure 5.9.

(R1) Each 2-vertex takes 2 from its sponsor and 2 from its incident $4^+$-face.

(R2) Each 3-vertex takes 2 from its sponsor and $\frac{1}{2}$ from each other neighbor.

(R3) Each 4-vertex and 5-vertex takes $\frac{1}{2}$ from each neighbor.

We must show that all vertices and faces end happy.



(R1) (R2) (R3)

Figure 5.9: The three discharging rules in the proof of Lemma 5.23. Here $\longrightarrow$ denotes a charge of $1/2$ and $\twoheadrightarrow\!\!\twoheadrightarrow$ denotes a charge of $4/2 = 2$.

Each 3-face starts and ends with 0. Each 4-face has at most one incident 2-vertex, so it ends with at least $2(4)-6-2=0$. Each 5-face has at most two incident 2-vertices, so ends with at least $2(5)-6-2(2)=0$. The maximality of $\|G\|$ implies that $G$ has no $6^+$-faces.

Each 2-vertex gets 2 from its sponsor and 2 from its incident $4^+$-face, so ends with $2-6+2+2=0$. Each 3-vertex gets 2 from its sponsor and $\frac{1}{2}$ from each other neighbor, so ends with $3-6+2+2(\frac{1}{2})=0$. Each 4-vertex and 5-vertex get $\frac{1}{2}$ from each neighbor. So 4-vertices end with $4-6+4(\frac{1}{2})=0$ and 5-vertices end with $5-6+5(\frac{1}{2})>0$. Each 6-vertex starts and ends with 0. For $7\leqslant s\leqslant 10$, each $s$-vertex starts and ends with $s-6>0$.

Since $H$ is a triangulation, and each edge of $G$ has weight at least 14, each $11^+$-vertex in $G$ gives charge to at most half of its neighbors in $H$. So each 11-vertex ends with at least $11-6-2-(\frac{1}{2})5>0$. Each 12-vertex sponsors at most two $3^-$-vertices. And if exactly two, then one is a 3-vertex. So each 12-vertex ends with at least $12-6-2(2)-(\frac{1}{2})4=0$. Finally, for $s\geqslant 13$, each $s$-vertex ends with at least $s-6-2(2)-(\frac{1}{2})(\frac{1}{2}(s-2))=\frac{3}{4}s-9.5>0$. □

**Theorem 5.24.** *Every simple planar graph $G$ with $\Delta\geqslant 12$ satisfies $\chi'_\ell(G)=\Delta$.*

*Proof.* We prove a more general statement, which implies the theorem: Every simple planar graph $G$ satisfies $\chi'_\ell(G)\leqslant\max\{12,\Delta\}$. Suppose this statement is false. Choose a counterexample $G$ that minimizes $\|G\|$ and an edge $k$-assignment $L$ such that $G$ is not $L$-edge-colorable, where $k:=\max\{12,\Delta\}$. Lemma 5.22 implies that each edge has weight at least $\Delta+2$, so $\delta(G)\geqslant 2$. Since $G$ is simple, no 2-vertex separates two 3-faces. So $G$ satisfies the hypotheses of Lemma 5.23. Thus $G$ contains either a 2-alternating cycle, a 3-alternator, or an edge with weight at most 13. However, each of these subgraphs is reducible, by Lemma 5.22, which contradicts the minimality of $\|G\|$. □

### 5.4.3 Bounded Mad

Recall that an $i$-*alternating subgraph* is a bipartite subgraph $H$ with parts $U$ and $W$ such that $d_H(u)=d_G(u)\leqslant i$ for all $i\in U$ and $d_H(w)\geqslant i$ for all $w\in W$.

Next we generalize results from Section 5.4.2 to sparse graphs that need not be planar.

**Theorem 5.25.** *If $G$ is a graph with mad$(G)<\lfloor\sqrt{2\Delta}\rfloor$, then $G$ contains either (a) an edge $uw$ with $d(u)+d(w)\leqslant\Delta+1$ or (b) an $i$-alternating subgraph, for some $i\leqslant\frac{1}{2}\Delta$. Thus, if $G$ has mad$(G)<\lfloor\sqrt{2\Delta}\rfloor$, then $\chi'_\ell(G)=\Delta$.*

The second statement follows directly from the first, by Lemma 5.22. The proof of the first statement is similar to that of Theorem 5.23. The absence of (a) implies $\delta(G)\geqslant 2$. Since our hypothesis is in terms of $\text{mad}(G)$, we give each vertex $v$ initial charge $d(v)$. Let $r:=\lfloor\sqrt{2\Delta}\rfloor$. Since $\text{mad}(G)<r$, to reach a contradiction we discharge so that each vertex ends with at least $r$. As before, low degree vertices are assigned sponsors, which send them their needed charge.

The main difference is that now $r-\delta(G)$ may be unbounded. As a result, vertices of low degree need to receive lots of charge. Our solution is to use multiple rounds of discharging, rounds 2 through $r-1$. On round $i$, each $i^-$-vertex $v$ is assigned a sponsor that sends 1 to $v$. Thus, each $r^-$-vertex finishes with exactly $r$. The absence of $i$-alternating subgraphs allows us to assign sponsors so high degree vertices do not lose too much charge.

*Proof.* Assume the first statement is true. If the second is false, then a minimal counterexample contradicts either the first statement or Lemma 5.22. This proves the second.

Now we prove the first statement. We give each vertex $v$ initial charge $d(v)$, and we use $r$ multiple rounds of discharging. Let $r := \lfloor\sqrt{2\Delta}\rfloor$. On round $i$, for each $i \in [r-1]$, let $H_i$ denote the subgraph induced by edges incident to $i^-$-vertices, and let $U$ and $W$ be its parts. Since each edge has weight at least $\Delta+2$, graph $H_i$ is bipartite whenever $i \leqslant \frac{1}{2}\Delta$. By assumption, $G$ has no $i$-alternating subgraph. Since $d_{H_i}(u) = d_G(u) \leqslant i$ for each $u \in U$, some $w \in W$ has $d_{H_i}(w) \leqslant i-1$. Let $w$ send 1 to each neighbor in $U$. Now delete from $H_i$ vertex $w$ and its neighbors in $U$. By repeating this process, each vertex in $U$ receives 1, while each vertex in $W$ sends at most $i-1$.

By assumption, $d(u)+d(w) \geqslant \Delta+2$ for each edge $uw$. Thus, an $s$-vertex $v$ sends charge only on rounds $\Delta+2-s$ through $r-1$. An $(s+1)$-vertex has initial charge 1 more than a $s$-vertex, but on round $\Delta+1-s$ it may send an additional $\Delta-s$. So, it suffices to check that $\Delta$-vertices finish with enough charge. That is, we need $\Delta-(1+\ldots+(r-2)) \geqslant r$. This inequality is satisfied when $2\Delta-2 \geqslant r^2-r$, which holds because $r \leqslant \sqrt{2\Delta}$. □

### 5.4.4 The Borodin–Kostochka Conjecture for Line Graphs of Multigraphs

Borodin and Kostochka conjectured that if a graph $G$ satisfies $\omega(G) < \Delta$ and $\Delta \geqslant 9$, then $\chi(G) \leqslant \Delta-1$. To conclude this chapter, we prove the choosability analogue of the their conjecture for the class of line graphs (of multigraphs), when $\Delta$ is sufficiently large.

**Theorem 5.26.** *Let $G$ be the line graph of some multigraph $H$. If $\omega(G) < \Delta(G)$ and $\Delta(G) \geqslant 135$, then $\chi_\ell(G) \leqslant \Delta(G)-1$.*

Before proving this theorem we need a definition and an easy lemma.

$d_1$-choosable BK-free $f_J(v)$ **Definition 5.27.** A graph is $d_1$-*choosable* if it has an $L$-coloring whenever $|L(v)| = d(v)-1$ for all $v$. A connected graph $G$ is *BK-free* if it contains no induced subgraph $J$ that is $f_J$-choosable, where $f_J(v) := \Delta(G)-1-(d_G(v)-d_J(v))$ for all $v \in V(J)$. In particular, if $G$ is BK-free, then it contains no $d_1$-choosable subgraph.

Intuitively, $f_J(v)$ is a lower bound on the number of colors remaining available for $v$ if we start with lists of size $\Delta(G)-1$ and color all vertices in $G-V(J)$. This definition is motivated by the following lemma. (We note the similarities between the proof below and that of Lemma 10.8.)

**Lemma 5.28.** *If $G$ is a minimal counterexample to Theorem 5.26, then $G$ is BK-free. In particular, $\delta(G) \geqslant \Delta(G)-1$.*

*Proof.* The first statement implies the second, as follows. If any vertex $v$ has degree at most $\Delta(G)-2$, then by letting $J := G[\{v\}]$, we have $f_J(v) \geqslant \Delta(G)-1-(\Delta(G)-2-0) = 1$. Since $J$ is choosable from every non-empty list, we contradict that $G$ is BK-free.

To prove the first statement, suppose it is false. Let $G$ be a minimal counterexample to Theorem 5.26 and let $J$ be an induced subgraph of $G$ that is $f_J$-choosable, with $f_J$ as in Definition 5.27. Let $G' := G - V(J)$. Since $J$ is $f_J$-choosable, we can extend any coloring of $G'$ to $G$. So it suffices to color $G'$.

If $\Delta(G') = \Delta(G)$, then $G'$ is $(\Delta(G)-1)$-choosable, by the minimality of $G$. If $\Delta(G') = \Delta(G)-1$, then $\omega(G) \leqslant \Delta(G)-1 = \Delta(G')$, so $G'$ is $(\Delta(G)-1)$-choosable by the list-coloring version of Brooks' Theorem, that is, Theorem 1.37. Finally, if $\Delta(G') \leqslant \Delta(G)-2$, then $G'$ is $(\Delta(G)-1)$-choosable by greedy coloring. Thus, $G$ is $(\Delta(G)-1)$-choosable, contradicting that $G$ is a counterexample to Theorem 5.26. □

The key step in proving Theorem 5.26 is to show that $H$ is 8-degenerate, which implies that $\text{mad}(H) < 16$. We prove the theorem now under this assumption, and justify the assumption in the three lemmas that follow.

*Proof of Theorem 5.26.* Suppose the theorem is false, and let $G$ be a counterexample minimizing $\|G\|$. Choose $H$ such that $G$ is the line graph of $H$. If $\Delta(H) \geqslant 128$, then Theorem 5.25 (with Lemma 5.28) shows that $\chi'_\ell(H) = \Delta(H) \leqslant \omega(G) \leqslant \Delta(G)-1$. So we will show that $\Delta(G) \geqslant 135$ implies $\Delta(H) \geqslant 128$. Let $u$ be a vertex of minimum degree in $H$ and $w$ be a neighbor of $u$. Lemma 5.28 implies that $\Delta(G)-1 \leqslant \delta(G) \leqslant d_H(u)+d_H(w)-\mu_H(uw)-1 \leqslant d_H(u)+\Delta(H)-\mu_H(uw)-1$. Since $G$ is 8-degenerate, $d_H(u) \leqslant 8$ which gives $\Delta(H) \geqslant \Delta(G)-7 = 135-7 = 128$. □

By using more reducible configurations, and analyzing the argument more carefully, the value 135 can be reduced to 69. The main savings in the proof of this strengthened version come in showing that $H$ is actually 6-degenerate.

We first prove that $H$ is 8-degenerate under the assumption that $\mu(H) \leqslant 3$. We will justify this assumption soon, in Lemma 5.32.

**Lemma 5.29.** *Let* $G$ *be the line graph of some multigraph* $H$. *If* $\delta(H) \geqslant 9$ *and* $\mu(H) \leqslant 3$, *then* $G$ *is not BK-free. Thus, if* $G$ *is BK-free and* $\mu(H) \leqslant 3$, *then* $H$ *is* 8*-degenerate.*

*Proof.* The second statement is implied by the first, as follows. If $G$ is BK-free and $\mu(H) \leqslant 3$, but $H$ is not 8-degenerate, then some subgraph of $H$ has minimum degree at least 9, which contradicts the first statement of the lemma.

Now we prove the first statement. Suppose to the contrary that $\delta(H) \geqslant 9$. Let $A, B$ be a partition of $V(H)$ chosen to maximize $\|A, B\|$, the number of edges between $A$ and $B$, and let $Q$ be the subgraph induced by these edges. Now $d_Q(u) \geqslant \lceil d_H(u)/2 \rceil$ for all $u \in V(H)$, since otherwise moving $u$ to the other part increases $\|A, B\|$. Let $R$ denote the line graph of $Q$. To reach a contradiction, we apply Theorem 5.17 to show that $R$ is $f_R$-choosable, as in Definition 5.27. For each edge $uw$ in $Q$, it suffices to show that $\max\{d_Q(u), d_Q(w)\} \leqslant$

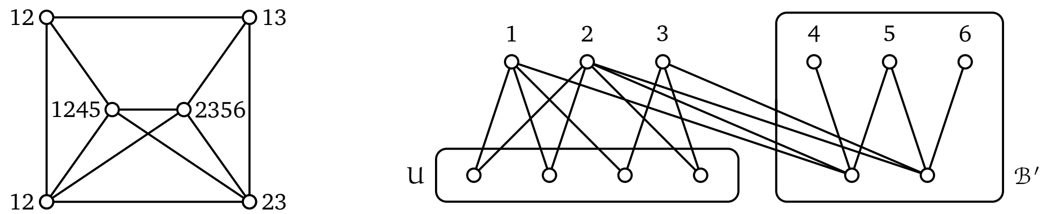



Figure 5.10: Left: A graph $G$ and list assignment $L$ for $G$, as in Lemma 5.30 (Small Pot Lemma). Right: The bipartite (vertex/color) incidence graph $\mathcal{B}(G, L)$, as well as the vertex subset $U$ and the subgraph $\mathcal{B}'$ of $\mathcal{B}(G, L)$.

$d_R(uw)-1 = d_Q(u)+d_Q(w)-2-(\mu(uw)-1)-1$. Since $\mu(H) \leqslant 3$, we have $d_R(uw)-1 \geqslant d_Q(u) + d_Q(w) - 5 \geqslant \max\{d_Q(u), d_Q(w)\}$ because $d_Q(u) \geqslant \lceil d_H(u)/2 \rceil \geqslant \lceil 9/2 \rceil = 5$, and similarly for $d_Q(w)$. $\square$

pot For a list assignment $L$, the *pot* is $\cup_{v \in V(G)} L(v)$, the set of all colors that appear in one or more lists. The Small Pot Lemma says that list-coloring is hardest when the pot has size less than $|G|$. This lemma is often useful when proving choosability results for graphs that are highly structured or that have small order. See Exercises 6 and 8.

**Lemma 5.30 (Small Pot Lemma).** *Let $G$ be a graph and $f : V(G) \to [|V(G)| - 1]$ be a list size assignment function. Now $G$ is $f$-choosable if and only if $G$ has an $L$-coloring for every list assignment $L$ such that $|L(v)| = f(v)$ for all $v \in V(G)$ and $| \cup_{v \in V(G)} L(v)| < |V(G)|$.*

$V$ *Proof.* Let $V := V(G)$. The "only if" direction is clear. Now we prove the "if" direction.

Let $L$ be a list assignment such that $|L(v)| = f(v)$ for all $v \in V$ and $| \cup_{v \in V} L(v)| \geqslant |V|$ and $G$ is not $L$-colorable. Assume that $G$ is $\tilde{L}$-colorable for each list assignment $\tilde{L}$ such that
$L(U)$ $|\tilde{L}(v)| = f(v)$ for all $v$ and $| \cup_{v \in V} \tilde{L}(v)| < |V|$. For every $U \subseteq V$, let $L(U) := \cup_{v \in U} L(v)$.

We construct a bipartite graph $\mathcal{B}$, where one part consists of vertices in $V$, the other part consists of colors in $L(V)$, and a vertex $v$ is adjacent to a color $\alpha$ if $\alpha \in L(v)$. For each $U \subseteq V$,
def$(U)$ let $\mathrm{def}(U) := |U| - |L(U)|$. Since $G$ is not $L$-colorable, $\mathcal{B}$ has no matching saturating $V$. So Hall's Theorem implies there exists a vertex subset $U$ with $\mathrm{def}(U) > 0$. Choose $U$ to maximize $\mathrm{def}(U)$. See Figure 5.10.

We construct an $f$-assignment $L'$ as follows. Let $A$ be an arbitrary set of $|V| - 1$ colors containing $L(U)$. For each $v \in U$, let $L'(v) := L(v)$. For each $v \notin U$, let $L'(v)$ be an arbitrary subset of $A$ of size $f(v)$. Now $|L'(V)| < |V|$, so by hypothesis $G$ has an $L'$-coloring. This gives an $L$-coloring of $U$. By the maximality of $\mathrm{def}(U)$, for all $W \subseteq (V \setminus U)$, we have $|L(W) \setminus L(U)| \geqslant |W|$. Let $\mathcal{B}' := \mathcal{B} \setminus (\cup_{u \in U} \{u\} \cup N_{\mathcal{B}}(u))$. Thus, by Hall's Theorem, $\mathcal{B}'$ has a matching saturating $V \setminus U$; so we can extend the $L$-coloring of $U$ to all of $V$. This contradicts that $G$ is not $L$-colorable. $\square$

$H_1 \vee H_2$ Let $H_1 \vee H_2$ denote the join of $H_1$ and $H_2$, formed from their disjoint union by adding all edges with one endpoint in each of $H_1$ and $H_2$.

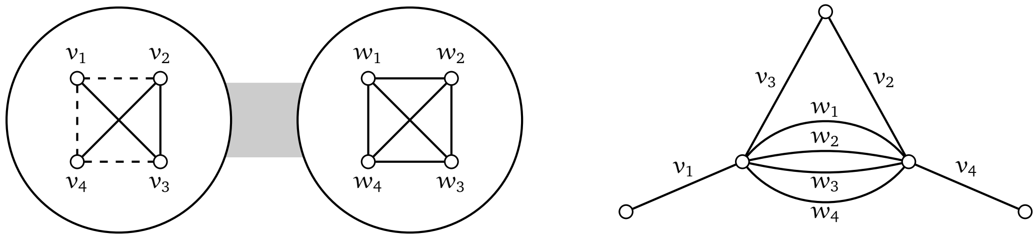



Figure 5.11: Left: The proof of Lemma 5.31 when S induces at least two edges. Right: The proof of Lemma 5.32 when $d_G(w) = \Delta(G)$; this corresponds to the case on the left.

**Lemma 5.31.** *If* $B$ *contains two disjoint pairs of non-adjacent vertices, then* $K_4 \vee B$ *is* $d_1$*-choosable.*

*Proof.* Suppose the lemma is false, and let $G$ and $L$ be a minimal counterexample, where $G := K_4 \vee B$ and $L$ is a $d_1$-assignment. By minimality[2], $|V(B)| = 4$. Let $v_1, \ldots, v_4$ denote the vertices of $B$ such that $v_1v_2, v_3v_4 \notin E(B)$. For short, let $S := V(B)$, let $T := V(K_4)$, and denote $T$ by $\{w_1, \ldots, w_4\}$. The left side of Figure 5.11 shows an example. $G, L$ $v_1, \ldots, v_4$ $S, T$ $w_1, \ldots, w_4$

By definition, $|L(z)| = d(z) - 1$ for all $z \in V(G)$; specifically, $|L(v_i)| = d_S(v_i) + 3$ and $|L(w_j)| = 6$ for all $v_i \in S$ and $w_j \in T$. When we have $i, j, k$ with $v_i \not\leftrightarrow v_j$ and $|L(v_i)|+|L(v_j)| > |L(w_k)|$, we often use the following technique, called *saving a color* on $w_k$ via $v_i$ and $v_j$. If there exists $c \in L(v_i) \cap L(v_j)$, then use $c$ on $v_i$ and $v_j$. Otherwise, color just one of $v_i$ and $v_j$ with some $c \in (L(v_i) \cup L(v_j)) \setminus L(w_k)$. For each $U \subseteq V(G)$, let $L(U) := \cup_{v \in U} L(v)$. By the Small Pot Lemma, assume that $|L(V(G))| \leqslant 7$. saving a color

Suppose $S$ induces at least two edges, so $|L(v_1)| + |L(v_2)| \geqslant 8$. Now $L(v_1) \cap L(v_2) \neq \emptyset$. Color $v_1$ and $v_2$ with a common color $c$. If $|L(w_1) \setminus \{c\}| \leqslant 5$, then save a color on $w_1$ via $v_3$ and $v_4$. Now finish greedily, ending with $w_1$.

Suppose instead that $S$ induces exactly one edge; by symmetry, say it is $v_1v_3$. Suppose that $L(v_1) \cap L(v_2) \neq \emptyset$. Similar to the previous argument, use a common color on $v_1$ and $v_2$, possibly save on $w_1$ via $v_3$ and $v_4$, then finish greedily. So instead, assume that $L(v_1) \cap L(v_2) = \emptyset$. Since $|L(V(G))| \leqslant 7$ and $L(v_1) \cap L(v_2) = \emptyset$, by symmetry (between $v_1$ and $v_3$ and also between $v_2$ and $v_4$), we may assume that $L(v_1) = L(v_3) = \{a, b, c, d\}$ and $L(v_2) = L(v_4) = \{e, f, g\}$. Also by symmetry, $a$ or $e$ is missing from $L(w_1)$. So color $v_1$ with $a$ and $v_2$ and $v_4$ with $e$ and $v_3$ arbitrarily; this saves one color on each $w_i$ and a second color on $w_1$. Now finish greedily, ending with $w_1$.

Assume instead that $G[S] = \overline{K_4}$. If a common color appears on 3 vertices of $S$, use it there, then finish greedily. If not, then by Pigeonhole, at least 5 colors appear on pairs of vertices in $S$; so, two colors appear on disjoint pairs. Color two such disjoint pairs, each with a common color. Now finish the coloring greedily. □

[2]Otherwise, let $B'$ denote the subgraph induced by the four vertices of the non-adjacent pairs. We can greedily $L$-color $B \setminus B'$. The resulting list assignment $L'$ for $K_4 \vee B'$ is a $d_1$-assignment. Since $B'$ is smaller than a minimal counterexample, we can extend the $L$-coloring of $B \setminus B'$ to $K_4 \vee B$.

**Lemma 5.32.** *If* $G$ *is BK-free with* $\omega(G) < \Delta(G)$ *and* $G$ *is the line graph of some multigraph* $H$*, then* $\mu(H) \leqslant 3$.

$e$ *Proof.* Suppose, to the contrary, that $H$ has some edge $e$ of multiplicity at least 4. Let $v_1 \in V(G)$
$v_1$ be a vertex corresponding to $e$. Lemma 5.28 implies $d_G(v_1) \in \{\Delta(G) - 1, \Delta(G)\}$.

**Case 1: $\mathbf{d_G(v_1) = \Delta(G) - 1}$.** Since $e$ has multiplicity at least 4, there exists some graph $B$ such that $G[N(v_1) \cup \{v_1\}] = K_4 \vee B$. Because $\omega(G) < \Delta(G)$, we know that $B$ is not a clique; in particular, there exist vertices $w_1, w_2 \in V(B)$ with $w_1 \not\leftrightarrow w_2$. Denote the vertex set of the $K_4$ by $\{v_1, v_2, v_3, v_4\}$. Let $W := \{v_1, v_2, v_3, v_4, w_1, w_2\}$ and $J := G[W]$, and note that $J \cong K_6 - e$. We show that $J$ is $f$-choosable, where $f(w_1) = f(w_2) = 3$, $f(v_2) = f(v_3) = f(v_4) = 4$, and $f(v_1) = 5$. Fix a list assignment $L$ with $|L(x)| = f(x)$ for each $x \in S$. By the Small Pot Lemma, we assume that $|\cup_{x \in S} L(x)| < |S| = 6$. By Pigeonhole, there exists a color $\alpha \in L(w_1) \cap L(w_2)$. After using $\alpha$ on $w_1$ and $w_2$, we color greedily in the order $v_4, v_3, v_2, v_1$.

**Case 2: $\mathbf{d_G(v_1) = \Delta(G)}$.** As above, there exists $B$ such that $G[N(v) \cup \{v\}] = K_4 \vee B$. Since $\omega(G) < \Delta(G)$, we conclude that $\omega(B) \leqslant |B| - 2$. Because $G$ is a line graph, $B$ has independence number 2. Thus, $B$ contains two disjoint pairs of non-adjacent vertices. Now Lemma 5.31 implies that $K_4 \vee B$ is $d_1$-choosable, which contradicts Lemma 5.28. □

## Notes

The Kernel Lemma was proved by Bondy, Boppana, and Siegel (see [20, Remark 2.4] and [170, Lemma 2.1]). Richardson's Theorem was proved in [342]. Lemma 5.4 holds more generally, when each vertex has its own prescribed bound on outdegree; see Exercise 1. The technique of reversing a directed path helps prove many results about orienting a graph subject to constraints on in-degrees and outdegrees, and is also useful when working with nowhere-zero flows.[3]

The Stable Matching Theorem is due to Gale and Shapley [168]. It has been used in numerous applications, such as matching organ donors with recipients and matching medical residents with residency programs. In 2012, Shapley (along with Alvin Roth) was awarded the Nobel prize in economics.[4] It is intriguing to study the maximum number of stable matchings admitted by fixed preference lists (for $n$ men and $n$ women); call this number $f(n)$. (See Exercise 4.) This problem was posed by Knuth in 1976 [263]. When $n$ is a power of 2, Irving and Leather [223] showed that $f(n) = \Omega(2.28^n)$. Thurber [387] extended their construction to prove the slightly weaker bound $f(n) = \Omega(2.28^n / c^{\log n})$, for all $n$. Trivially, $f(n) = O(n!)$. Despite significant effort, the first simply exponential upper bound was proved only in 2018. Karlin, Oveis Gharan, and Weber [238] proved $f(n) = O(2^{17n})$. In 2021, Palmer and Pálvölgyi [327] improved this upper bound to $f(n) \leqslant 3.55^n + O(1)$.

Theorem 5.11 is due to Galvin [170]. Theorem 5.13 is due to Cambie, Cames van Batenburg, Davies, and Kang [77], but the proof we present is due to Mudrock [314]. Theorem 5.14 is due to Peterson and Woodall [329, 330] and Theorem 5.15 is due to Trotter [388]. All results in

[3] For example, we reuse this idea in the proof of Theorem 6.10.
[4] Gale passed away in 2008, making him ineligible for the prize.

Sections 5.4.1, 5.4.2, and 5.4.3 are due to Borodin, Kostochka, and Woodall [69]. The results in Section 5.4.4 are due to Cranston and Rabern [102], except for the Small Pot Lemma; versions of it were proved independently by Kierstead [245] and Reed and Sudakov [341].

To simplify the presentation we stated our coloring results in terms of list-coloring, rather than paintability. However, the proofs yield the same bounds for paintability; see Exercise 9. The one exception is our proof of Lemma 5.31, which is valid only for list coloring. Nonetheless, the analogous statement for paintability is also true. This is proved directly in [101], and via the Alon–Tarsi Theorem in [102], using computers to verify that the numbers of even and odd Eulerian subgraphs differ.

Slivnik [363] gave a short self-contained proof of Theorem 5.11 that avoids the use of orientations and kernels. Its core ideas are essentially the same as those in Galvin's proof. But the presentation is streamlined, directly constructing the matching where each color is used (these matchings are indeed the kernels of the orientation of the line graph). In fact, this approach can be combined with Lemma 5.18 to give a shorter proof of Theorem 5.17.

## Exercises

5.1. Generalize Lemma 5.4 as follows. Fix a graph $G$, and for each vertex $v \in V(G)$, let $f(v)$ be a prescribed bound on the outdegree of $v$. Show that $G$ has an orientation $D$ with $d^+_D(v) \leqslant f(v)$ for all $v$ if and only if for each $S \subseteq V(G)$ we have $\sum_{v \in S} f(v) \geqslant \|G[S]\|$. This condition is obviously necessary, since each edge in $G[S]$ contributes to the outdegree of a vertex in $S$.

5.2. Run the Proposal Algorithm with the preference lists in Figure 5.3, but with the women proposing to the men.

5.3. Prove that the Proposal Algorithm yields a stable matching in which each man is at least as happy as he is in any other stable matching. [168]

5.4. (a) Construct preference lists (for $n$ men and $n$ women) that admit at least two distinct stable matchings. (b) Improve the lower bound in part (a) to $2^{n/2}$.

5.5. Consider the more general version of the stable matching problem, where women or men can leave some of the other sex "unranked". Given a set of preference lists (where every woman ranks every man and vice versa), show how the women, by working together, can shorten their lists so that each ends up as happy as she does in any stable matching (if the men don't modify their lists).

5.6. The graph $K_{2\star n}$ is complete $n$-partite with each part of size 2. Use the Small Pot Lemma to prove that $\chi_\ell(K_{2\star n}) = n$. [152]

5.7. (a) Show that the graph on the left in Figure 5.10 is $f$-choosable, where $f$ is given by the sizes of the lists in the picture (2 for 3-vertices and 4 for 4-vertices). (b) Show that if we form $f'$ by decreasing $f(x)$ for any $x \in V(G)$, then there exists an $f'$-assignment $L'$ such that $G$ has no $L'$-coloring.

5.8. Use the Small Pot Lemma to show that $\chi_\ell(K_{2,2,3}) = 3$, where $K_{2,2,3}$ is the complete tripartite graph with parts of sizes 2, 2, and 3.

5.9. (a) Adapt proofs in this chapter to work for paintability (assuming an analogous version of Lemma 5.31). (b) Prove the analogue of Lemma 5.31 for paintability.

5.10. A *total coloring* gives colors to both vertices and edges so that two elements must receive distinct colors whenever they are incident or adjacent. The *total choice number*, denoted $\chi''_\ell(G)$ is defined analogously. Adapt the proofs of Theorems 5.24 and 5.25 to give the bound $\chi''_\ell(G) \leqslant \Delta(G) + 1$ for the graphs in these theorems.

# Chapter 6

# Deletion and Contraction: Nowhere-Zero Flows

If you can't solve a problem, then there is an easier problem you can't solve: find it.

—attributed to George Pólya

In this chapter we generalize face coloring to graphs without faces. More precisely, we study "nowhere-zero flows" (see Definition 6.1) which are equivalent to face colorings for plane graphs, but which exist for many more graphs. Throughout this chapter we **always allow both loops and parallel edges**, unless stated otherwise.

## 6.1 Background

**Definition 6.1.** Fix a graph $G$, an orientation $D$ of $G$, and an abelian group $H$. We will mainly study the case when $H = \mathbb{Z}$ or $H = \mathbb{Z}_k$ for some integer $k$, but we start more generally. For each $W \subseteq V(G)$, let $\partial(W)$ denote those edges with exactly one endpoint in $W$; this is the *boundary* of $W$. We write $\partial^+(W)$ and $\partial^-(W)$ for the subsets of $\partial(W)$ with their tails and (respectively) heads in $W$. For each vertex $x$, we typically write $\partial(x)$, $\partial^+(x)$, and $\partial^-(x)$ rather than $\partial(\{x\})$, $\partial^+(\{x\})$, and $\partial^-(\{x\})$. An $H$-*flow* (or simply flow) on $G$ is a weight function $f : E(G) \to H$ such that "flow in" equals "flow out" at each vertex $w$; formally $\sum_{e\in\partial^-(w)} f(e) = \sum_{e\in\partial^+(w)} f(e)$. A *nowhere-zero* $H$-*flow* is an $H$-flow where $f(e) \neq 0$ for each edge $e$. A *nowhere-zero* $k$-*flow* is a $\mathbb{Z}$-flow where $0 < |f(e)| < k$ for each edge $e$. We often abbreviate nowhere-zero as *NZ*. For any map $f : E(G) \to H$ (not necessarily a flow), and $W \subseteq V(G)$, the *net flow* into $W$, denoted $\partial f(W)$, is equal to $\sum_{e\in\partial^-(W)} f(e) - \sum_{e\in\partial^+(W)} f(e)$. Figure 6.1 shows an NZ 4-flow.

$\partial(W)$

$\partial^+(W)$, $\partial^-(W)$

nowhere-zero $H$-flow

nowhere-zero $k$-flow

NZ

net flow, $\partial f$

**Observation 6.2.** If a graph $G$ has an NZ $H$-flow for some orientation, then it has one for every orientation $D$. This is because, given one NZ $H$-flow, we get one for $D$ by repeatedly reversing an edge and negating its flow value. This proof also works for NZ k-flows. Typically we assume that each graph G has a fixed orientation, D, but hereafter we will not say much about D.

**Observation 6.3.** For a graph G and NZ H-flow f, the net flow into any vertex set $W$ is 0. (That is, $\partial f(W) = \sum_{e\in\partial^-(W)} f(e) - \sum_{e\in\partial^+(W)} f(e) = 0$.) So if G has an NZ flow, then G has no cut-edge. The fact that $\partial f(V(G)) = 0$ is called the *zero-sum rule*.

zero-sum rule

*Proof.* For the first statement, we have

$$\sum_{e\in\partial^-(W)} f(e) - \sum_{e\in\partial^+(W)} f(e) = \sum_{w\in W}\left(\sum_{e\in\partial^-(w)} f(e) - \sum_{e\in\partial^+(w)} f(e)\right) = \sum_{w\in W} 0 = 0.$$

Here the first equality holds because edges with both endpoints in $W$ appear in both a positive term and a negative term in the second sum, so they cancel. For the second statement, suppose G has a cut-edge $e$, and let $W$ be the vertex set of one component of $G - e$. By the first statement, $\partial f(W) = 0$. But $\partial f(W) = f(e)$, which contradicts that f is nowhere-zero. □

bridge
bridgeless
cubic
dual graph $G^*$

**Definition 6.4.** A cut-edge is also called a *bridge*, and a graph with no bridge is *bridgeless*. (In view of Observation 6.3, we will only consider bridgeless graphs.) A graph is *cubic* if it is 3-regular. In a bridgeless plane graph G, a *face*-k-*coloring* assigns each face a color in [k] so that faces sharing an edge get distinct colors. The *dual graph* $G^*$ of a plane graph G has as its vertices the faces of G; two vertices of $G^*$ are joined by an edge for every edge shared by the boundaries of their corresponding faces in G. We can check that if G is a plane graph, then so is $G^*$, and that $(G^*)^* = G$.

The following observation is easy to verify.

**Observation 6.5.** A planar graph is face-k-colorable if and only if its planar dual is k-colorable.

(The 4 Color Theorem is often stated in terms of vertex coloring, but the original formulation was for face coloring.) Tutte was interested in generalizing face coloring to graphs that are non-planar. The following theorem shows that NZ flows accomplish this.

**Theorem 6.6.** *A planar graph* G *has an NZ* k*-flow if and only if it has a face-*k*-coloring.*

Given a face-k-coloring of G, the idea in the proof of Theorem 6.6 is to orient each edge so the larger of its two adjacent colors is on its right and to assign as its flow the difference of these two colors. Given an NZ k-flow, we can essentially reverse this process. However, verifying the details is a bit tedious and distracts us from the flow of this chapter, so we defer the proof to the appendix; see Theorem A.5.

Recall from Chapter 4 the 5 Color Theorem, 4 Color Theorem, and 3 Color Theorem (Grötzsch's Theorem). Every planar graph is 5-colorable, and proving this is easy. Every planar graph is also 4-colorable, but proving this is hard. Finally, every triangle-free planar graph is

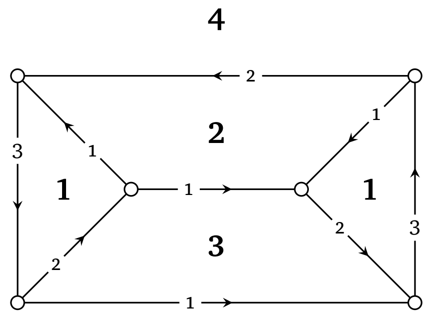



Figure 6.1: A face-4-coloring of a planar graph G and its associated NZ 4-flow.

3-colorable, and proving this is somewhere between easy and hard. To restate these results in terms of NZ flows, for a plane graph $G$, we apply the vertex coloring result to $G^*$, get a face-coloring of $G$ by Observation 6.5, and get an NZ flow by Theorem 6.6. Thus, every bridgeless planar graph $G$ has an NZ 5-flow and, in fact, has an NZ 4-flow. ($G$ must be bridgeless, since otherwise $G^*$ has a loop, and thus has no proper vertex coloring.) Further, every 4-edge-connected planar graph has an NZ 3-flow. Here $G$ must be 4-edge-connected[1], since any 3-edge-cut in $G$ becomes a 3-cycle in $G^*$. Aiming to generalize these results led Tutte to the following three conjectures.

**Conjecture 6.7 (5-Flow Conjecture).** *Every bridgeless graph has an NZ 5-flow.*

**Conjecture 6.8 (4-Flow Conjecture).** *Every bridgeless graph with no Petersen subdivision has an NZ 4-flow.*

**Conjecture 6.9 (3-Flow Conjecture).** *Every 4-edge-connected graph has an NZ 3-flow.*

Most of this chapter will study progress made on these conjectures. Both the 5-Flow Conjecture and the 4-Flow Conjecture are best possible, since the Petersen graph has no NZ 4-flow. To see this, recall that (a) the Petersen graph has no 3-edge-coloring (see Theorem A.1) and (b) a cubic graph has a 3-edge-coloring if and only if it has an NZ 4-flow (see Exercise 4). Since the Petersen graph is non-planar[2], the class of graphs with no Petersen subdivision properly contains the class of all planar graphs. By Theorem 6.6, the 4 Color Theorem is equivalent to the statement that every planar graph has an NZ 4-flow. Thus, the 4-Flow Conjecture strengthens the 4 Color Theorem.

The following theorem shows that looking for NZ $H$-flows reduces to looking for NZ $k$-flows. But it is also helpful in attacking Tutte's flow conjectures, since it allows us to choose our favorite $H$ with order $k$. For example, we will prefer to work with $\mathbb{Z}_2 \times \mathbb{Z}_2$ rather than $\mathbb{Z}_4$, and we will prefer $\mathbb{Z}_2 \times \mathbb{Z}_3$ rather than $\mathbb{Z}_6$.

[1]Actually, 2-edge-cuts are manageable, as we we prove in Lemma 6.17.
[2]Since the Petersen graph is cubic and has girth 5, this follows from Lemma 1.6.

**Theorem 6.10.** *For any graph* $G$ *and positive integer* $k$*, the following three statements are equivalent.*

*(i)* $G$ *has an NZ* $k$*-flow.*

*(ii)* $G$ *has an NZ* $\mathbb{Z}_k$*-flow.*

*(iii)* $G$ *has an NZ* $H$*-flow, for any finite abelian group* $H$ *of order* $k$*.*

$\Phi(G,H)$

$G/e$

contracting $e$

*Proof.* First we prove that (ii) and (iii) are equivalent. Fix two abelian groups $H_1$ and $H_2$, each of order $k$. For a graph $G$, let $\Phi(G,H)$ denote the number of NZ $H$-flows on $G$. By induction on $\|G\|$, we will prove that $\Phi(G,H_1) = \Phi(G,H_2)$. This implies that (ii) and (iii) are equivalent. (Recall that throughout this chapter we allow both loops and parallel edges.) For the base case, suppose that every edge of $G$ is a loop. Now $\Phi(G,H_1) = (k-1)^{\|G\|} = \Phi(G,H_2)$. Suppose instead that $G$ has a non-loop edge $e = vw$. Let $G/e$ denote the graph formed from $G$ by *contracting* $e$, that is identifying its endpoints, preserving any loops this creates.

Suppose we are given an $H$-flow $f$ of $G/e$. When we view $f$ as a flow on $G-e$, we have $\partial f(x) = 0$ for all $x \in V(G) \setminus \{v,w\}$. Further, $\partial f(v) = -\partial f(w)$. So if $\partial f(v) \neq 0$, then we can extend $f$ to an NZ $H$-flow of $G$ in exactly one way. And if $\partial f(v) = 0$, then $f$ is an NZ $H$-flow of $G-e$. This shows that $\Phi(G,H) = \Phi(G/e,H) - \Phi(G-e,H)$. Since $\|G/e\| = \|G-e\| < \|G\|$, by hypothesis $\Phi(G/e,H_1) = \Phi(G/e,H_2)$ and also $\Phi(G-e,H_1) = \Phi(G-e,H_2)$. Thus

$$\Phi(G,H_1) = \Phi(G/e,H_1) - \Phi(G-e,H_1) = \Phi(G/e,H_2) - \Phi(G-e,H_2) = \Phi(G,H_2).$$

So (ii) and (iii) are equivalent.

Now we show that (i) and (ii) are equivalent. To see that (i) implies (ii), given an NZ $k$-flow, we simply take the value of each flow modulo $k$. This gives an NZ $\mathbb{Z}_k$-flow. Now we show that (ii) implies (i). Suppose $f$ is an NZ $\mathbb{Z}_k$-flow. If we view the flow values from $f$ as elements of $\mathbb{Z}$, then $\partial f(v) \equiv 0 \bmod k$ for each $v \in V(G)$. We assume each flow value is positive, by possibly reversing edges and negating their values. Further, among all such possibilities for $f$, choose one to minimize the sum $\sum_{v \in V(G)} |\partial f(v)|$. If this sum is 0, then $\partial f(v) = 0$ for all $v \in V(G)$, so $f$ is an NZ $k$-flow. Assume instead that this sum is non-zero.

Since $\sum_{v \in V(G)} \partial f(v) = 0$, there exists a vertex $w$ such that $\partial f(w) < 0$. Let $W$ be the set of all vertices reachable from $w$ by a directed path (including $w$). Since each edge starting in $W$ also ends in $W$, we conclude that $\sum_{y \in W} \partial f(y) \geqslant 0$. Since $\partial f(w) < 0$, there exists $x \in W$ such that $\partial f(x) > 0$. By the definition of $W$, there exists a directed $w,x$-path $P$. For each edge $e$ on $P$, we reverse the direction of $e$ and change its flow to $k - f(e)$. Call these new flow values $f'$. For each vertex $y \in V(G) \setminus \{w,x\}$, we have $\partial f'(y) = \partial f(y)$. However $\partial f'(w) = \partial f(w) + k$ and $\partial f'(x) = \partial f(x) - k$. This contradicts our choice of $f$ to minimize $\sum_{v \in V(G)} |\partial f(v)|$. So when $f$ is minimal, we have $\partial f(y) = 0$ for all $y$, which means that $f$ is an NZ $k$-flow. This shows that (i) and (ii) are equivalent. $\square$

Our proof that (ii) and (iii) are equivalent actually shows something stronger.

**Corollary 6.11.** *For every multigraph* $G$*, there exists a polynomial* $\Phi_G$ *such that for every finite abelian group* $H$*, with order* $|H|$*, the number of NZ* $H$*-flows on* $G$ *is* $\Phi_G(|H|)$.

*Proof.* What we denote by $\Phi_G(|H|)$ here is what we called $\Phi(G, H)$ in the proof of the previous theorem. We already showed that $\Phi_G(H)$ depends only on $|H|$, rather than the structure of $H$. To see that $\Phi_G$ is always a polynomial, we note that it is in the base case, when $G$ has only loops, and that the induction step just takes the difference of two polynomials; thus, this difference is again a polynomial. □

Suppose a group $H$ is the direct product of groups $H_1$ and $H_2$; that is, $H = H_1 \oplus H_2$. If $f_1$ and $f_2$ are $H_1$- and $H_2$-flows on $G$, then $f := (f_1, f_2)$ is also an $H$-flow on $G$ (here $f(e) = (f_1(e), f_2(e))$ for each edge $e$. Further, if each $f_i$ is an NZ $H_i$-flow, then $(f_1, f_2)$ is also an NZ $H$-flow. But we can actually weaken our hypothesis to require only that for each edge $e$ either $f_1(e) \neq 0$ or $f_2(e) \neq 0$. This motivates the following lemma.

**Lemma 6.12.** *Let* $G$ *be a graph with* $t$ *spanning trees* $T_1, \ldots, T_t$*. If each* $e \in E(G)$ *is omitted from at least one* $T_i$*, then* $G$ *has an NZ* $\mathbb{Z}_2^t$*-flow. (Figure 6.2 shows an example.)*

*Proof.* For each $T_i$ we construct below a $\mathbb{Z}_2$-flow $f_i$ that is non-zero on $E(G) \setminus E(T_i)$. (Note that for a $\mathbb{Z}_2$-flow, orientation is irrelevant.) Our $\mathbb{Z}_2^t$-flow $f$ is formed by giving each edge $e$ the value $f(e) = (f_1(e), \ldots, f_t(e))$. Now $f$ is a $\mathbb{Z}_2^t$-flow because each $f_i$ is a $\mathbb{Z}_2$-flow, and $f$ is nowhere-zero because each edge $e$ is omitted from some $T_i$, so has $f_i(e) \neq 0$.

Let $T_i$ be a spanning tree of $G$, and let $v$ be a leaf of $T_i$, with $vw \in E(T_i)$. We use induction on $|G|$ to construct our $\mathbb{Z}_2$-flow that is non-zero on $E(G) \setminus E(T_i)$. Form $G'$ from $G$, and $T_i'$ from $T_i$, by contracting $vw$, preserving any loops or parallel edges this creates. By hypothesis, $G'$ has a $\mathbb{Z}_2$-flow $f'$ that is non-zero on $E(G') \setminus E(T_i')$. When we view $f'$ on $E(G)$, we have $\partial f'(x) \equiv 0 \pmod 2$ for all $x \in V(G) \setminus \{v, w\}$, and $\partial f'(v) \equiv \partial f'(w) \pmod 2$. If $\partial f'(v) \equiv 1 \pmod 2$, then we give $vw$ flow value 1; otherwise, we give $vw$ value 0. Call this new flow $f$. Note that $\partial f(x) \equiv 0 \pmod 2$ for all $x \in V(G)$, as desired. □

To apply Lemma 6.12, we prefer that our graph $G$ have edge-disjoint spanning trees. To ensure this we use Corollary 6.14, which follows easily from the so-called Tree-Packing Theorem of Tutte and (independently) Nash-Williams. We defer its proof to the appendix (Theorem A.11). Given a graph $G$ and a partition $\mathcal{P}$ of $V(G)$, let $E_{\mathcal{P}}$ denote the set of edges with their endpoints in distinct parts of the partition. $\mathcal{P}$, $E_{\mathcal{P}}$

**Theorem 6.13 (Tree-Packing Theorem).** *A graph* $G$ *has* $t$ *edge-disjoint spanning trees if and only if for every partition* $\mathcal{P}$ *of* $V(G)$*, we have* $|E_{\mathcal{P}}| \geqslant t(|\mathcal{P}| - 1)$.

The condition $|E_{\mathcal{P}}| \geqslant t(|\mathcal{P}| - 1)$ is clearly necessary, as follows. Form $G_{\mathcal{P}}$ from $G$ by contracting the vertices in each part of $\mathcal{P}$ to a single vertex (deleting loops and suppressing multiple edges). Each spanning tree in $G$ contracts to a spanning tree in $G_{\mathcal{P}}$, possibly with extra edges, so has at least $|G_{\mathcal{P}}| - 1 = |\mathcal{P}| - 1$ edges. Since the $t$ spanning trees in $G$ are edge-disjoint, so are their contractions in $G_{\mathcal{P}}$. This means that $|E_{\mathcal{P}}| = \|G_{\mathcal{P}}\| \geqslant t(|\mathcal{P}| - 1)$. So the hard part is showing that this necessary condition is also sufficient; see Theorem A.11.

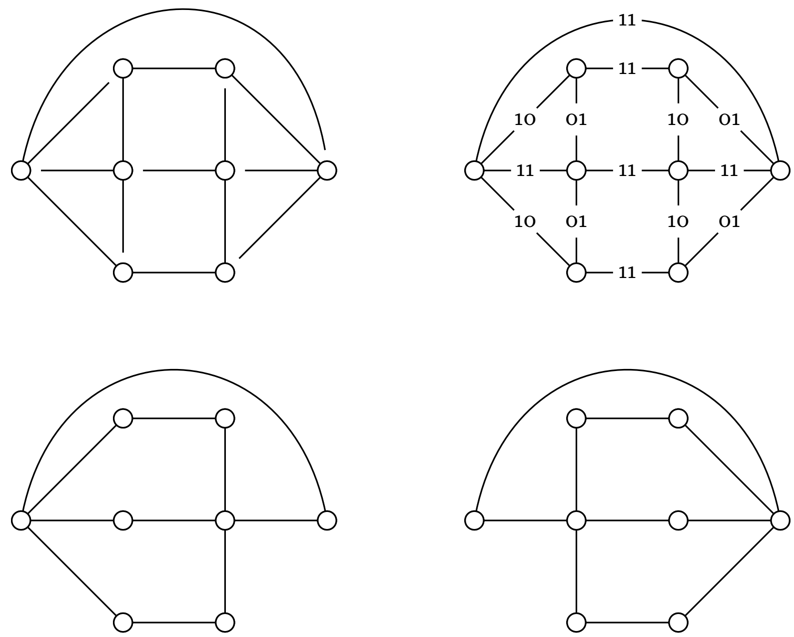



Figure 6.2: Top left: An 8-vertex graph with 14 edges, and a decomposition into two edge-disjoint spanning trees. Bottom left and right: $\mathbb{Z}_2$-flows that are non-zero on all edges outside of one of these two spanning trees. (We depict each $\mathbb{Z}_2$-flow as the subgraph induced by edges with flow value 1.) Top right: An NZ $\mathbb{Z}_2^2$-flow given by these $\mathbb{Z}_2$-flows.

**Corollary 6.14.** *Every* $2t$*-edge-connected graph* $G$ *has* $t$ *edge-disjoint spanning trees.*

*Proof.* Fix a partition $\mathcal{P}$ of $V(G)$ with parts $\mathcal{P}_1, \ldots, \mathcal{P}_s$. Let $d(\mathcal{P}_i)$ be the number of edges with exactly one endpoint in $\mathcal{P}_i$. Since G is 2t-edge-connected, $d(\mathcal{P}_i) \geqslant 2t$ for every $i$. So

$$|E_{\mathcal{P}}| = \frac{1}{2}\sum_{i=1}^{s} d(\mathcal{P}_i) \geqslant \frac{1}{2}s(2t) = st = t|\mathcal{P}|.$$

Now the result follows from Theorem 6.13. □

**Theorem 6.15.** *Every* 4*-edge-connected graph has an NZ* 4*-flow.*

*Proof.* By Corollary 6.14, G has two edge-disjoint spanning trees. By Lemma 6.12, with $t = 2$, G has an NZ $\mathbb{Z}_2^2$-flow. So Theorem 6.10 implies the result. □

Corollary 6.15 is due to Jaeger. He also used the same approach to show that every bridgeless graph has an NZ 8-flow (equivalently, by Theorem 6.10, an NZ $\mathbb{Z}_2^3$-flow). First he showed that it suffices to consider 3-edge-connected graphs, which we prove in Lemma 6.17. Given such a G, we form 2G by replacing each edge with two parallel edges. Since 2G is 6-edge-connected, it contains three edge-disjoint spanning trees. Each edge of G is omitted from at least one of these trees. So Lemma 6.12 gives the desired NZ $\mathbb{Z}_2^3$-flow.

## 6.2 The Nowhere-Zero 6-Flow Theorem

In this section we prove that every bridgeless graph has an NZ 6-flow. Before embarking on that journey we need some preparation.

When a vertex $v$ has $d(v) \geqslant 4$, we will often *lift* two of its incident edges. That is, if $wv, vx \in E(G)$, then we delete $wv$ and $vx$ and add the new edge $wx$. Call this new graph $G'$. (If $d_{G'}(v) = 2$, then we also lift its two remaining edges, and delete $v$ altogether.) lift

Given any NZ k-flow in $G'$, we get an NZ k-flow in G by assigning both $wv$ and $vx$ the value of $wx$ in $G'$. Since $|G'| < |G|$, we often can proceed by induction. Because our theorems frequently require certain edge-connectivity, we want to lift edges in a way that preserves this. We use the following lemma of Mader. Its proof is long and subtle, so we defer it to the appendix: Theorem A.16.

**Lemma 6.16 (Mader's Splitting Off Theorem).** *Suppose that $v$ is a non-cut-vertex in a graph $G$, and $d(v) \geqslant 4$. We can lift some pair of edges incident to $v$, so that for every pair of distinct vertices $w, x \in V(G) \setminus \{v\}$, the maximum number of edge-disjoint $w, x$-paths does not decrease.*

Next we show that it suffices to prove our main result for all 3-connected cubic graphs.

**Lemma 6.17.** *Fix $k \geqslant 3$. If there exists a bridgeless graph $G$ with no NZ $k$-flow, then, when we take $G$ to minimize $|G| + \|G\|$, the graph $G$ is simple, cubic, and 3-connected.*

It is easy to check that $K_4$ has no NZ 3-flow, which verifies the lemma for $k = 3$. For $k = 4$, recall[3] that the Petersen graph has no NZ 4-flow. However, it requires more work to check that the Petersen graph minimizes $|G| + \|G\|$.

*Proof.* Let G be bridgeless with no NZ k-flow and, subject to that, $|G| + \|G\|$ is minimum.

Suppose G has a 2-edge-cut $\{e_1, e_2\}$. By minimality, $G/e_1$ has an NZ k-flow $f$. As in the proof of Theorem 6.10 this corresponds to an NZ k-flow either on G or on $G - e_1$. But in $G - e_1$ edge $e_2$ is a bridge, so no NZ k-flow exists. Thus, $f$ gives an NZ k-flow on G, a contradiction. So G is 3-edge-connected. (In particular, $\delta(G) \geqslant 3$.)

Similarly, suppose G has parallel edges, $e_1$ and $e_2$, and let $G' := G - e_1$. Since $G'$ is 2-edge-connected, by minimality, $G'$ has an NZ k-flow $f_1$. Let $f_2$ be an NZ 2-flow on the subgraph induced by $e_1, e_2$ (with $e_2$ oriented as in $f_1$). Now either $f_1 - f_2$ or $f_1 + f_2$ is an NZ k-flow of G. Thus, G is simple.

[3]We sketch a proof of this in the paragraph following Conjecture 6.9.

Suppose there exists $v \in V(G)$ with $d(v) \geqslant 4$. By Lemma 6.16, we can form a new graph $G'$ by lifting two edges incident to $v$ so that for all distinct $w, x \in V(G) - v$, the maximum number of edge-disjoint $w, x$-paths does not decrease. So if $G'$ has a bridge, $e$, then $e$ separates $v$ from $V(G) - v$. But this is impossible, since $d_{G'}(v) \geqslant d_G(v) - 2 \geqslant 2$. Thus, $G'$ is bridgeless. But now $G'$ contradicts the minimality of $G$. So $G$ must be cubic.

If $v$ has a cut-vertex $v$, then some edge incident to $v$ is a bridge, since $d(v) = 3$. So $G$ is 2-connected. Suppose $G$ has a cut-set $\{v_1, v_2\}$. For each $v_i$ some component of $G - \{v_1, v_2\}$ has in $G$ only a single incident edge $e_i$ that is also incident to $v_i$. But now $\{e_1, e_2\}$ is a 2-edge-cut, contradicting above. Thus, $G$ is 3-connected. □

Now we can prove that every bridgeless graph $G$ has an NZ 6-flow. Before presenting the details, we outline and motivate our approach. By Lemma 6.17, it suffices to consider 3-connected, cubic graphs. And by Theorem 6.10, it is equivalent to show that every cubic graph has an NZ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow. Similar to how we proved Lemma 6.12, we will find a $\mathbb{Z}_2$-flow $f_2$ and a $\mathbb{Z}_3$-flow $f_3$ such that every edge $e \in E(G)$ has $(f_2(e), f_3(e)) \neq (0, 0)$. Our proof will use induction on $\|G\|$. Since deleting edges makes the graph no longer cubic, we consider the larger class of 2-edge-connected *subcubic* graphs, those with $\Delta \leqslant 3$. Suppose we find an NZ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow in $G - e$ for some edge $e$. If we add a non-zero flow on $e$, this results in a non-zero net flow into each endpoint of $e$, which is not what we want. So instead we consider a larger class of pseudo-flows.

subcubic

pseudo-flow

Similar to flows, a *pseudo-flow* orients each edge $e$ of a subcubic graph $G$ and prescribes a flow value $f(e)$, so that the net flow into each 3-vertex is 0. The difference from flows is that now each 2-vertex may have non-zero net flow. This has the following benefit. When we delete an edge $e$, we pick $f_3(e) \in \{1, 2\}$ and in $G - e$ we require a pseudo-flow with prescribed net flows in $f_3$ at both endpoints of $e$, so that when we restore $e$, with flow $f_3(e)$, the net flows into its endpoints each become 0. Something similar works if we delete a 2-vertex that has two 3-neighbors. An unusual feature of the induction is that when we delete an edge $e$, we pick the value of $f_3(e)$ *before* invoking the induction hypothesis, but pick $f_2(e)$ *afterward*.

To proceed by induction, we must ensure that our smaller graph $G'$ is also 2-edge-connected. To guarantee this, whenever $G$ has a non-trivial 2-edge-cut $\partial(W)$ we use a different induction step. We form $G_W$ from $G$ by contracting $W$ to a single new vertex $w$. Similarly, we contract $\overline{W}$ to get $G_{\overline{W}}$ with a new vertex $\overline{w}$.[4] We want to find good pseudo-flows for $G_W$ and $G_{\overline{W}}$ and take their union to get a pseudo-flow for $G$. For this to work, these pseudo-flows must agree on $\partial W$; see Figure 6.4. This motivates our final wrinkle, which is allowing a single 2-vertex $z$ to have flow values prescribed on its incident edges.[5] By symmetry, we assume that $z \in \overline{W}$. By induction we get a good pseudo-flow for $G_W$. Now in $G_{\overline{W}}$ we take $\overline{w}$ to be the new instance of $z$, and prescribe the flow values on its incident edges (which are the edges of $\partial W$). This allows us to merge the pseudo-flows for $G_W$ and $G_{\overline{W}}$, as desired.

[4]These contractions are similar to those we used when proving Hadwiger's Conjecture for line graphs of multigraphs (Theorem 3.16) and also when proving Menger's Theorem (Theorem A.6).

[5]This is similar to the precoloring we use to prove Theorems 4.5 and 4.26. Also, see Chapter 11.

To formalize the outline above, we use the following technical lemma.

**Lemma 6.18.** *Let $G$ be a directed (loopless) 2-edge-connected subcubic graph, with a specified 2-vertex $z$ as its root. Fix functions $\mu : V(G) \to \mathbb{Z}_3$ and $f_k^*(z) : \partial(z) \to \mathbb{Z}_k$, for each $k \in \{2, 3\}$. If the properties (i)–(v) below hold, then we can extend $f_k^*$ to $f_k : E(G) \to \mathbb{Z}_k$, for each $k \in \{2, 3\}$ that satisfies properties (1)–(4) below.*

*(i)* $\sum_{v \in V(G)} \mu(v) \equiv 0 \pmod 3$.

*(ii)* $\mu(v) = 0$ *for each* $3$*-vertex* $v$.

*(iii) The net flow into $z$ specified by $f_3^*$ equals $\mu(z)$.*

*(iv) If $\mu(z) = 0$, then the net flow into $z$ specified by $f_2^*$ equals 0.*

*(v)* $(f_2(e), f_3(e)) \neq (0, 0)$ *for each* $e \in \partial(z)$.

*The four guaranteed properties of $f_2$ and $f_3$ are as follows.*

*(1)* $f_k(e) = f_k^*(e)$ *for each* $e \in \partial(z)$ *and each* $k \in \{2, 3\}$.

*(2) For each $v \in V(G)$, the net flow into $v$ from $f_3$ equals $\mu(v)$.*

*(3) For each $v \in V(G)$, if $\mu(v) = 0$, then the net flow into $v$ from $f_2$ equals 0. (In particular, this is true for every $3$-vertex.)*

*(4)* $(f_2(e), f_3(e)) \neq (0, 0)$ *for each* $e \in E(G)$.

Before proving the lemma, we use it to prove the NZ 6-flow theorem.

**Theorem 6.19 (6-Flow Theorem).** *Every bridgeless graph has an NZ 6-flow.*

*Proof.* Figure 6.3 shows an example. By Lemma 6.17, it suffices to consider cubic graphs. And by Theorem 6.10, it is equivalent to show that every cubic graph has an NZ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow. Let $G$ be a bridgeless cubic graph, and form $G'$ from $G$ by subdividing a single edge, and call this new 2-vertex $z$. Orient $G'$ arbitrarily, let $\mu$ be identically 0, and let $f_2(e) = 1$ for each $e \in \partial(z)$. Also fix $f_3(e)$ for each $e \in \partial(z)$ so that $z$ has net flow 0 in $f_3$. It is easy to check that $G'$ is 2-edge-connected and $\mu$, $f_2^*$, $f_3^*$ satisfy properties (i)–(v). By Lemma 6.18, $G'$ has pseudo-flows $f_2$ and $f_3$ satisfying (1)–(4).

By possibly reversing one edge (and negating its flow values), we assume that $z$ has indegree 1, so both of its incident edges have the same flow value. By suppressing $z$ we get a $\mathbb{Z}_2 \times \mathbb{Z}_3$-pseudoflow $f$ for $G$. Property (4) ensures that $f(e) \neq (0, 0)$ for each edge $e$, so $f$ is nowhere zero. And property (3) ensures that each 3-vertex has net flow 0. Thus, $f$ is an NZ $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow for $G$, as desired. $\square$

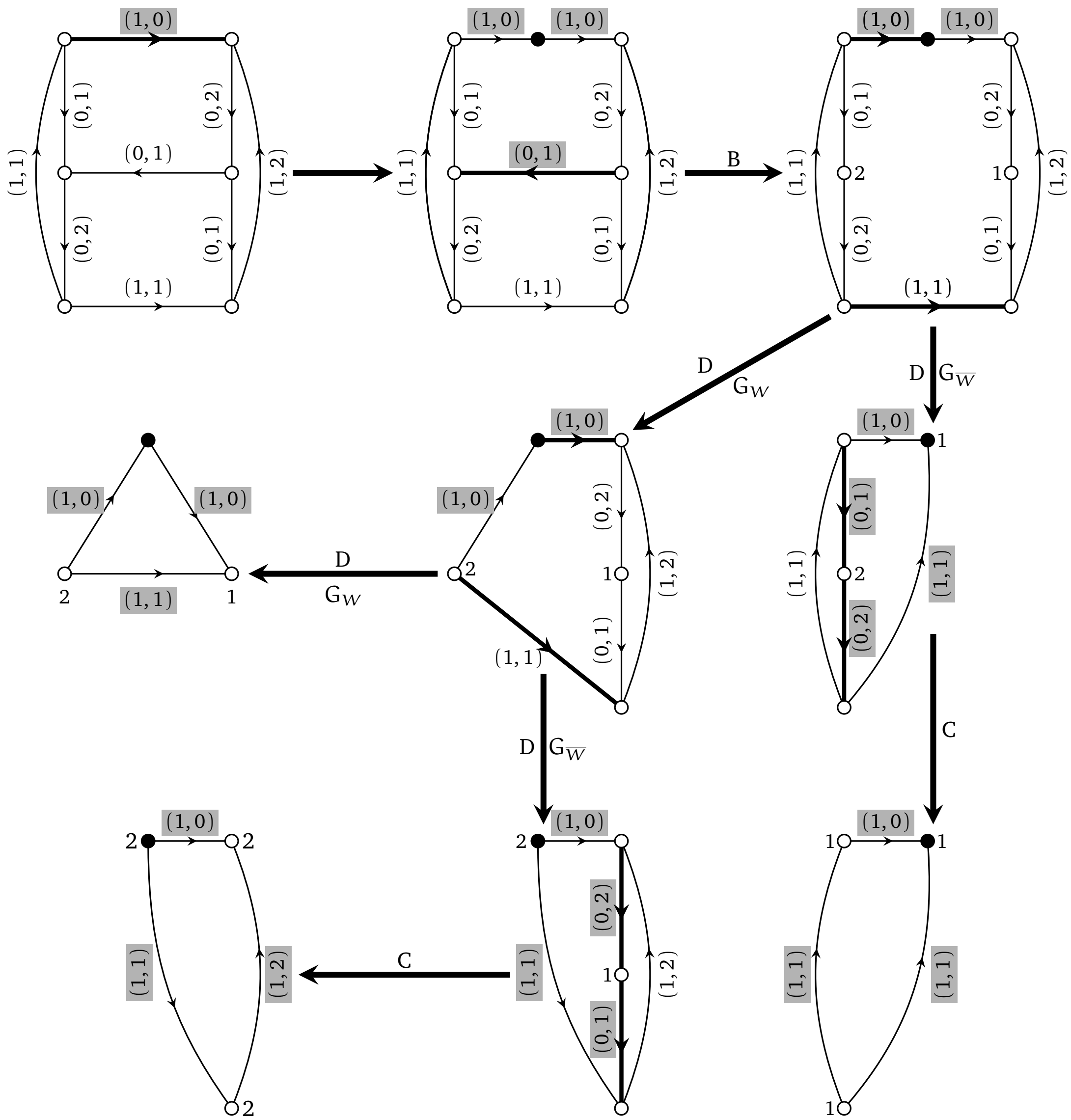



Figure 6.3: An example of the proof of Theorem 6.19, that every bridgeless graph has an NZ 6-flow. The diagrams should be read using a depth-first search (branching left), from top left. Flow values that are highlighted in gray are known the first time a diagram is reached; if a flow value is highlighted for the first time when an edge is bold, and that bold edge is deleted in the next diagram, then only the second coordinate of that flow value is known initially. Flow values that are not highlighted are only known when we return to the diagram after visiting more diagrams.

The value $\mu(v)$ in Lemma 6.18 should be viewed as capturing the flow into each 2-vertex $v$ that will be added later, when we restore an edge incident to $v$ that we previously deleted. (In particular, $\mu(v) = 0$ whenever $d(v) = 3$.) Now we prove the lemma.

*Proof of Lemma 6.18.* Let $G$ be a loopless, 2-edge-connected subcubic graph, and $\mu$, $f_2^*$, $f_3^*$ satisfy the hypotheses of the lemma. We use induction on $\|G\|$. We first present the induction steps, of which we have four, since they are more interesting. We defer to the end the base cases, which are a bit tedious to verify.

(A) Suppose $\mu(v) = 0$ for some 2-vertex $v$ with $v \neq z$, and $|G| \geqslant 3$. We form $G'$ from $G$ by contracting some edge $e$ incident to $v$ (but not $z$), and get the desired $\mathbb{Z}_2 \times \mathbb{Z}_3$ pseudo-flow for $G'$. When we uncontract $e$ we give it the same flow values as the other edge incident to $v$. By (iii) and (iv), this gives the desired $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow for $G$. Henceforth we assume that $\mu(v) \neq 0$ for every 2-vertex other than $z$.

(B) Suppose $G$ has no 2-vertex other than $z$, so $\mu$ is identically 0. Let $G' := G - v_1v_2$, for some edge $v_1v_2 \in E(G-z)$. Note that $G'$ is 2-edge-connected, since $G$ has only the single trivial 2-edge-cut $\partial(z)$, and $v_1v_2 \notin \partial(z)$, since $d(v_1) = d(v_2) = 3$. Let $\mu'(v_1) := 1$, $\mu'(v_2) := -1$, and $\mu'(x) := \mu(x)$ for all $x \in V(G) \setminus \{v_1, v_2\}$. By induction on $G'$, with $\mu'$, $f_2^*$, and $f_3^*$, there exists a pseudo-flow $(f_2', f_3')$ satisfying properties (1)–(4). To extend $(f_2', f_3')$ to $G$, we pick $f_3(v_1v_2) \in \{1, 2\}$ and $f_2(v_1v_2) \in \{0, 1\}$ so that $\partial f_k(v_i) = 0$ for each $k \in \{2, 3\}$ and $i \in \{1, 2\}$. (This is possible by (2) and (3), since $\mu$ is identically 0 in $G$.)

(C) Suppose $G$ has a 2-vertex $v$, distinct from $z$, and denote its neighbors by $v_1$ and $v_2$; suppose also that $G$ has no non-trivial 2-edge cut. (We can further assume that $vv_1$ and $vv_2$ are both oriented away from $v$.) Pick $\hat{f}_3 : \partial(v) \to \{1, 2\}$ so that $\partial\hat{f}_3(v) \equiv \mu(v) \pmod 3$. Let $\mu'(v_i) := \mu(v_i) - \hat{f}_3(vv_i)$, and $\mu'(x) := \mu(x)$ for all $x \in V(G) \setminus \{v, v_1, v_2\}$. Let $G' := G - v$. Since $G$ has only trivial 2-edge-cuts, $G'$ is 2-edge-connected. By induction on $G'$, with $\mu'$, $f_2^*$, and $f_3^*$, there exists a pseudo-flow $(f_2', f_3')$ for $G'$ satisfying properties (1)–(4). To extend this to the desired pseudo-flow for $G$, for each $i \in [2]$, let $f_3(vv_i) := \hat{f}_3(vv_i)$ and let $f_2(vv_i) := \partial f_2'(v_i)$.

(D) Suppose $G$ has a non-trivial 2-edge-cut $\partial(W)$, as in Figure 6.4. By symmetry we assume $z \in \overline{W}$. Form $G_W$ from $G$ by contracting $W$ to a single vertex $w$, and let $\mu(w) := \sum_{v \in W} \mu(v)$, $w, \overline{w}$
and all other vertices inherit $\mu$ from $G$. Form $G_{\overline{W}}$, and its $\mu$ analogously. By induction, $G_W$ $G_W, G_{\overline{W}}$
has a pseudo-flow $(f_2', f_3')$ satisfying properties (1)–(4). For $G_{\overline{W}}$, let $\overline{w}$ be the new $z$ (since the $f_2', f_3'$
original $z$ is contracted away) and let $f_k''^*$ specify values on $\partial(W)$ to agree with $f_k'$. We must show that $G_{\overline{W}}$ with $f_k''^*$ satisfies hypotheses (i)–(v).

This is clear for all but hypothesis (iv), and (iv) holds trivially if $\mu(\overline{w}) \neq 0$. So suppose $\mu(\overline{w}) = 0$. Hypothesis (i) implies that $\mu(w) \equiv -\mu(\overline{w}) \pmod 3$; so $\mu(w) = 0$. Thus, if $\mu(\overline{w}) = 0$, then $\mu(w) = 0$. By (3) for $G_W$ the net flow into $w$ from $f_2'$ is 0. But $\partial_{G_{\overline{W}}}(\overline{w}) = \partial_G(W) = \partial_{G_W}(w)$, so the net flow into $\overline{w}$ by $f_2''^*$ is also 0. That is, (iv) holds for $G_{\overline{W}}$. By induction, $G_{\overline{W}}$ has a pseudo-flow $(f_2'', f_3'')$ satisfying properties (1)–(4). Since $(f_2', f_3')$ and $(f_2'', f_3'')$ agree on $\partial W$, they combine to give the desired pseudo-flow for $G$.

Now we consider the base cases. Each of (A), (B), and (C) decreases $\|G\|$ by at most 2 before its recursive call. And (D) may decrease $\|G\|$ arbitrarily much, but ensures that $\|G_W\| \geqslant 3$ and

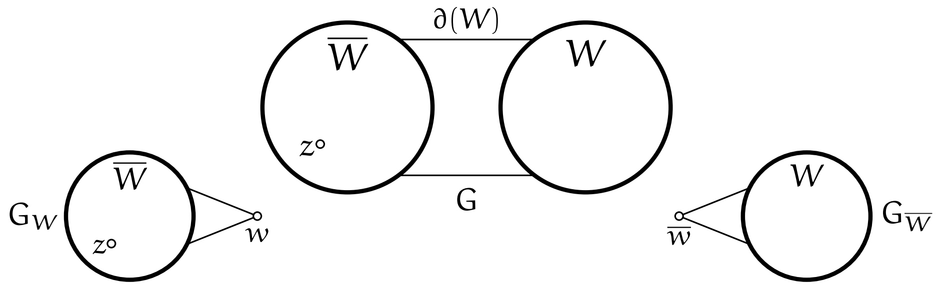



Figure 6.4: A non-trivial 2-edge-cut $\partial(W)$ and the graphs $G_W$ and $G_{\overline{W}}$ formed from G by contracting $W$ and $\overline{W}$ (respectively).

$\|G_{\overline{W}}\| \geqslant 3$. Thus, for the base cases, we may assume that $\|G\| \in \{2, 3\}$.

Suppose $\|G\| = 2$, and $V(G) = \{v, z\}$. Since each edge of G is in $\partial(z)$, we simply let $f_k := f_k^*$ for each $k \in \{2, 3\}$. Now (1) is immediate, and hypothesis (v) implies (4). By the zero-sum rule: $\partial f_k(v) = -\partial f_k(z)$, for each $k \in \{2, 3\}$. So properties (2) and (3) follow from (iii) and (iv), together with (i).

Now suppose instead that $\|G\| = 3$, and $V(G) = \{v, w, z\}$. All flows are prescribed by $f_k^*$ except for those on edge $vw$. Choose $f_3(vw)$ so that $\partial f_3(v) = \mu(v)$ and $\partial f_3(w) = \mu(w)$. This is possible by (i) and the zero-sum rule. Let $f_2(vw) = 1$, which ensures that $f(vw) \neq (0, 0)$. This choice cannot violate (3), since (A) implies that $\mu(v) \neq 0$ and $\mu(w) \neq 0$. □

## 6.3 Exponentially Many Nowhere-Zero $\mathbb{Z}_k$-flows

In Theorem 6.15 we proved that every 4-edge-connected graph has an NZ $\mathbb{Z}_4$-flow. In Theorem 6.19 we proved that every 2-edge-connected graph has an NZ $\mathbb{Z}_6$-flow. In this section we prove that such graphs have exponentially many NZ flows. But we must be careful. A cycle of any length has only 5 NZ $\mathbb{Z}_6$-flows. More generally, subdividing an edge does not change the number of NZ flows. So, if we want exponentially many NZ $\mathbb{Z}_6$-flows in a 2-edge-connected graph, then this count must be exponential in something other than the graph's order. This motivates the following 2 results.

**Theorem 6.20.** *If* G *is* 4*-edge-connected, then it has at least* $2^{|G|/3}$ *NZ* $\mathbb{Z}_4$*-flows.*

**Theorem 6.21.** *If* G *is* 2*-edge-connected, then it has at least* $2^{(\|G\|-|G|)/3}$ *NZ* $\mathbb{Z}_6$*-flows.*

By Corollary 6.11, counting NZ $\mathbb{Z}_4$-flows (resp. NZ $\mathbb{Z}_6$-flows) is equivalent to counting NZ $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows (resp. NZ $\mathbb{Z}_3 \times \mathbb{Z}_2$-flows). To find many $\mathbb{Z}_k \times \mathbb{Z}_2$-flows, we start with a single one, say $(\varphi_1, \varphi_2)$, where $\varphi_1$ is a $\mathbb{Z}_k$-flow and $\varphi_2$ is a $\mathbb{Z}_2$-flow; we allow the possibility of $\varphi_i(e) = 0$ for some edges $e$, as long as $(\varphi_1(e), \varphi_2(e)) \neq (0, 0)$. Let $E_1$ be the set of edges with $\varphi_1(e) \neq 0$, and let $\varphi_3$ be a $\mathbb{Z}_2$-flow such that $\varphi_3(e) = 0$ for all $e \in E(G) \setminus E_1$. Consider $(\varphi_1, \varphi_2 + \varphi_3)$. Clearly it is a $\mathbb{Z}_k \times \mathbb{Z}_2$-flow. Further, it is NZ, since each $e \in E_1$ has $\varphi_1(e) \neq 0$ and each $e \notin E_1$ has $\varphi_2(e) + \varphi_3(e) = \varphi_2(e) \neq 0$. Thus, to find many NZ $\mathbb{Z}_k \times \mathbb{Z}_2$-flows, it suffices to find many $\mathbb{Z}_2$-flows like $\varphi_3$ above. We do this in Lemma 6.25. But first we need a little preparation.

**Lemma 6.22.** *Let $G$ be a directed graph and let $H$ be an abelian group. Fix $F \subseteq E(G)$ such that $F$ induces a (directed) forest. If $\varphi_1$ and $\varphi_2$ are $H$-flows on $G$ and $\varphi_1(e) = \varphi_2(e)$ for all $e \in E \setminus F$, then $\varphi_1 = \varphi_2$.*

*Proof.* Since $\varphi_1$ and $\varphi_2$ are $H$-flows, so is $\varphi_1 - \varphi_2$. But now $\varphi_1 - \varphi_2$ must be identically 0, since the net flow into every vertex is 0, by Observation 6.3. □

We will only need the following lemma in the case $k = 2$, but we prove it more generally, since the proof is nearly identical.

**Lemma 6.23.** *The number of $\mathbb{Z}_k$-flows (perhaps not NZ) in a connected graph $G$ is exactly $k^{\|G\|-|G|+1}$.*

*Proof.* Fix an arbitrary orientation of $G$ and a spanning tree $T$. We show that every map $\varphi : E(G) \setminus E(T) \to \mathbb{Z}_k$ extends to a $\mathbb{Z}_k$-flow of $G$ in exactly one way. Such a $\varphi$ extends in at most one way by Lemma 6.22. So now we prove that it extends in at least one way.

Let[6] $\mathrm{fd}(G) := \|G\| - |G| + 1$. We use induction on $\mathrm{fd}(G)$. If $\mathrm{fd}(G) = 0$, then $G$ is a tree, so the only $\mathbb{Z}_k$-flow on $G$ is identically 0. Suppose instead that $\mathrm{fd}(G) = 1$. Pick $e \in E(G) \setminus E(T)$. Let $C$ be the single cycle in $G$. We can assign any value to $e$, and must then also assign the same value to each other edge of $C$, and assign 0 to each edge outside $C$. (We assume that $C$ is oriented consistently, by the comment following Definition 6.1.)

For the induction step, suppose that $\mathrm{fd}(G) = s$. Choose $e \in E(G)\setminus E(T)$, and let $G' := G - e$. Fix a map $\varphi : E(G) \setminus E(T) \to \mathbb{Z}_k$ and let $\varphi'$ denote its restriction to $E(G') \setminus E(T)$. Since $\mathrm{fd}(G') = s - 1$, by hypothesis $\varphi'$ extends to a $\mathbb{Z}_k$-flow $\varphi_1$ on $G'$. Again, let $C$ denote the unique cycle contained in $E(T) \cup e$. Let $\varphi_2$ be the $\mathbb{Z}_k$-flow that assigns $\varphi(e)$ to each edge of $C$ and assigns 0 elsewhere. Now $\varphi_1 + \varphi_2$ is a $\mathbb{Z}_k$-flow in $G$. □

**Definition 6.24.** The *support* of a flow $\varphi$, denoted $\mathrm{supp}(\varphi)$, is the set of edges where $\varphi$ is nonzero. That is, $\mathrm{supp}(\varphi) := \{e : \varphi(e) \neq 0\}$. $\mathrm{supp}(\varphi)$

**Lemma 6.25.** *Fix a graph $G$. Let $\varphi_1 : E(G) \to \mathbb{Z}_k$ and $\varphi_2 : E(G) \to \mathbb{Z}_2$ be flows with $\mathrm{supp}(\varphi_1) \cup \mathrm{supp}(\varphi_2) = E(G)$. Let $t := |\mathrm{supp}(\varphi_2)|$. Now $G$ has at least $2^{\|G\|-|G|-t/k}$ NZ $\mathbb{Z}_k \times \mathbb{Z}_2$-flows.*

We follow the outline after the statement of Theorem 6.21. By Lemma 6.23, we want the support of the $\mathbb{Z}_k$-flow to be large. So we begin by modifying it to ensure this.

[6]It is easy to check that the collection of $\mathbb{Z}_k$-flows on $G$ is a vector space over $\mathbb{Z}_k$. So fd is short for flow dimension, the dimension of that vector space.

*Proof.* Let $\varphi_1$, $\varphi_2$, and $t$ be as in the statement of the lemma. We first modify $\varphi_1$ to get a new $\mathbb{Z}_k$-flow $\varphi_1'$ such that $(\varphi_1', \varphi_2)$ is still an NZ $\mathbb{Z}_k \times \mathbb{Z}_2$-flow, but now also $|\mathrm{supp}(\varphi_1')| \geqslant \|G\| - t/k$. We can write $\varphi_2$ as the disjoint union of edge sets of cycles $C_1, \ldots, C_s$ (we assume that each $C_i$ is oriented consistently). For each $C_i$, let $\phi_i$ be a $\mathbb{Z}_k$-flow with value 1 on each edge of $C_i$ and value 0 elsewhere. For each $e \in E(C_i)$, there is exactly one value $j \in \{0, \ldots, k-1\}$ such that $\varphi_1(e) + j\phi_i(e) = 0$. By Pigeonhole, there exists $j_i$ such that $\varphi_1(e) + j_i\phi_i(e) = 0$ for at most $|E(C_i)|/k$ edges $e \in E(C_i)$. So let $\varphi_1' := \varphi_1 + \sum_{i=1}^{s} j_i\phi_i$. Clearly, $|\mathrm{supp}(\varphi_1')| \geqslant \|G\| - \sum_{i=1}^{s} \|C_i\|/k = \|G\| - t/k$.

Let $G' := G[\mathrm{supp}(\varphi_1')]$ and note that $\|G'\| \geqslant \|G\| - t/k$. By Lemma 6.23, $G'$ has at least $2^{\|G\|-t/k}$ $\mathbb{Z}_2$-flows. Let $\varphi_3$ be one of these $\mathbb{Z}_2$-flows. Clearly, $(\varphi_1', \varphi_2 + \varphi_3)$ is a $\mathbb{Z}_k \times \mathbb{Z}_2$-flow. Further, it is NZ, since $\varphi_1'(e) \neq 0$ for each $e \in \mathrm{supp}(\varphi_1')$ and $\varphi_2(e) + \varphi_3(e) = \varphi_2(e) \neq 0$ for each $e \notin \mathrm{supp}(\varphi_1')$. □

Now we can prove Theorem 6.20. For easy reference, we restate it.

**Theorem 1.19.** If $G$ is 4-edge-connected, then $G$ has at least $2^{|G|/3}$ NZ $\mathbb{Z}_4$-flows.

*Proof.* Let $G$ be a 4-edge-connected graph. By Corollary 6.11, counting NZ $\mathbb{Z}_4$-flows is equivalent to counting NZ $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows. By Theorem 6.15, $G$ has an NZ $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Let $E_1$, $E_2$, $E_3$ denote the sets of edges with flow values $(0,1)$, $(1,0)$, and $(1,1)$, respectively. Because of the structure of $\mathbb{Z}_2 \times \mathbb{Z}_2$, at each vertex the numbers of incident edges in each $E_i$ are either all even or all odd. So swapping the flow values on the edges in any two $E_i$ yields another NZ $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Thus, we can assume that $|E_2| \geqslant \|G\|/3$. Call this flow $(\varphi_1, \varphi_2)$. Since $G$ has minimum degree at least 4, $\|G\| \geqslant 4|G|/2 = 2|G|$. Now we apply Lemma 6.25, with $\mathrm{supp}(\varphi_2) = \|G\| - |E_2| \leqslant 2\|G\|/3$. Thus, the number of NZ $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows in $G$ is at least $2^{\|G\|-|G|-(2\|G\|/3)/2} = 2^{2\|G\|/3-|G|} \geqslant 2^{|G|/3}$. □

Next, we turn to proving Theorem 6.21. The main extra complication is getting an NZ $\mathbb{Z}_3 \times \mathbb{Z}_2$-flow $(\varphi_1, \varphi_2)$ with a good upper bound on $|\mathrm{supp}(\varphi_2)|$. Our solution is to show that we can restrict the problem to cubic graphs. If $G$ is cubic, then $\varphi_2$ is a disjoint union of cycles, so $|\mathrm{supp}(\varphi_2)| \leqslant |G|$. Now the number of $\mathbb{Z}_3 \times \mathbb{Z}_2$-flows guaranteed by Lemma 6.25 is $2^{\|G\|-|G|-|G|/3} = 2^{|G|/6} = 2^{(\|G\|-|G|)/3}$.

$\ell$ **Lemma 6.26.** *Let $G$ be a graph and let $\ell := \|G\| - |G|$. If $G$ is 3-edge-connected, then there exists a*
$G'$ *3-edge-connected cubic graph $G'$ with $|G'| = 2\ell$ such that $G$ can be formed from $G'$ by contracting a set of edges $F \subseteq V(G)$ that induces a forest. (See the right of Figure 6.5.)*

*Proof.* Let $f(G) := \sum_{v \in V(G)} (d(v) - 3)$. We use induction on $f(G)$. If $f(G) = 0$, then $G$ is cubic, so let $G' := G$. Instead assume $f(G) \geqslant 1$, and choose $v \in V(G)$ with $d(v) \geqslant 4$.

We will use the following operation. To *expand at $v$*, we pick edges $vw_1$ and $vw_2$ and form $G''$ from $G - \{vw_1, vw_2\}$ by adding a new vertex $v'$ and edges $vv'$, $v'w_1$, and $v'w_2$; see the left of Figure 6.5. Clearly, $f(G') = f(G) - 1$. Also, $G'/vv' \cong G$. So, to complete the induction step,

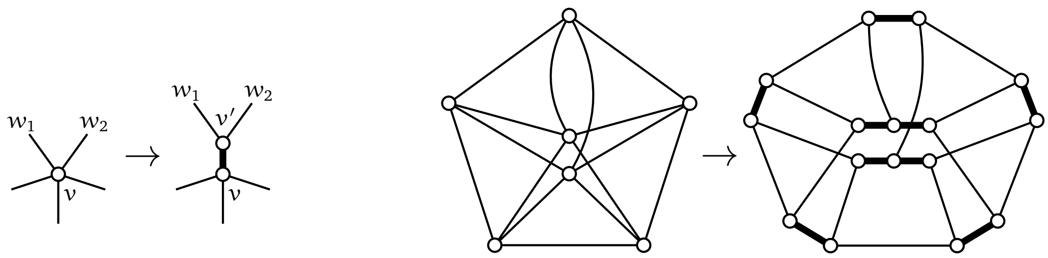



Figure 6.5: Forming $G'$ from $G$, as in Lemma 6.26. Left: Expanding at a vertex $v$. Right: A 3-edge-connected graph $G$ (no bold edges) with $|G| = 7$ and $\|G\| = 15$, so $\ell := 15 - 7 = 8$. And a 3-edge-connected cubic graph $G'$ (9 bold edges) with $|G'| = 2\ell = 16$. Note that $G'$ is formed from $G$ by expanding at each vertex $v$ exactly $d_G(v) - 3$ times.

it suffices to show that we can choose edges $vw_1$ and $vw_2$ so that $G''$ is 3-edge-connected, since by the induction hypothesis such a $G''$ has the desired graph $G'$ and $F \subseteq E(G')$. (Since $vv'$ is not a loop, the edge set $F \cup \{vv'\}$ induces a forest in $G'$.) To show that $G''$ is 3-edge-connected, it is helpful to note that, for every choice of edges $vw_1$ and $vw_2$, if $G''$ has an edge-cut $E'$ with $|E'| \leqslant 2$, then $vv' \in E'$.

Consider the edge-connectivity of $G - v$. If $G - v$ is disconnected, then $v$ has at least 3 edges to each of its components. So we choose $w_1$ and $w_2$ in distinct components of $G - v$, and $G''$ is 3-edge-connected. Suppose instead that $G - v$ is connected, but has a cut-edge $e$. Since $G$ is 3-edge-connected, $v$ has at least 2 neighbors in each component of $G - v - e$; let $w_1$ and $w_2$ be neighbors of $v$ in distinct components. Note that $G'$ is 3-edge-connected. Finally, suppose that $G - v$ is 2-edge-connected. Now let $w_1$ and $w_2$ be arbitrary neighbors of $v$. Again, $G''$ is 3-edge-connected. This concludes the induction step. □

**Theorem 6.27.** *If $G$ is 2-edge-connected, then $G$ has at least $2^{(\|G\|-|G|)/3}$ NZ $\mathbb{Z}_6$-flows.*

We can assume that $\delta(G) \geqslant 3$. If not, then we form $G'$ from $G$ by contracting an edge incident to a 2-vertex. Now proving the result for $G'$ also proves it for $G$. As explained above, we want to reduce to the case when $G$ is cubic. Since Lemma 6.26 requires that $G$ is 3-edge-connected, we begin by handling 2-edge-cuts.

*Proof.* We use induction on $|G|$. If $|G| = 1$, then each edge is a loop, and can be assigned any nonzero value in $\mathbb{Z}_3 \times \mathbb{Z}_2$. The number of NZ flows is thus $5^{\|G\|} \geqslant 2^{(\|G\|-1)/3} = 2^{(\|G\|-|G|)/3}$.

Now we consider the induction step. First suppose that $G$ contains a 2-edge-cut $\{e_1, e_2\}$. Let $G' := G/e_1$. By hypothesis, the theorem holds for $G'$. Furthermore, each NZ $\mathbb{Z}_3 \times \mathbb{Z}_2$-flow in $G'$ naturally maps to an NZ $\mathbb{Z}_3 \times \mathbb{Z}_2$-flow in $G$, by giving $e_1$ the same value as $e_2$ (assuming $e_1$ and $e_2$ are oriented oppositely). This proves the theorem for $G$, since $\|G\| - |G| = \|G'\| - |G'|$.

Suppose instead that $G$ is 3-edge-connected. Form $G'$ from $G$ as in Lemma 6.26. By Theorem 6.19, $G'$ has an NZ $\mathbb{Z}_3 \times \mathbb{Z}_2$-flow $(\varphi_1, \varphi_2)$. Since $G'$ is cubic, $|\text{supp}(\varphi_2)| \leqslant |G'|$. By Lemma 6.25, the number of $\mathbb{Z}_3 \times \mathbb{Z}_2$-flows in $G'$ must be at least $2^{\|G'\|-|G'|-|G'|/3} =$

$2^{3|G'|/2-|G'|-|G'|/3} = 2^{|G'|/6} = 2^{\ell/3}$. Recall that there exists $F \subseteq E(G')$ such that $F$ induces a forest and contracting $F$ in $G'$ yields $G$. Given any NZ $\mathbb{Z}_3 \times \mathbb{Z}_2$-flow on $G'$, contracting $F$ yields an NZ $\mathbb{Z}_3 \times \mathbb{Z}_2$-flow on $G$. We only need to check that distinct flows in $G'$ map to distinct flows in $G$. Since $F$ induces a forest, this follows directly from Lemma 6.22. □

## 6.4 The Weak 3-Flow Conjecture

In this section we make progress toward's Tutte's 3-Flow Conjecture. In fact, we prove a much more general statement, given in Theorem 6.28.

$\mathbb{Z}_k$ Fix a graph $G$ and an odd integer $k \geqslant 3$. Recall that $\mathbb{Z}_k = \{0, 1, \dots, k-1\}$. Fix $\beta : V(G) \to$
$\mathbb{Z}_k$-boundary $\mathbb{Z}_k$ such that $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{k}$. We call $\beta$ a $\mathbb{Z}_k$-*boundary* of $G$. Any orientation $D$
$\beta$-orientation of $G$ such that $d^+_D(v) - d^-_D(v) \equiv \beta(v) \pmod{k}$ for all $v \in V(G)$ is a $\beta$-*orientation*.

**Theorem 6.28.** *Fix a graph $G$, an odd integer $k \geqslant 3$, and a $\mathbb{Z}_k$-boundary of $G$. If $G$ is $(3k-3)$-edge-connected, then $G$ has a $\beta$-orientation.*

This theorem has the following result as a special case.

**Theorem 6.29 (Weak 3-Flow Theorem).** *If $G$ is 6-edge-connected, then $G$ has an NZ 3-flow.*

*Proof.* Let $k = 3$ and consider the $\mathbb{Z}_3$-boundary $\beta$ with $\beta(v) = 0$ for all $v \in V(G)$. By Theorem 6.28, graph $G$ has a $\beta$-orientation $D$. Further, $D$ naturally yields an NZ $\mathbb{Z}_3$-flow (if we give each edge flow value 1). So Theorem 6.10 implies that $G$ has an NZ 3-flow. □

lift Before we prove Theorem 6.28, we need some preparation. Recall what it means to *lift* edges $vw$ and $wx$ incident to a vertex $w$. We delete $vw$ and $wx$ and add $vx$. And if either $vw$ is oriented as $\overrightarrow{vw}$ or else $wx$ is oriented as $\overrightarrow{wx}$, then the new edge is oriented as $\overrightarrow{vx}$.

To prove Theorem 6.28, as if often the case, we use induction (equivalently, minimal counterexample) to prove something stronger. Our induction is on the number of edges. Typically we proceed by deleting an edge, contracting a vertex subset, or (occasionally) lifting a pair of edges. Thus, a key step is phrasing our hypotheses so that they continue to hold when we perform any one of these operations. Before stating the theorem, we introduce a function $\tau$, which allows us to state our hypotheses.

### 6.4.1 The Definition and Properties of $\tau$

The definitions and properties in this subsection are essential to the proof of Theorem 6.33 in Subsection 6.4.2.

Fix a graph $G$, an odd integer $k \geqslant 3$, and a $\mathbb{Z}_k$-boundary $\beta$ of $G$. Define $\tau : V(G) \to \{0, \pm 1, \pm 2, \dots, \pm k\}$ such that, for each vertex $v \in V(G)$, we have

$$\tau(v) = \begin{cases} \beta(v) & \pmod{k} \\ d(v) & \pmod{2}. \end{cases} \tag{6.1}$$

If $\beta(v) = 0$ and $d(v)$ is odd, then $|\tau(v)| = k$, so we can let either $\tau(v) := k$ or $\tau(v) := -k$ (for now we chose $\tau(v)$ arbitrarily, but we will revisit this case in Claim 5). Otherwise, (6.1) is equivalent to the following:

$$\tau(v) := \begin{cases} \beta(v) & \text{if } d(v) - \beta(v) \text{ is even} \\ \beta(v) - k & \text{if } d(v) - \beta(v) \text{ is odd.} \end{cases} \tag{6.2}$$

**Example 6.30.** The table below shows the value of $\tau$, when $k = 7$, for each pair $(a, \beta(v))$, where $a := d(v) \bmod 2$. Note that each value in $\{0, \pm 1, \ldots, \pm k\}$ maps bijectively to a pair $(a, \beta(v))$, except that $k$ and $-k$ both map to the pair $(1, 0)$. ♢

| $d(v)\backslash\beta(v)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | −6 | 2 | −4 | 4 | −2 | 6 |
| 1 | ±7 | 1 | −5 | 3 | −3 | 5 | −1 |

Figure 6.6: Values of $\tau$ for pairs $(a, \beta(v))$ where $a := d(v) \bmod 2$ and $\beta(v) \in \{0, \ldots, 6\}$.

Suppose $v \in V(G)$ and $d(v) \geqslant |\tau(v)|$. Now (6.2) implies that $d(v) - |\tau(v)|$ is even. So a natural way to achieve $d^+(v) - d^-(v) \equiv \beta(v) \pmod{k}$ is to first orient $(d(v) - |\tau(v)|)/2$ incident edges into $v$ and also $(d(v) - |\tau(v)|)/2$ incident edges out of $v$. Next we direct all remaining edges of $\partial(v)$ either out of $v$ (if $\tau(v)$ is positive) or into $v$ (if $\tau(v)$ is negative). Our challenge is to orient the edges as above *simultaneously* for every vertex $v$. However, if $d(v) - |\tau(v)|$ is large, then we have significant freedom in how we orient each set $\partial(v)$.

We will often want to contract vertex subsets, so we extend our definition of $\tau$ as follows. For each $W \subseteq V(G)$, let $\beta(W) := \sum_{w \in W} \beta(w) \pmod{k}$ and $d(W) := |\partial(W)|$. Analogous to (6.1), for each $W \subseteq V(G)$, we define $\tau(W)$ to be the element of $\{0, \pm 1, \pm 2, \ldots, \pm k\}$ such that $\beta(W), d(W)$

$$\tau(W) = \begin{cases} \beta(W) & \pmod{k} \\ d(W) & \pmod{2}. \end{cases} \tag{6.3}$$

Note that $\tau(W)$ also satisfies a statement analogous to (6.2). We close this subsection with two easy observations about $\tau$.

**Observation 6.31.** Form a graph $G'$ from $G$ by deleting some edge $e$ incident to a vertex $v$. Further, let $\beta'(v) := \beta(v) + 1$ or $\beta'(v) := \beta(v) - 1$. Now $|\tau'(v)| - |\tau(v)| \in \{-1, 1\}$. Also, if $e \in \partial(W)$, then $|\tau'(W)| - |\tau(W)| \in \{-1, 1\}$.

*Proof.* Suppose $d(v) - \beta(v)$ is even. Now $d'(v) - \beta'(v)$ is also even. So $\tau(v) = \beta(v)$ and $\tau'(v) = \beta'(v) = \beta(v) \pm 1$, as desired. Assume instead that $d(v) - \beta(v)$ is odd. Now $d'(v) - \beta'(v)$ is also odd. Thus, $|\tau'(v)| - |\tau(v)| = (k - \beta'(v)) - (k - \beta(v)) \in \{-1, 1\}$.

Note that $d'(W) - \beta'(W) \equiv d(W) - \beta(W) \pmod 2$. So either $\tau(W) = \beta(W)$ and $\tau'(W) = \beta'(W)$ or else $\tau(W) = \beta(W) - k$ and $\tau'(W) = \beta'(W) - k$. Thus, either $|\tau'(W)| - |\tau(W)| = \beta'(W) - \beta(W) = \beta'(v) - \beta(v) \in \{-1, 1\}$ or else $|\tau'(W)| - |\tau(W)| = (k - \beta'(W)) - (k - \beta(W)) = \beta(W) - \beta'(W) = \beta(v) - \beta'(v) \in \{-1, 1\}$. □

**Observation 6.32.** If $d(W) \geqslant 3k - 3$, then also $d(W) \geqslant (2k - 2) + |\tau(W)|$.

*Proof.* If $|\tau(W)| \leqslant k - 1$, then we are done. So assume $|\tau(W)| = k$. Since $k$ is odd, equation (6.1) implies that $d(W)$ is also odd. Thus, $d(W) \geqslant 3k - 3$ implies the stronger inequality $d(W) \geqslant 3k - 2$, which yields the desired inequality. □

### 6.4.2 The Main Result

Now we can state the main result of this section.

$D_0$, $V_0$, $v_0$

**Theorem 6.33.** *Fix an odd integer $k \geqslant 3$. Let $G$ be a (loopless) multigraph with a $\mathbb{Z}_k$-boundary $\beta$. Fix $z \in V(G)$ and an orientation $D_0$ of $\partial(z)$. Let $V_0 := \{v \in V(G) \setminus \{z\} \mid \tau(v) = 0\}$. If $V_0 \neq \emptyset$, then let $v_0$ be a vertex of $V_0$ with smallest degree. We can extend $D_0$ to a $\beta$-orientation $D$ of all of $G$ provided that the following two conditions hold.*

*(i)* $d(W) \geqslant (2k - 2) + |\tau(W)|$ *whenever* $\emptyset \subsetneq W \subsetneq V(G) \setminus \{z\}$ *and* $W \neq \{v_0\}$.

*(ii)* $d(z) \leqslant (2k - 2) + |\tau(z)|$ *and* $d^+_{D_0}(z) - d^-_{D_0}(z) \equiv \beta(z) \pmod k$.

Whenever we write $d(z)$ or $d(W)$, both in Theorem 6.33 and below, we refer to degrees in $G$ (which are unaffected by the orientation $D_0$).

It is easy to prove Theorem 6.28 from Theorem 6.33, so we start with that.

*Proof of Theorem 6.28 from Theorem 6.33.* Suppose we are given an odd integer $k \geqslant 3$, a $(3k - 3)$-edge-connected graph $G$, and a $\mathbb{Z}_k$-boundary $\beta$ for $G$. Choose an arbitrary edge $e \in E(G)$ and subdivide it by adding a vertex $z$; call this new graph $G'$. Let $\beta'(z) := 0$ and $\beta'(v) := \beta(v)$ for all $v \in V(G)$. Orient $\partial(z)$ so that $d^+(z) = d^-(z) = 1$.

Since $G$ is $(3k - 3)$-edge-connected, for each $W \subseteq V(G') \setminus \{z\}$ with $|V(G') \setminus W| > 1$, we have $d(W) \geqslant 3k - 3$. So Observation 6.32 implies that $d(W) \geqslant 2k - 2 + |\tau(W)|$. Thus, Theorem 6.33 guarantees that $G'$ has a $\beta'$-orientation $D'$ that extends the orientation of $\partial(z)$. Now suppressing $z$ gives the desired $\beta$-orientation of $G$. □

Before proving Theorem 6.33, we give some intuition. A crucial step in the proof is Claim 1, which shows that $d(W) \geqslant 2k + |\tau(W)|$ for every $W \subseteq V(G) \setminus \{z\}$ such that $1 < |W| < |V(G) \setminus \{z\}|$. Note that this is $d(W) \geqslant 2k + |\tau(W)|$, rather than $d(W) \geqslant (2k - 2) + |\tau(W)|$, as in hypothesis

(i) of the theorem. This claim plays the role of a gap lemma in proofs using the potential method (see Chapter 12). It allows us to slightly modify $G$ (by deleting an edge or lifting an edge pair) and get the desired orientation by induction, since the modified graph still satisfies the hypotheses of the theorem. The key is that our modification can decrease $d(W)$ by at most 2, for every set $W$.

It is our need to prove Claim 1 that motivates us to introduce vertex $z$, in hypothesis (ii). If Claim 1 fails for some set $W$, then the edge cut $\partial(W)$ is "small". We contract each of $W$ and $V(G) \setminus W$ in turn, getting good orientations for each of the two resulting smaller graphs. To get a good orientation for $G$, we want to combine these good orientations for these smaller graphs. For this idea to work, these orientations must agree on $\partial(W)$. It is this need to have the orientation agree on $\partial(W)$ that motivates our choice in the statement of Theorem 6.33 to allow a prescribed orientation[7] $D_0$ of $\partial(z)$.

In Claim 2 we show that $V_0 = \emptyset$. So we include $V_0$ and $v_0$ in the statement of Theorem 6.33 largely for technical reasons. Algorithmically, we think of choosing a vertex of $V_0$ of smallest degree and repeatedly lifting pairs of incident edges until $v_0$ has degree 0. (This works precisely because $\beta(v_0) = 0$ and $d(v_0)$ is even.) At this point we delete $v_0$ and repeat this process for the vertex in $V_0$ with current smallest degree. Eventually we delete all vertices in $V_0$, which justifies our claim that $V_0 = \emptyset$. (We must also consider the case that $v_0$ has only a single neighbor, particularly if it is $z$, but this is not too difficult.) This algorithmic view helps explain why we require $W \neq \{v_0\}$ in hypothesis (i). As we repeatedly lift edge pairs incident to $v_0$, eventually $d(v_0)$ gets arbitrarily small, culminating with $d(v_0) = 0$, just before we delete $v_0$.

*Proof of Theorem 6.33.* We can assume $|G| \geqslant 3$, as follows. The case $|G| = 1$ is trivial, since $G$ is loopless. So assume $|G| = 2$. Now $E(G) = \partial(z)$, so all edges are oriented by $D_0$. Further, since $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{k}$, the condition in (ii) that $d^+_{D_0}(z) - d^-_{D_0}(z) \equiv \beta(z) \pmod{k}$ implies that the other vertex of $G$, call it $x$, satisfies $d^+_{D_0}(x) - d^-_{D_0}(x) \equiv d^-_{D_0}(z) - d^+_{D_0}(z) \equiv -\beta(z) \equiv \beta(x) \pmod{k}$. Thus, we assume $|G| \geqslant 3$.

Assume the theorem is false. We partially order the graphs (with $z$ specified) by the value of $|G| + \|G - z\|$. Let $\mathcal{M}$ be the set of of all counterexamples $(G, \beta, z)$ to the theorem that are $\mathcal{M}$
smallest in this partial order. For most of the proof we prove certain properties of every element of $\mathcal{M}$. Near the end we chose an element of $\mathcal{M}$ that also minimizes $\|G\|$ and show that it is not a counterexample. That is, we show that $\mathcal{M} = \emptyset$.

**Claim 1.** *If* $W \subseteq V(G) \setminus \{z\}$ *and* $1 < |W| < |V(G) \setminus \{z\}|$, *then* $d(W) \geqslant 2k + |\tau(W)|$.

Proof. Fix $W$ as in the hypothesis and suppose $d(W) < 2k + |\tau(W)|$. Since $d(W) \equiv |\tau(W)| \pmod{2}$, by (6.3) we have $d(W) \leqslant (2k-2) + |\tau(W)|$. Form $G_W$ from $G$ by contracting $W$ to a new vertex $w$, similar to Figure 6.4 except that now we only have $|\partial(W)| \leqslant (2k-2) + |\tau(W)|$.

[7] Akin to what we mentioned in Section 6.2, this prescribed orientation $D_0$ plays the same role as (a) the two precolored adjacent vertices on the outer face in the proof of Theorem 11.1 (that planar graphs are 5-choosable) and (b) the precolored vertices on the outer face in the proof of Theorem 4.26 (that planar graphs with no cycle of length 4 to 8 are correspondence 3-colorable).

Since $|G_W| < |G|$ and $\|G_W - z\| \leqslant \|G - z\|$, graph $G_W$ has a $\beta$-orientation $D_W$ that extends $D_0$ (here $\beta(w) := \sum_{v \in W} \beta(v) \pmod{k}$). Note that all edges of $\partial(W)$ are oriented by $D_W$. Similarly, we can contract $\overline{W}$ to get $G_{\overline{W}}$ with a new vertex $\overline{w}$. Now we let $\overline{w}$ play the role of $z$; this is possible, since $d(\overline{w}) = |\partial(W)| \leqslant 2k - 2 + |\tau(W)| = 2k - 2 + |\tau(\overline{w})|$. Further, we orient each edge of $\partial(\overline{w})$ in $G_{\overline{W}}$ as the corresponding edge of $\partial(W)$ in $D_W$. Again, $G_{\overline{W}}$ has a $\beta$-orientation $D_{\overline{W}}$ that extends the orientation of $\partial(W)$. Since $\partial(W) = \partial(\overline{w})$, orientations $D_W$ and $D_{\overline{W}}$ agree on $\partial(W)$, so they combine to give the desired $\beta$-orientation D of G. $\diamondsuit$

**Claim 2.** $V_0 = \emptyset$.

Proof. Suppose $V_0 \neq \emptyset$ and choose $v_0 \in V_0$ to minimize $d(v_0)$. By definition, $\tau(v_0) = 0$, so $d(v) \equiv 0 \pmod{2}$. If $v_0$ has at least two distinct neighbors, then we lift one pair of edges incident to $v_0$ (that are not parallel); call the resulting graph $G'$. Clearly $\|G' - z\| < \|G - z\|$, and (ii) holds trivially for $G'$, since $d'(z) = d(z)$ and $|\tau'(z)| = |\tau(z)|$. (This is where we use the condition $W \neq \{v_0\}$ in hypothesis (i), since perhaps $d(v_0) = 2k-2+|0|$ and $d'(v_0) = 2k-4$.) Note that $d'(W) \in \{d(W), d(W)-2\}$ for all $W \subseteq V(G) \setminus \{z\}$. Since $d'(W) \equiv d(W) \pmod{2}$, also $\tau'(W) = \tau(W)$. So hypothesis (i) also holds for $G'$, since Claim 1 implies $d'(W) \geqslant d(W) - 2 \geqslant 2k - 2 + |\tau(W)| = 2k - 2 + |\tau'(W)|$.

Assume instead that $v_0$ has only a single neighbor $x$. If $x = z$, then we let $W := V(G)\setminus\{v_0, z\}$, so that $\tau(W) = -\tau(\{z, v_0\}) = -\tau(z)$. Now hypothesis (i) gives $d(z) = d(W) + d(v_0) \geqslant (2k-2) + |\tau(W)| + 2 = 2k + |\tau(z)|$, contradicting hypothesis (ii). So $x \neq z$.

Now let $G' := G - v_0$ and $\beta' := \beta$ for all $v \in V(G')$. By minimality, we will show that $G'$ has a $\beta'$-orientation $D'$. To extend $D'$ to a $\beta$-orientation of G, we orient $d(v_0)/2$ edges into $v_0$ and $d(v_0)/2$ edges out of $v_0$. Clearly, $G'$ is smaller than G. Thus, it suffices to show that $G'$ satisfies the hypotheses of the theorem. Hypothesis (ii) holds, since $d'(z) = d(z)$ and $\tau'(z) = \tau(z)$. So we consider (i). Suppose $W \subseteq V(G') \setminus \{z\}$ and $|V(G') \setminus W| > 1$. Now $d'(W) \in \{d(W), d(W \cup \{v_0\})\}$ and $d(W) \equiv d(W \cup \{v_0\}) \pmod{2}$. Since $\beta'(W) = \beta(W) = \beta(W \cup \{v_0\})$, Definition (6.3) gives $\tau'(W) = \tau(W) = \tau(W \cup \{v_0\})$. Thus (i) holds for $G'$. So, by minimality $G'$ has a $\beta'$-orientation, which we can extend to a $\beta$-orientation for G. $\diamondsuit$

**Claim 3.** $G - z$ *is connected.*

Proof. Suppose that $G - z$ is disconnected, and let U and W be the vertex sets of two of its components. Recall, from Claim 2, that $V_0 = \emptyset$. So hypothesis (i) gives $d(U) \geqslant 2k - 2$ and $d(W) \geqslant 2k - 2$. Summing these inequalities gives $d(z) \geqslant d(U) + d(W) \geqslant 2(2k-2) > 3k-2 \geqslant (2k-2) + |\tau(z)|$, which contradicts hypothesis (ii); here the strict inequality uses that $k \geqslant 3$. See the left of Figure 6.7. $\diamondsuit$

**Claim 4.** $d(z) \geqslant k$.

Proof. Suppose instead that $d(z) \leqslant k-1$. Form $G'$ from G by replacing some edge $vw$ of $G-z$ with two directed edges $\overrightarrow{vz}$ and $\overrightarrow{zw}$, and letting $\beta' := \beta$. See the right of Figure 6.7. Clearly $|G'| = |G|$ and $\|G'-z\| < \|G-z\|$. Now $d'(z) \leqslant k-1+2 \leqslant 2k-2 \leqslant (2k-2)+|\tau'(z)|$. So $G'$

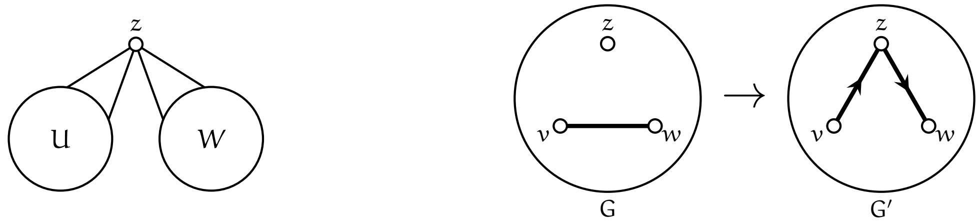



Figure 6.7: Left: In Claim 3, we assume (to get a contradiction) that $G - z$ is disconnected. Right: In Claim 4, we replace undirected edge $vw$ with directed edges $\overrightarrow{vz}$ and $\overrightarrow{zw}$.

satisfies hypothesis (ii). Finally, $d'(W) = d(W)+2$ if $v, w \in W$ and otherwise $d'(W) = d(W)$. This implies that $\tau'(W) = \tau(W)$ for all $W$. Thus, $G'$ satisfies hypothesis (i). Since $G'$ is smaller than $G$, by minimality $G'$ has a $\beta'$-orientation $D'$. Now lifting the edge pair $vz, zw$ gives the desired $\beta$-orientation of $G$. $\diamondsuit$

It is convenient now to define 2 vertex subsets. Let $V^+ := \{v \in V(G)\setminus\{z\} : 1 \leqslant \tau(v) \leqslant k-1\}$ and $V^- := \{v \in V(G) \setminus \{z\} : 1-k \leqslant \tau(v) \leqslant -1\}$. $V^+, V^-$

**Claim 5.** *Either* $V(G) \setminus \{z\} = V^+$ *or else* $V(G) \setminus \{z\} = V^-$.

Proof. Suppose that $\tau$ must satisfy

$$\tau(v)\tau(w) > 0 \text{ for all } v, w \in V(G) \setminus \{z\}. \tag{6.4}$$

By Claim 2, $V_0 = \emptyset$. So $\tau(v) \neq 0$ for every vertex $v \neq z$. If there exists $w \in V(G) \setminus \{z\}$ with $\beta(w) = 0$ and $d(w)$ odd, then we can take either $\tau(w) = k$ or $\tau(w) = -k$. So, for any edge $vw$, we can choose $\tau(w)$ so that $vw$ violates (6.4). Thus, we assume that $1 \leqslant |\tau(v)| \leqslant k-1$ for every $v \in V(G) \setminus \{z\}$. That is, $V(G) = V^+ \cup V^-$. By Claim 3, $G \setminus \{z\}$ is connected, so if $V^+ \neq \emptyset$ and also $V^- \neq \emptyset$, then there is some edge $vw$ with $v \in V^+$ and $w \in V^-$, and this edge violates (6.4). So to prove the claim, it suffices to prove Condition (6.4). We do this now.

Suppose instead that there exist $v, w \in V(G) \setminus \{z\}$ with $\tau(v)\tau(w) \leqslant 0$; among all such pairs $v, w$, choose one to minimize the distance from $v$ to $w$. By Claim 2, we know $V_0 = \emptyset$, so we assume $\tau(v) > 0$ and $\tau(w) < 0$. By Claim 3, we know $G - z$ is connected, so it contains a $v, w$-walk. At some point along this walk, $\tau$ changes sign. Since we chose $v$ and $w$ at minimum distance, we must have $vw \in E(G)$. Let $G' := G - vw$. Let $\beta'(v) := \beta(v) - 1$, let $\beta'(w) := \beta(w) + 1$, and $\beta'(x) := \beta(x)$ for all $x \in V(G) \setminus \{v, w\}$. See the left of Figure 6.8. $v, w$ $G', \beta'$

We will show that $(G', \beta', z)$ satisfies the hypothesis of the theorem. Since $G'$ is smaller than $G$, this means $G'$ has a $\beta'$-orientation $D'$ that extends $D_0$. By restoring edge $\overrightarrow{vw}$, we get the desired $\beta$-orientation of $G$. So it suffices to show that $(G', \beta', z)$ satisfies hypotheses (i) and (ii). Hypothesis (ii) holds trivially, since $d'(z) = d(z)$ and $\beta'(z) = \beta(z)$, so $\tau'(z) = \tau(z)$.

Now we consider hypothesis (i). If $v, w \notin W$ or $v, w \in W$, then $d'(W) = d(W)$ and $\beta'(W) = \beta(W)$, so $\tau'(W) = \tau(W)$; thus, (i) holds. Instead assume $|W \cap \{v, w\}| = 1$ and $1 < |W| < |G - z|$. For $v$, we have $d'(v) = d(v) - 1$ and $\beta'(v) = \beta(v) - 1$. Thus,

$\tau'(v) = \tau(v) - 1$ and $|\tau'(v)| = |\tau(v)| - 1$, since $\tau'(v) > 0$. Similarly, $d'(w) = d(w) - 1$ and $\beta'(w) = \beta(w) + 1$. So $\tau'(w) = \tau(w) + 1$, but $|\tau'(w)| = |\tau(w)| - 1$. Thus, $|\tau'(x)| = |\tau(x)| - 1$ for each $x \in \{v, w\}$. Note that $|\tau'(x)| = |\tau(x)| - 1$ even when $|\tau(x)| = k$. Now $d'(W) = d(W) - 1$, and $\beta'(W) = \beta(W) \pm 1$. By Observation 6.31, also $\tau'(W) = \tau(W) \pm 1$. By Claim 1, $d'(W) = d(W) - 1 \geqslant 2k + |\tau(W)| - 1 = (2k-2) + |\tau(W)| + 1 \geqslant (2k-2) + |\tau'(W)|$. So hypothesis (i) holds. Thus, $\tau(v)\tau(w) > 0$, as desired. $\diamondsuit$

**Remark.** Note that Claims 1–5 above hold for every member of $\mathcal{M}$. For the rest of the proof we choose $(G, \beta, z)$ to be an element of $\mathcal{M}$ that minimizes $\|G\|$.

**Claim 6.** *We can assume, for all $v \in V(G) \setminus \{z\}$, that $d(v) \geqslant (2k-2) + \tau(v)$ and $1 \leqslant \tau(v) = \beta(v) \leqslant k - 1$.*

Proof. By Claim 5, we assume that $V(G) \setminus \{z\} = V^+$. If this is not the case, then we form $G'$ from $G$ by reversing all edges in $\partial(z)$, and we let $\beta'(v) := k - \beta(v)$ for all $v \in V(G)$. It is easy to check that $(G', \beta', z)$ is also an element of $\mathcal{M}$ that minimizes $\|G\|$. For all $v \in V(G) \setminus \{z\}$, we know $v \in V^+$; so $1 \leqslant \tau(v) \leqslant k - 1$, which implies $\tau(v) = \beta(v)$. Finally, hypothesis (i) gives $d(v) \geqslant (2k-2) + \tau(v)$. $\diamondsuit$

**Claim 7.** $d(z) = k + \beta(z)$*, and all edges in* $\partial(z)$ *are oriented away from* $z$*.*

Proof. Claim 4 gives $d(z) \geqslant k$, so $z$ has a neighbor, $x$. And Claim 6 gives $1 \leqslant \tau(x) \leqslant k-1$. If $xz$ is directed as $\overrightarrow{xz}$, then let $G' := G - \overrightarrow{xz}$, let $\beta'(x) := \beta(x) - 1$, and let $\beta'(z) := \beta(z) + 1$. Now $(G', \beta', z)$ satisfies the hypothesis of the theorem (the proof follows that of Claim 5). Since $G'$ is smaller than $G$, by the remark before Claim 6, we know $G'$ has a $\beta'$-orientation $D'$, which we can extend to a $\beta$-orientation of $G$. So $G$ is not a counterexample.

Thus, all edges in $\partial(z)$ are oriented away from $z$. So $\beta(z) = d(z) - ck$, where $c$ is the largest integer such that $d(z) - ck \geqslant 0$. Hypothesis (ii) gives $d(z) \leqslant (2k-2) + |\tau(z)| \leqslant 3k - 2$, and Claim 4 gives $d(z) \geqslant k$. So $1 \leqslant c \leqslant 2$. If $2k \leqslant d(z) \leqslant 3k-2$, then $c = 2$, so $d(z) - \beta(z) = 2k$, which is even. Now (6.2) implies $\tau(z) = \beta(z)$. Thus, $d(z) = \beta(z) + 2k = |\tau(z)| + 2k$, which contradicts hypothesis (ii). Hence, $c = 1$, and $d(z) = k + \beta(z)$. $\diamondsuit$

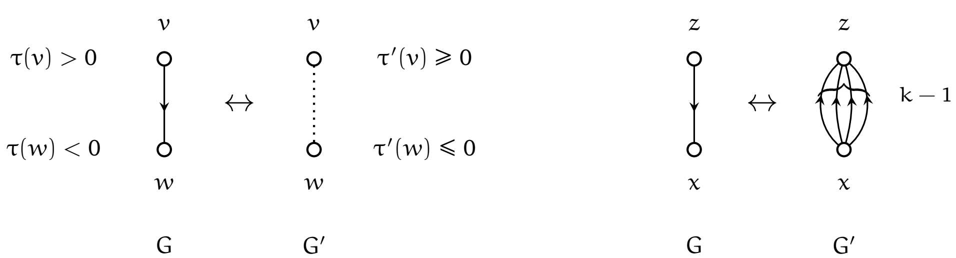



Figure 6.8: Left: In Claim 5, $G' := G - \overrightarrow{vw}$. Right: In Claim 8, $G'$ is formed from $G - \overrightarrow{zx}$ by adding $k-1$ parallel edges from $x$ to $z$.

**Claim 8.** $(G, \beta, z)$ *is not a counterexample.*

Proof. By Claim 7, vertex $z$ has a neighbor $x$ and an incident edge $\overrightarrow{zx}$. Form $G'$ from $G$ by deleting $\overrightarrow{zx}$ and adding $k-1$ edges directed from $x$ to $z$; see the right of Figure 6.8. Let $\beta' := \beta$. If $G'$ has a $\beta'$-orientation $D'$, then in $D'$ we replace $k-1$ edges oriented from $x$ to $z$ with the edge $\overrightarrow{zx}$. The resulting orientation is a $\beta$-orientation of $G$, which contradicts that $(G, \beta, z)$ is in $\mathcal{M}$. Thus, $G'$ must have no $\beta'$-orientation.

If $(G', \beta', z)$ satisfies hypotheses (i) and (ii) of the theorem, then $(G', \beta', z)$ is in $\mathcal{M}$, since $|G'| = |G|$ and $\|G' - z\| = \|G - z\|$. So $(G', \beta', z)$ satisfies Claims 1–5 above. To reach a contradiction, we now show that $(G', \beta', z)$ violates Claim 5. Since $x \in V^+$, by (6.2) we know that $d(x) - \beta(x)$ is even. Since $d'(x) = d(x) + k - 2$ and $\beta'(x) = \beta(x)$, we get $\tau'(x) = \beta'(x) - k$. Since $\beta'(x) = \beta(x) \leqslant k-1$, we have $\tau'(x) < 0$. That is $x \in V'^-$. However $V'^+ \subseteq V^+ \setminus \{x\} = V(G') \setminus \{x, z\}$, which contradicts Claim 5. Thus, to reach a contradiction, it suffices to show that $(G', \beta', z)$ satisfies hypotheses (i) and (ii) of the theorem.

We start with hypothesis (ii). By Claim 7, $d(z) = k + \beta(z)$. Since $\beta'(z) = \beta(z)$, we know $d'(z) - \beta'(z) = d(z) - 1 + (k-1) - \beta(z) = d(z) - \beta(z) + k - 2 = k + (k-2)$, which is even. So $\tau'(z) = \beta'(z)$. Also since $\beta'(z) = \beta(z)$, we have $d'(z) = k - 2 + d(z) = k - 2 + k + \beta(z) = 2k - 2 + \beta'(z) = 2k - 2 + |\tau'(z)|$. Also $d'^+(z) - d'^-(z) = d^+(z) - 1 - (d'(z) + (k-1)) = d^+(z) - d^-(z) - k \equiv \beta(z) \equiv \beta'(z) \pmod k$. So (ii) holds for $(G', \beta', z)$.

Now we consider hypothesis (i). Since it holds for $(G, \beta, z)$, we only need to consider subsets $W$ of $V(G')$ that contain $x$. By Claim 6 above, $d(x) \geqslant 2k - 2 + \tau(x) \geqslant 2k - 1$. Thus, $d'(x) = d(x) + (k-2) \geqslant (2k-1) + (k-2) = 3k - 3$. So Observation 6.32 implies $d'(x) \geqslant (2k-2) + |\tau'(x)|$. For any $W \subseteq V(G')$ in hypothesis (i), by Claim 1 we have $d'(W) = d(W) + (k-2) \geqslant 2k + (k-2) = (2k-2) + k \geqslant (2k-2) + |\tau'(W)|$. So hypothesis (i) holds. Thus, $G'$ has a $\beta'$-orientation, which yields a $\beta$-orientation of $G$. $\diamondsuit$

This completes the proof. $\square$

### 6.4.3 Odd-edge-connectivity and Modulo k-Orientations

To conclude this chapter, we use ideas similar to those in the proof of Theorem 6.33, to prove one more result. But first we need a definition. Recall that an *edge-cut* in a connected graph $G$ is a set $E(V_1, V_2)$, all those edges with one endpoint in each of $V_1$ and $V_2$, such that $V_1, V_2$ is a partition of $V(G)$. An edge-cut $E(V_1, V_2)$ is *odd* if $|E(V_1, V_2)|$ is odd. A graph $G$ is *odd-s-edge-connectivity* if the smallest odd edge-cut in $G$ has size at least $s$. odd-s-edge-connectivity

**Theorem 6.34.** *Fix an odd integer* $k \geqslant 1$ *and a graph* $G$. *If* $G$ *is odd-*$(3k-2)$*-edge-connected, then* $G$ *has a modulo* $k$*-orientation.*

Our final theorem only requires high *odd*-edge-connectivity, but it only guarantees the specific $\beta$-orientation when $\beta(v) = 0$ for all $v$. As in the previous case, most of our work goes into proving a more technical lemma, which gives the desired result as an easy corollary. $\beta$

**Theorem 6.35.** *Let $k$ be a positive odd integer, and let $G$ be an odd-$(3k-2)$-edge-connected graph. Let $\partial(W)$ be some smallest odd edge-cut in $G$, and assume $|\partial(W)| = 3k-2$. If we orient $\partial(W)$ so it is balanced (that is, the number of edges oriented in one direction across the cut equals the number oriented in the other, modulo $k$), then we can extend this orientation of $\partial(W)$ to a modulo $k$-orientation of $G$.*

*Proof.* We use induction on $\|G\|$. First suppose $|W| \geqslant 2$ and $|\overline{W}| \geqslant 2$. By induction we get good orientations for $G_W$ and $G_{\overline{W}}$ (these are formed from $G$, respectively, by contracting $W$ and $\overline{W}$ to single vertices). Since these orientations agree on $\partial(W)$, they combine to give a good orientation of $G$. So we assume instead $|W| = 1$, and define $z$ such that $W = \{z\}$.

We claim that if $X \subseteq V(G) \setminus \{z\}$ and $1 < |X| < |V(G) \setminus \{z\}|$ and $d(X)$ is odd, then $d(X) \geqslant 3k$. Suppose, to the contrary, that there exists such an $X$ with $d(X) \leqslant 3k-2$. Since $G$ is odd-$(3k-2)$-edge-connected, we know $d(X) = 3k-2$. Now form $G_X$ and $G_{\overline{X}}$. By induction $G_X$ has an orientation $D_X$ that extends the orientation of $\partial(z)$ and is a modulo $k$-orientation. Further, $D_X$ gives a balanced orientation of $\partial(X)$, since $D_X$ is a modulo $k$-orientation. By induction, we also get an orientation $D_{\overline{X}}$ of $G_{\overline{X}}$ that extends this orientation of $\partial(X)$. Since $D_X$ and $D_{\overline{X}}$ agree on $\partial(X)$, they give a modulo $k$-orientation of $G$ that extends the orientation of $\partial(W)$. Thus, $d(X) \geqslant 3k$.

As in Claim 2 in our proof of Theorem 6.33, we show that no vertex $v$ of $G$ has even degree. Suppose not. (We know $v \neq z$ since, by assumption, $d(z) = 3k-2$, which is odd.) If possible, we lift a pair of edges in $\partial(v)$ going to distinct vertices and finish by induction. Otherwise $v$ has only a single neighbor, so we delete $v$ and finish by induction. Any modulo $k$-orientation of $G - v$ extends to a modulo $k$-orientation of $G$, by orienting half of the edges in $\partial(v)$ into $v$ and orienting the other half out. Thus, every vertex of $G$ has odd degree.

Suppose each $X \subseteq V(G) \setminus \{z\}$ with $d(X)$ even has $d(X) \geqslant 2k$. Since $d(X) - \beta(X) = d(X)$ is even, $\tau(X) = \beta(X) = 0$, so $d(X) \geqslant 2k + \tau(X)$. (And if $d(X)$ is odd, then $d(X) \geqslant 3k-2 \geqslant (2k-2) + |\tau(X)|$, since $G$ is odd-$(3k-2)$-edge-connected.) Recall the remark following (6.1): since $d(z) = 3k-2$, which is odd, and $\beta(z) = 0$, we have $|\tau(z)| = k$. Since $d(z) = 3k-2 = 2k-2+|\tau(z)|$, graph $G$ satisfies the hypotheses of Theorem 6.33. Thus, $G$ has a modulo $k$-orientation, as desired.

Now assume instead that some $X \subseteq V(G)$ with $d(X)$ even has $d(X) \leqslant 2k-2$, and choose $X$ to be minimal. By induction, $G_X$ has a modulo $k$-orientation $D_X$ (that extends the orientation of $\partial(z)$). We can also get a modulo $k$-orientation of $G_{\overline{X}}$ that extends the orientation of $\partial(x)$ given by $D_X$. To do so, we use Theorem 6.33 with $\overline{x}$ playing the role of $z$, as follows.

By assumption $d(\overline{x}) = d(X) \leqslant 2k-2$. Since we chose $X$ to be minimal, there does not exist $Y \subseteq V(G_{\overline{X}}) \setminus \{\overline{x}\}$ with $d(Y)$ even and $d(Y) \leqslant 2k-2$. Similarly, if there exists $Y \subseteq V(G_{\overline{X}}) \setminus \{\overline{x}\}$ with $d(Y)$ odd and $d(Y) < 3k-2$, then such a $Y$ exists in $G$, which violates the hypothesis. So $G_{\overline{X}}$ has a modulo $k$-orientation $D_{\overline{X}}$ that extends the orientation of $\partial(X)$ given by $D_X$. Combining $D_{\overline{X}}$ and $D_X$ gives a modulo $k$-orientation of $G$ that extends the prescribed orientation of $\partial(W)$, as desired. □

Now we use Theorem 6.35 to prove Theorem 6.34.

*Proof of Theorem 6.34.* Fix an odd integer $k \geqslant 1$, and let $G$ be a graph that is odd-$(3k-2)$-edge-connected. We can handle each component separately, so assume $G$ is connected. Suppose $|G| = 2$. If $\|G\|$ is even, then we orient half of the edges in each direction. If $\|G\|$ is odd, then we first orient $k$ edges in a single direction, and afterward we orient half of the remaining edges in each direction.

Now assume $|G| \geqslant 3$. We use induction on $\|G\|$. If each vertex $v$ has even degree, then it has an Eulerian tour $D$ with $d_D^+(v) = d_D^-(v)$, so $G$ has a modulo $k$-orientation. Instead assume that some vertex has odd degree (so the odd-edge-connectivity is finite).

We pick an arbitrary vertex $v$ with $d(v)$ odd and with distinct neighbors $w$ and $x$, and lift the edge pair $wv, vx$ to form a new graph $G'$. (Again, if $v$ has a unique neighbor, then we proceed by induction on $V(G) \setminus \{v\}$.) Consider an edge-cut in $G$, denoted $\partial_G(W)$ and its corresponding edge-cut $\partial_{G'}(W)$ in $G'$. If $w, x \in W$ and $v \notin W$ (or $w, x \in \overline{W}$ and $v \notin \overline{W}$), then $|\partial_G(W)| - |\partial_{G'}(W)| = 2$; otherwise $|\partial_{G'}(W)| = |\partial_G(W)|$. So by repeatedly lifting edge pairs incident to $v$, we eventually reach a graph $G^*$ with odd-edge-connectivity exactly $3k-2$. We choose an arbitrary smallest odd edge-cut $\partial(W)$ in $G^*$ and give it a balanced orientation. By Theorem 6.35, we can extend this orientation of $\partial(W)$ to a modulo $k$-orientation of $G^*$, which translates to the desired modulo $k$-orientation of $G$. □

## Notes

Nowhere-zero flows were introduced by Tutte [389, 390] to generalize the face-coloring problem to non-planar graphs. Much of the work in this area has been motivated by Tutte's 5-Flow Conjecture [390], 4-Flow Conjecture [392], and 3-Flow Conjecture (see [48, Open Problem #48]). Theorem 6.10 is due to Tutte [389, 390], and we follow his original proof that statements (ii) and (iii) are equivalent. But to show that (i) and (ii) are equivalent we follow Younger [424]; notice the similarity between this proof and that of Lemma 5.4.

After the proof of Theorem 6.10, we noted the following. For every multigraph $G$ there exists a polynomial $\Phi_G$ such that for every finite abelian group $H$, with order $|H|$, the number of NZ $H$-flows on $G$ is $\Phi_G(|H|)$. The key observation needed to prove this fact is that $\Phi_G(H) = \Phi_{G/e}(H) - \Phi_{G-e}(H)$ for every group $H$ and every non-loop edge $e$. This important identity is called a *deletion/contraction equation*. Tutte's investigations into other parameters satisfying similar recurrences led him to define (what is now called) the Tutte polynomial, shown below.

deletion/ contraction equation

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-|V|},$$

where $k(A)$ denotes the number of components of the graph with vertex set $V$ and edge set $A$. (It is straightforward to check that $T_G = T_{G-e} + T_{G/e}$.) The Tutte polynomial has connections with many areas of mathematics. [148]

The key step in proving Lemma 6.12 is the observation that for each spanning tree $T$, the graph $G$ has a 2-flow that is non-zero on each edge outside $T$; see Exercise 1. This was observed by Jaeger [224]. The Tree-Packing Theorem was proved by Tutte [391] and Nash-Williams [318]. More generally, Edmonds gave necessary and sufficient conditions for a matroid to have k disjoint bases. (For graphic matroids, the independent sets are the acyclic edge sets, so if a graph is connected, then its bases are precisely its spanning trees. Thus, we immediately recover the Tree-Packing Theorem. West [412, Corollary 8.2.59] gives a nice presentation.)

As a special case of his 4-Flow Conjecture, Tutte also conjectured the following [393].

**Conjecture 6.36 (Tutte's 3-Edge-Coloring Conjecture).** *Every bridgeless cubic graph with no subdivision of the Petersen graph is* 3*-edge-colorable.*

Recall that a cubic graph is 3-edge-colorable if and only if it has an NZ 4-flow (see Exercise 4). Tait proved that a bridgeless cubic planar graph is 3-edge-colorable if and only if it is 4-face colorable. Thus, Tutte's 3-Edge-Coloring Conjecture implies the 4 Color Theorem, since the Petersen graph is non-planar.

In 1998, Robertson, Sanders, Seymour, and Thomas announced a proof of Tutte's 3-Edge-Coloring Conjecture. Seymour described the proof: "Repeat the proof of the 4 Color Theorem, twice." Their general approach is to classify all cubic bridgeless graphs with no Petersen subdivision. The non-planar ones comprise "single-cross" graphs (which can be drawn in the plane with a single edge crossing on the outer face, and which reduce to the planar case[8]), "double-cross" graphs, and "apex" graphs (which become planar after deleting some single vertex). This classification is done by hand (without computer) in [346, 347, 345]. In contrast, the proofs that the double-cross graphs [145] and apex graphs[9] are 3-edge-colorable each reuse computer code developed by Robertson, Sanders, Seymour, and Thomas for their proof of the 4 Color Theorem.

We digress briefly to discuss the complexity of deciding, for each fixed integer $k \geqslant 2$, whether an arbitrary input graph has an NZ k-flow. (It is easy to check whether a connected graph is bridgeless; see [357, Section 15.3].) For $k = 2$, the answer is yes if and only if every vertex degree is even. And for $k \geqslant 6$, the answer is simply yes. For $k = 3$, when G is cubic the problem is equivalent to G being 3-edge-colorable, which Holyer showed to be NP-complete; see Theorem 2.8. For $k = 5$, if the 5-Flow Conjecture is true, then the answer is again yes. But Kochol [264] showed that if the conjecture is false, then the problem is NP-Complete.

Jaeger studied the problem of NZ $\mathbb{Z}_{2k+1}$-flows where the flow values are restricted to be either 1 or $-1$. Equivalently, these are orientations D of G such that $d^+_D(v) - d^-_D(v) \equiv 0 \pmod{2k+1}$ for each vertex $v$. Recall that we call such a D a modulo $(2k+1)$-orientation. Jaeger posed the following conjecture [226].

**Conjecture 6.37 (Jaeger's Circular Flow Conjecture).** *For each* $k \geqslant 1$*, every* $4k$*-edge-connected graph has a modulo* $(2k+1)$*-orientation.*

[8]Single-cross graphs were shown to be 3-edge-colorable by Jaeger [225]. In the introduction to [145], Edwards, Sanders, Seymour, and Thomas give a short proof, which they attribute to Jaeger. The idea is to modify the graph slightly to get a bridgeless cubic planar graph G. Now G is 3-edge-colorable by the 4 Color Theorem, and we can modify this coloring to 3-edge-color the original graph. We leave the details to Exercise 11.

[9]The proof that cubic apex graphs are 3-edge-colorable is still in preparation.

It is easy to check that the case $k = 1$ is equivalent to Tutte's 3-Flow Conjecture. Now we observe that the case $k = 2$ implies Tutte's 5-Flow Conjecture. Recall, from Lemma 6.17, that we can assume $G$ is 3-edge-connected. Form $G'$ from $G$ by replacing each edge with 3 parallel edges. Now $G'$ is 9-edge-connected, since $G$ is 3-edge-connected. By assumption, $G'$ has a modulo 5-orientation $D$, since it is 8-edge-connected. To get an NZ $\mathbb{Z}_5$-flow for $G$, we do the following. For each three parallel edges in $G'$ that arose from a single edge $e$ in $G$, we give $e$ flow value equal to the "net orientation" of those edges: 3 (resp. 1, $-1$, $-3$) if exactly 3 (resp. 2, 1, 0) edges are oriented by $D$ in the direction of $e$ in $G$. It is easy to check that this produces the desired NZ $\mathbb{Z}_5$-flow in $G$.

The Circular Flow Conjecture inspired many partial results (particularly in the case when $G$ is planar); the most notable of these is Theorem 6.28. Ultimately, however, this conjecture was disproved [201]. Now work has shifted to determining the minimum edge-connectivity that guarantees a graph $G$ has a modulo $(2k+1)$-orientation, and that guarantees $G$ has a $\beta$-orientation for every $\mathbb{Z}_{2k+1}$-boundary $\beta$. Since all known counterexample are non-planar, some research focuses on proving the Circular Flow Conjecture restricted to planar graphs.

Jaeger also posed [224] a "Weak 3-Flow Conjecture": There is some integer $t$ such that every $t$-edge-connected graph has an NZ 3-flow. This was confirmed by Thomassen [383], with $t = 8$. More generally, Thomassen showed that, for each odd integer $k \geqslant 3$, if $G$ is $(2k^2+k)$-edge-connected and $\beta$ is a $\mathbb{Z}_{2k+1}$-boundary, then $G$ has a $\beta$-orientation. Theorem 6.28, which was proved by Lovász, Thomassen, Wu, and Zhang [291], weakens this hypothesis to $(3k-3)$-edge-connected, and its proof draws heavily on Thomassen's work. Kochol showed [265] that it suffices to prove Tutte's 3-Flow Conjecture for graphs that are 5-edge-connected. So Theorem 6.28 is only "one step" away.

The definitive reference on nowhere-zero flows is the monograph of Zhang [427]. It contains many enlightening exercises, and we recommend it to the reader. Our presentation in Section 6.1 follows the introduction of an unpublished essay by Lovász [290]. The NZ 6-Flow Theorem has at least five distinct proofs: [361] and [116] each give two; see also [114] and [115]. The proof we presented is from [116], and we chose it because it most closely mirrors the proof of Theorem 6.28. The results in Section 6.3 are due to DeVos, Langhede, Mohar, and Šámal [113]. They improve on similar results of Dvořák, Mohar, and Šámal [137] which, under the same hypotheses, only yielded smaller exponents.

To conclude this section, we mention two generalizations of conjectures discussed above. An *odd cut* is an edge-cut $\partial(W)$ such that $W$ contains an odd number of vertices of odd degree. An $r$-*graph* is an $r$-regular graph in which each odd cut has size at least $r$. As we mentioned at the end of the Notes for Chapter 3, Seymour conjectured that every planar $r$-graph is $r$-edge-colorable. The case $r = 3$ is equivalent to the 4 Color Theorem and the conjecture has also been verified when $r \in \{4, 5\}$ [326], $r = 6$ [131], $r = 7$ [86], and $r = 8$ [87].

A stronger conjecture is that an $r$-graph is $r$-edge-colorable whenever it has no Petersen subdivision; now the case $r = 3$ is Tutte's 3-Edge-Coloring Conjecture. This stronger version remains open when $r \geqslant 4$. Jaeger conjectured that every cubic bridgeless graph $G$ has a coloring of its edges with the edges of the Petersen graph, $P$, so that any 3 edges incident to a common vertex in G are colored with 3 edges incident to a common vertex in P. Jaeger described this problem in the language of nowhere-zero flows, in his excellent survey [227]. This is now known as the Petersen Coloring Conjecture. If it is true, it implies both the Berge–Fulkerson Conjecture (that every bridgeless cubic graph has 6 perfect matchings, with each edge appearing in exactly two of them) and the Cycle Double Cover Conjecture (that every bridgeless cubic graph has a set of cycles with each edge appearing in exactly two of them).

## Exercises

6.1. Given a spanning tree $T$ in a graph $G$, for each edge $e \in E(G) \setminus E(T)$, let $C_e$ denote the unique cycle in $T+e$. Show that $\sum_{e \in E(G)\setminus E(T)} C_e \pmod 2$ is an eulerian subgraph of $G$ containing every edge of $E(G) \setminus E(T)$; here we write modulo 2 sum to denote symmetric difference. Show that this subgraph is identical to that in the proof of Lemma 6.12.

6.2. Determine the minimum $k$ such that $K_n$ has an NZ $k$-flow, for each integer $n \geqslant 3$.

6.3. Show that a cubic graph has an NZ 3-flow if and only if it is bipartite. [389]

6.4. Show that a cubic graph has an NZ 4-flow if and only if it is 3-edge-colorable.

6.5. Show that a cubic graph has an NZ $\mathbb{Z}_2^2$-flow if and only if it is 3-edge-colorable. By Theorem 6.10, this gives an alternate solution to the previous problem.

6.6. Show that every graph with a Hamiltonian cycle admits an NZ 4-flow.

6.7. If each edge of $G$ is in a triangle, show that $G$ has an NZ 4-flow.

6.8. (a) Show that a wheel graph (the join of a cycle and $K_1$) has an NZ 3-flow if and only if its order is odd. (b) Moreover, it is a contractible configuration for a 3-flow. (c) Apply this fact to show that the 3-flow conjecture is true for graphs with a dominating vertex (one adjacent to all other vertices in $G$).

6.9. Show that if $G$ has at most two vertices of odd degree, then $G$ has an NZ 3-flow.

6.10. Show that if $G - e$ has an NZ 4-flow, then $G$ has an NZ 5-flow.

6.11. A bridgeless cubic graph is "single-cross" if it can be drawn in the plane with a single edge crossing. Tait proved that a bridgeless planar graph is 3-edge-colorable if and only if it is 4-face-colorable. (By Exercise 4, Tait's Theorem is the case $k = 4$ of Theorem 6.6.) Use the 4 Color Theorem (and Tait's Theorem) to show that every single-cross graph is 3-edge-colorable.

# Chapter 7

# Rosenfeld Counting

> There is no problem in all mathematics that cannot be solved by direct counting. But with the present implements of mathematics many operations of counting can be performed in a few minutes which without mathematical methods would take a lifetime.
>
> —Ernst Mach

In this chapter we consider a wide variety of coloring problems. Generally, the colorings that we seek will be proper colorings that also satisfy additional constraints (for example every 2 color classes might be required to induce a forest, or to induce a forest in which each tree is a star). We typically prove upper bounds on these chromatic numbers, for all graphs, in terms of maximum degree. Many results of this type were first proved using the Local Lemma and later reproved using Entropy Compression. Both of these techniques are powerful, and we describe them a bit more in the Notes. But here we focus on a clever type of inductive counting argument, recently discovered by Matthieu Rosenfeld. When such arguments apply, they often give rise to strikingly simple proofs, avoiding many technical details required by the two alternate methods mentioned above.

## 7.1 Introduction

The key idea in this chapter is to repeatedly extend a partial coloring, showing that each time that we color an additional vertex $v$ the number of valid partial colorings increases by at least a constant factor. Given a coloring of a subgraph $H$, some candidate colors for $v$ will lead to invalid colorings of $H + v$. But we will injectively map these invalid extensions to $v$ onto valid colorings of some smaller subgraph $H'$. Since $H'$ has exponentially fewer valid colorings, by the induction hypothesis, some of the possible extensions to $H + v$ must be valid. In fact, enough are valid that we can finish the induction step. These arguments guarantee not only one coloring of $G$, but exponentially many!

### 7.1.1 Nonrepetitive List-coloring of Paths

square

nonrepetitive

**Definition 7.1.** A *square* in a coloring $\varphi$ of a graph $G$ is a path $v_1 \cdots v_{2k}$ such that $\varphi(v_{i+k}) = \varphi(v_i)$ for all $i \in [k]$. That is, the colors on the first half of the path are repeated on the second half, in exactly the same order; see the left of Figure 7.1. A coloring is *nonrepetitive* if it does not contain any square. In particular, every nonrepetitive coloring is proper.

Figure 7.1: A 3-coloring of $P_8$ containing a square (left) and a square-free 3-coloring of $P_8$ (right).

Using only 3 colors, Thue constructed nonrepetitive colorings of $P_n$, with $n$ arbitrarily large (Exercise 1 presents the construction). This motivated the following conjecture.

**Conjecture 7.2.** *For each* 3*-assignment* $L$ *to the vertices of* $P_n$*, for each positive integer* $n$*, there exists a nonrepetitive coloring* $\varphi$ *such that* $\varphi(v) \in L(v)$ *for all* $v$*.*

The conjecture posits that Thue's result on nonrepetitive 3-coloring of paths generalizes to list 3-coloring. This problem remains open, but our first theorem nearly solves it.

**Theorem 7.3.** *For each* 4*-assignment* $L$ *to the vertices of* $P_n$*, for each positive integer* $n$*, there exists a nonrepetitive coloring* $\varphi$ *such that* $\varphi(v) \in L(v)$ *for all* $v$*.*

Theorem 7.3 follows immediately from our next lemma, which actually proves that $P_n$ has exponentially many nonrepetitive $L$-colorings.

$\mathcal{C}_i$, $P$

**Lemma 7.4.** *Let* $L$ *be a* 4*-assignment to the vertices of a path* $P$*. For each* $i \geqslant 1$*, let* $\mathcal{C}_i$ *be the set of nonrepetitive* $L$*-colorings of the first* $i$ *vertices of* $P$*. For all* $i < |V(P)|$*, we have*

$$|\mathcal{C}_{i+1}| \geqslant 2|\mathcal{C}_i|.$$

Since $|\mathcal{C}_1| = 4$, the lemma shows that $P$ has more than $2^{|V(P)|}$ nonrepetitive $L$-colorings.

$\mathcal{F}$

$\mathcal{F}_j$

*Proof.* Our proof is by induction on $i$. Let $\mathcal{F}$ be the set of $L$-colorings of the first $i+1$ vertices of $P$ that are nonrepetitive when restricted to the first $i$ vertices but that contain a square including vertex $i+1$. Clearly, $|\mathcal{C}_{i+1}| = 4|\mathcal{C}_i| - |\mathcal{F}|$. Let $\mathcal{F}_j$ be the subset of $\mathcal{F}$ that contains a square of length $2j$. So[1] $\mathcal{F} = \cup_{j \geqslant 1} \mathcal{F}_j$. Thus, we seek to bound $|\mathcal{F}_j|$ for all $j \geqslant 1$.

Each $\varphi \in \mathcal{F}_j$ restricts to a nonrepetitive $L$-coloring $\varphi'$ of the first $i - j + 1$ vertices of $P$. Furthermore, $\varphi$ is uniquely determined by this restriction $\varphi'$. (We say that $\varphi'$ is mapped to by $\varphi$, or that $\varphi$ is "charged" to $\varphi'$.) Thus, $|\mathcal{F}_j| \leqslant |\mathcal{C}_{i-j+1}|$. By the induction hypothesis,

[1]It is easy to check that $\mathcal{F}_j \cap \mathcal{F}_k = \emptyset$ whenever $j \neq k$, but our proof does not need this, so we omit the details.

$|\mathcal{C}_{i-j+1}| \leqslant 2^{-j+1}|\mathcal{C}_i|$, for each $j \geqslant 1$. Thus,

$$\begin{aligned}|\mathcal{C}_{i+1}| = 4|\mathcal{C}_i| - |\mathcal{F}| &\geqslant 4|\mathcal{C}_i| - \sum_{j\geqslant 1}|\mathcal{F}_j| \\ &\geqslant 4|\mathcal{C}_i| - \sum_{j\geqslant 1}|\mathcal{C}_{i-j+1}| \\ &\geqslant 4|\mathcal{C}_i| - \sum_{j\geqslant 1}2^{-j+1}|\mathcal{C}_i| \geqslant 2|\mathcal{C}_i|.\end{aligned}$$

□

The heart of Rosenfeld Counting is mapping all bad extensions to valid partial colorings with fewer vertices colored (the number of which is exponentially smaller). To better illuminate the proof of Lemma 7.4, we consider the following example.

**Example 7.5.** Let $P$ be a path on 5 vertices, $v_1 \cdots v_5$, and let $L(v_i) := \{1,2,3,4\}$ for all $i \in [5]$. $v_1 \cdots v_5$
Now $|\mathcal{C}_1| = 4$, since all possible colorings are valid. Next, $|\mathcal{C}_2| = 4|\mathcal{C}_1| - |\mathcal{C}_1| = 12$, since an extension $\varphi$ of $\varphi' \in \mathcal{C}_1$ is valid if and only if $\varphi(v_2) \neq \varphi(v_1)$. Similarly, $|\mathcal{C}_3| = 4|\mathcal{C}_2| - |\mathcal{C}_2| = 36$, since again an extension $\varphi$ of $\varphi' \in \mathcal{C}_2$ is valid if and only if $\varphi(v_3) \neq \varphi(v_2)$. Furthermore, $|\mathcal{C}_4| = 4|\mathcal{C}_3| - |\mathcal{C}_3| - |\mathcal{C}_2| = 96$, since an extension $\varphi$ of $\varphi' \in \mathcal{C}_3$ is valid if and only both (a) $\varphi(v_4) \neq \varphi(v_3)$ and (b) $(\varphi(v_4), \varphi(v_3)) \neq (\varphi(v_2), \varphi(v_1))$. Note that every coloring in $\mathcal{C}_2$ is mapped to by an invalid extension of a coloring in $\mathcal{C}_3$.

In contrast $|\mathcal{C}_5| \geqslant 4|\mathcal{C}_4| - |\mathcal{C}_4| - |\mathcal{C}_3| = 4(96) - 96 - 36 = 252$, but this inequality is strict. The reason is that not every coloring in $\mathcal{C}_3$ is mapped to by an invalid extension of a coloring in $\mathcal{C}_4$. Specifically, consider a coloring $\varphi'$ in $\mathcal{C}_3$ with $\varphi'(v_1) = \varphi'(v_3) = a$ and $\varphi'(v_2) = b$ for distinct $a, b \in [4]$. If $\varphi$ is a coloring of $v_1 \cdots v_5$ that both (i) restricts to $\varphi'$ and (ii) has a square of length 4 ending at $v_5$, then $\varphi(v_1) = \varphi(v_3) = \varphi(v_5) = a$ and $\varphi(v_2) = \varphi(v_4) = b$. However, $\varphi$ is still invalid when restricted to $v_1 \cdots v_4$. Thus, this restriction is excluded from $\mathcal{C}_4$. In total, 12 such colorings $\varphi' \in \mathcal{C}_3$ are not mapped to by candidate extensions of colorings in $\mathcal{C}_4$. Thus, we in fact have $|\mathcal{C}_5| = 252 + 12 = 264$. ♢

In the next section we develop a more general framework, with the same core approach. In the rest of the chapter, we apply these techniques to a wide variety of coloring problems.

## 7.2 A General Framework

Informally, we want a proper coloring of $G$ that also avoids on certain subgraphs $H$ specific proper colorings of $H$ that are forbidden. We call such a coloring of $G$ *good*; colorings of $G$ that are not good are *bad*. Given a coloring that is good (resp. bad), renaming a color always yields another coloring that is good (resp. bad). So, when specifying bad colorings for a subgraph $H$, we usually specify just one representative from each equivalence class. We call this representative a forbidden *template*. We often denote a template for a subgraph $H$ by $\tilde{H}$. template, $\tilde{H}$
For each subgraph $H$, let $\mathcal{B}_H$ be a set of templates forbidding certain colorings of $H$, before $\mathcal{B}_H$, $\mathcal{B}$

renaming colors. Let $\mathcal{B} := \{\mathcal{B}_H : H \subseteq G\}$; for each H, possibly $\mathcal{B}_H = \emptyset$, i.e., no colorings of H are explicitly forbidden. (Typically, we will have $|\mathcal{B}_H| \leqslant 1$ for all $H \subseteq G$, but Sections 7.3.3 and 7.4 offer exceptions to this general rule.) A coloring $\varphi$ of G is $\mathcal{B}_H$-*bad* if $\varphi$ gives H a forbidden coloring, and $\varphi$ is $\mathcal{B}$-*bad* if $\varphi$ is $\mathcal{B}_H$-bad for some H. If $\varphi$ is not $\mathcal{B}$-bad, then $\varphi$ is $\mathcal{B}$-good. A pair $(G, \mathcal{B})$ is an *instance*.

$\mathcal{B}_H$-bad

$\mathcal{B}$-bad, $\mathcal{B}$-good

instance

A few examples will clarify these definitions. In standard vertex coloring, we use a single type of template (for each edge $e \in E(G)$): $\tilde{H}_1$ is edge $e$ with endpoints colored 1,1. In *star coloring*, we require a proper coloring with no 2-colored path on 4 vertices; so any two color classes induce a star forest. For star coloring, we use two templates: $\tilde{H}_1$ is as above and $\tilde{H}_2$ is (for each 4-vertex path P in G) the path P colored 1,2,1,2.

star coloring

Fix an instance $(G, \mathcal{B})$ and consider a subgraph H of G. A subset $S \subseteq V(H)$ *determines* $\mathcal{B}_H$ if any two colorings that are both forbidden by templates in $\mathcal{B}_H$ and agree on S must be identical on H. For every $v \in V(H)$, we assume that $\mathcal{B}_H$ is determined by some (non-empty) subset of $V(H) \setminus \{v\}$; we consider this a part of the definition of 'instance'. The *weight* of a subgraph H is $\min_{v \in V(H)}\{|V(H)| - |S| - 1\}$, where S is a minimum-sized subset of $V(H) \setminus \{v\}$ that determines $\mathcal{B}_H$ (if $\mathcal{B}_H = \emptyset$, then H has no weight). For the edge $H_1$ in the previous paragraph with $\mathcal{B}_{H_1} = \{\tilde{H}_1\}$ note that $\tilde{H}_1$ is determined by each endpoint so $H_1$ has weight $2-1-1=0$. Similarly, for the 4-vertex path $H_2$ with $\mathcal{B}_{H_2} = \{\tilde{H}_2\}$, note that $\tilde{H}_2$ is determined by any two successive vertices, so $H_2$ has weight $4-2-1=1$.

determines

weight

For each $v \in V(G)$, let $N_i(v)$ be the number of subgraphs H with $v \in V(H)$ and with weight $i$. For a list-assignment L, let $P(G, \mathcal{B}, L)$ be the number of L-colorings of G that are $\mathcal{B}$-good.

$N_i(v)$

$P(G, \mathcal{B}, L)$

The notion of "instance" is very general. In particular, it captures acyclic coloring, star coloring, nonrepetitive coloring, frugal coloring, and acyclic edge-coloring, to name a few. But before we turn to those examples, we state and prove our general framework.

**Theorem 7.6.** *Let* $(G, \mathcal{B})$ *be an instance. Assume there exists a real number* $\beta$ *and an integer* $k \geqslant 1$ *such that every vertex* $v \in V(G)$ *satisfies*

$$k \geqslant \beta + \sum_{i \geqslant 0} \beta^{-i} N_i(v). \tag{7.1}$$

*For every* k*-assignment* L *to* $V(G)$*, we have*

$$P(G, \mathcal{B}, L) \geqslant \beta^{|V(G)|}.$$

The proof of Theorem 7.6 is not overly difficult (as we will see soon). But the impatient reader should feel no guilt about skipping it now, moving ahead to the abundance of applications in Section 7.3, and only returning to the proof after being convinced of the theorem's utility.

For an instance $(G, \mathcal{B})$ and $H \subseteq G$, we write $(H, \mathcal{B})$ for the instance $(H, \{\mathcal{B}_J : J \subseteq H\})$. And if L is a list assignment for G, then we also consider L restricted to $V(H)$ to be a list assignment for H. Theorem 7.6 will follow directly from Lemma 7.7.

**Lemma 7.7.** *Let* $(G, \mathcal{B})$ *be an instance. Fix a real number* $\beta \geqslant 1$ *and an integer* $k \geqslant 1$ *such that* (7.1) *holds for every vertex* $v$ *of* $G$*. Now for every* $k$*-assignment* $L$ *to* $G$*, for every induced subgraph* $H$ *of* $G$*, and for every vertex* $v$ *of* $H$ *we have*

$$P(H, \mathcal{B}, L) \geqslant \beta P(H - v, \mathcal{B}, L).$$

*For the subgraph* $H_\emptyset$ *with no vertices, we let* $P(H_\emptyset, \mathcal{B}, L) := 1$*, for the unique "empty"* $L$*-coloring.*

The proof of this lemma reuses many ideas from the proof of Lemma 7.4.

*Proof.* Our proof is by induction on $|V(H)|$. First we consider the base case, $|V(H)| = 1$. By definition, $P(H - v, \mathcal{B}, L) = P(H_\emptyset, \mathcal{B}, L) = 1$. In our definition of instance, we require that if $\mathcal{B}_J \neq \emptyset$, then $J$ contains at least two vertices. So all $k$ choices of colors in $L(v)$ yield $\mathcal{B}$-good colorings of $H$. Thus, we are done by (7.1) since $P(H, \mathcal{B}, L) = k \geqslant \beta = \beta P(H - v, \mathcal{B}, L)$.

Now we consider the induction step, when $|V(H)| > 1$. Fix an arbitrary vertex $v \in V(H)$. Let $\mathcal{F}$ be the set of $\mathcal{B}$-bad $L$-colorings of $H$ that are $\mathcal{B}$-good on $H - v$. By definition,

$$P(H, \mathcal{B}, L) = kP(H - v, \mathcal{B}, L) - |\mathcal{F}|.$$

For each $\varphi \in \mathcal{F}$, there exists $J \subseteq H$ such that $\varphi$ is $\mathcal{B}_J$-bad; we will "charge" this $L$-coloring $\mathcal{F}$
$\varphi$ to this subgraph $J$. (If $\varphi$ is $\mathcal{B}_J$-bad for more than one subgraph $J$, then we choose one arbitrarily.) Let $\mathcal{F}_i$ be the set of $L$-colorings in $\mathcal{F}$ that are charged to subgraphs $J$ with weight $\mathcal{F}_i$
$i$. So $|\mathcal{F}| = \sum_{i \geqslant 0} |\mathcal{F}_i|$. Given $\varphi$ charged to a subgraph $J$ with weight $i$, there exists a subset $S$ $S$
of $V(J) \setminus \{v\}$ that determines $\mathcal{B}_J$ and such that $|V(J)| - |S| - 1 = i$. Let $T := V(J) \setminus S$. Since $T$
$\varphi$ is $\mathcal{B}$-good for $H - v$, it is also $\mathcal{B}$-good for $H - T$. Since $\mathcal{B}_J$ is determined by $S$, each $\mathcal{B}$-bad $L$-coloring $\varphi$ charged to the subgraph $J$ maps injectively to a $\mathcal{B}$-good $L$-coloring of $H - T$ (this is simply the restriction of $\varphi$ to $H - T$). So we must bound $P(H - T, \mathcal{B}, L)$. Note that $v \in T$ and $|T| = |V(J)| - |S| = i + 1$. So $|T \setminus \{v\}| = i$. Using the induction hypothesis $i$ times gives

$$P(H - v, \mathcal{B}, L) \geqslant \beta^i P(H - T, \mathcal{B}, L).$$

That is, $P(H - T, \mathcal{B}, L) \leqslant \beta^{-i} P(H - v, \mathcal{B}, L)$. The number of subgraphs $J$ with weight $i$ that contain $v$ is $N_i(v)$. And the number of bad colorings charged to each such $J$ is at most $\beta^{-i} P(H - v, \mathcal{B}, L)$. Thus, $|\mathcal{F}_i| \leqslant N_i(v)\beta^{-i} P(H - v, \mathcal{B}, L)$. Using this and (7.1), we get:

$$\begin{aligned}
P(H, \mathcal{B}, L) &= kP(H - v, \mathcal{B}, L) - |\mathcal{F}| \\
&= kP(H - v, \mathcal{B}, L) - \sum_{i \geqslant 0} |\mathcal{F}_i| \\
&\geqslant kP(H - v, \mathcal{B}, L) - \sum_{i \geqslant 0} N_i(v)\beta^{-i} P(H - v, \mathcal{B}, L) \\
&= [k - \sum_{i \geqslant 0} N_i(v)\beta^{-i}] P(H - v, \mathcal{B}, L) \\
&\geqslant \beta P(H - v, \mathcal{B}, L).
\end{aligned}$$

□

In the rest of the chapter, the following easy proposition is often useful.

**Proposition 7.8.** *For each real number* $x > 1$*, we have*

$$\sum_{i \geqslant 1} ix^{-i} = \frac{x}{(x-1)^2}.$$

*Proof.* Recall that $(1-y)^{-1} = \sum_{i \geqslant 0} y^i$, whenever $|y| < 1$. Taking $x := y^{-1}$ gives $\sum_{i \geqslant 0} x^{-i} = \frac{x}{x-1}$. We differentiate each side, and afterwards multiply by $-x$. □

## 7.3 Easy Examples

In this section we present six applications of Theorem 7.6.

### 7.3.1 Star Coloring

star coloring

A proper coloring $\varphi$ is a *star coloring* if each pair of its color classes induces a star forest. Equivalently, under $\varphi$ every path on 4 vertices uses at least 3 colors.

**Theorem 7.9.** *Let* $G$ *be a graph with maximum degree* $\Delta$*, let* $\beta := (\Delta-1)\sqrt{2\Delta}$*, and let* $k := \lceil (\Delta-1)\sqrt{8\Delta} + \Delta \rceil$*. If* $L$ *is a* $k$*-assignment for* $G$*, then* $G$ *has at least* $\beta^{|V(G)|}$ *star* $L$*-colorings.*

*Proof.* Given a graph $G$ that we want to star color, we create an instance $(G, \mathcal{B})$ and apply Theorem 7.6 as follows. For each edge $vw$ of $G$, we let $\mathcal{B}_{vw}$ consist of edge $vw$ colored 1,1. For each path $vwxy$ of $G$, we let $\mathcal{B}_{vwxy}$ consist of $vwxy$ colored 1,2,1,2. (We consider a path to be an unordered subgraph of $G$, so the paths $vwxy$ and $yxwv$ are the same and have only a single set of forbidden templates.) Clearly, any $\mathcal{B}$-good coloring of $G$ is a star coloring of $G$. So it remains to find the optimal choices of $\beta$ and $k$.

For each edge $H$, the set $\mathcal{B}_H$ (which consists of a single template 1,1) is determined by each endpoint, so $H$ has weight $2-1-1=0$. For each subgraph $H$ which is $P_4$, the set $\mathcal{B}_H$ (which consists of a single template 1,2,1,2) is determined by any two successive vertices of $H$, so $H$ has weight $4-2-1=1$. Each vertex $v$ lies on at most $\Delta$ edges and

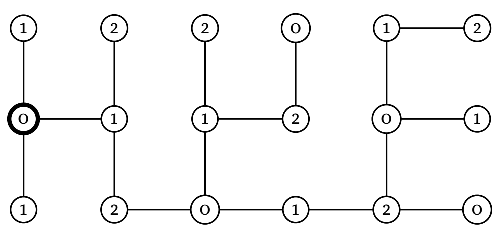



Figure 7.2: Every forest has a star 3-coloring. In each component we pick an arbitrary root $r$ and color vertices by distance from $r$ modulo 3.

lies on at most $2\Delta(\Delta-1)^2$ copies of $P_4$; so $N_0(v) \leqslant \Delta$ and $N_1(v) \leqslant 2\Delta(\Delta-1)^2$. Thus, we need $k \geqslant \beta + \Delta + 2\Delta(\Delta-1)^2\beta^{-1}$. The right side is minimized (using calculus) when $1-\beta^{-2}(2\Delta)(\Delta-1)^2 = 0$, which simplifies to $\beta = (\Delta-1)\sqrt{2\Delta}$. Thus, we take $\beta := (\Delta-1)\sqrt{2\Delta}$ and $k := \lceil \beta + \Delta + 2\Delta(\Delta-1)^2\beta^{-1} \rceil = \lceil (\Delta-1)\sqrt{8\Delta} + \Delta \rceil$. □

### 7.3.2 Acyclic Edge Coloring

A proper edge coloring $\varphi$ of a graph is *acyclic* if the subgraph induced by any two color classes is acyclic; see Figure 7.2. Equivalently, $\varphi$ gives the edges of each cycle at least 3 distinct colors.

acyclic edge coloring

**Theorem 7.10.** *Let* $G$ *be a graph, let* $\beta := \sqrt{3}(\Delta-1)$*, and let* $k := \lceil 4.6(\Delta-1) \rceil$*. If* $L$ *is a* $k$*-assignment to the edges of* $G$*, then* $G$ *has at least* $\beta^{|V(G)|}$ *acyclic* $L$*-edge-colorings.*

Technically, Theorem 7.6 only applies when we are coloring vertices. So, to be formal, we should color the vertices of the line graph of G. However, this formality only obscures understanding. So below we slightly abuse terminology and color the edges of G.

*Proof.* Let $G$ and $L$ be as stated above. We create an instance $(G, \mathcal{B})$ and apply Theorem 7.6 as follows. For each edge pair $e_1, e_2$ of adjacent edges in $G$, we let $\mathcal{B}_{e_1e_2}$ consist of $e_1$ and $e_2$ colored 1,1. For each even length cycle $e_1 \cdots e_{2i}$ in $G$, we let $B_{e_1\cdots e_{2i}}$ be $\varphi$ such that $\varphi(e_{2j-1}) = 1$ and $\varphi(e_{2j}) = 2$, for all $j \in [i]$. Each subgraph of the first type has weight 0; and always $N_0(e) \leqslant 2(\Delta-1)$. Each subgraph of the second type is determined by any two successive edges in the cycle, so has weight $2i-2-1 = 2i-3$. And $N_{2i-3}(e) \leqslant (\Delta-1)^{2i-2}$. Thus, to apply Theorem 7.6, we need

$$\begin{aligned} k &\geqslant \beta + 2(\Delta-1) + \sum_{i\geqslant 2} \beta^{-(2i-3)}(\Delta-1)^{2i-2} \\ &= \beta + 2(\Delta-1) + \beta \sum_{i\geqslant 1} ((\Delta-1)/\beta)^{2i} \\ &= \beta + 2(\Delta-1) + \beta\frac{(\Delta-1)^2}{\beta^2-(\Delta-1)^2} \\ &= (\Delta-1)[\sqrt{3}+2+\sqrt{3}/2] \approx (\Delta-1)4.598\ldots \end{aligned}$$

The third line uses the geometric sum formula and the fourth line comes by letting $\beta := \sqrt{3}(\Delta-1)$, which minimizes the expression on the third line (as is shown by calculus). □

### 7.3.3 Nonrepetitive Coloring

A *nonrepetitive coloring* of a graph $G$ is a (proper) coloring $\varphi$ such that $G$ does not contain any path $v_1 \cdots v_{2s}$ such that the colors on the first half of the path repeat on the second half of the path in the same order. That is, we cannot have $\varphi(v_{i+s}) = \varphi(v_i)$ for all $i \in [s]$.

nonrepetitive coloring

**Theorem 7.11.** *Let* $G$ *be a graph with maximum degree* $\Delta$*, let* $\beta := (\Delta-1)^2(1+2^{1/3}\Delta^{-1/3})$*, and let* $k := \lceil \beta + \Delta^{5/3}2^{-2/3}(1+2^{1/3}\Delta^{-1/3})^2 \rceil$*. If* $L$ *is a* $k$*-assignment, then the number of nonrepetitive* $L$*-colorings of* $G$ *is at least* $\beta^{|V(G)|}$*.*

*Proof.* Given a graph $G$ that we want to nonrepetitively color, we construct an instance $(G, \mathcal{B})$ as follows. For each vertex $v$, and each path $P$ of even order containing $v$, we let $\mathcal{B}_P$ consist of all colorings in which the colors on the second half of $P$ repeat the colors on the first half (in the same order). Given such a path $P$ of order $2i$, its weight is exactly $i-1$. And the number of such paths containing each vertex $v$ is at most $i\Delta(\Delta-1)^{2i-2}$. That is, $N_{i-1}(v) \leqslant i\Delta(\Delta-1)^{2i-2}$. To finish by Theorem 7.6, we want

$$k \geqslant \beta + \sum_{i \geqslant 1} i\Delta(\Delta-1)^{2i-2}\beta^{1-i}.$$

Let $\beta := (1+\varepsilon)(\Delta-1)^2$, where $\varepsilon$, which is small and positive, will be specified soon. Now we need $k \geqslant (1+\varepsilon)(\Delta-1)^2 + \sum_{i\geqslant 1} i\Delta(1+\varepsilon)^{1-i} = (1+\varepsilon)(\Delta-1)^2 + \varepsilon^{-2}(1+\varepsilon)^2\Delta$; the equality holds by Proposition 7.8. To get $k = \Delta^2(1+o(1))$, we simply need $\varepsilon^{-1} = o(\sqrt{\Delta})$. To approximately minimize $k$, we let $\varepsilon := 2^{1/3}\Delta^{-1/3}$. □

### 7.3.4 Frugal Coloring

t-frugal A proper coloring $\varphi$ is $t$-*frugal* if each color appears at most $t$ times in the neighborhood of each vertex. That is, $|w \in N(v) : \varphi(w) = i| \leqslant t$ for each vertex $v$ and each color $i$.

**Theorem 7.12.** *Fix an integer* $t \geqslant 2$*. Let* $G$ *be a graph with maximum degree* $\Delta$*, let* $\beta := ((t-1)\Delta\binom{\Delta-1}{t})^{1/t}$ *and let* $k := \lceil \frac{t\beta}{t-1} \rceil + \Delta$*. If* $L$ *is a* $k$*-assignment for* $G$*, then* $G$ *has at least* $\beta^{|V(G)|}$ $k$*-frugal* $L$*-colorings.*

*Proof.* Fix $t$, $G$, $\beta$, $k$, and $L$ as in the theorem. To construct frugal $L$-colorings of $G$, we build an instance $(G, \mathcal{B})$ and apply Theorem 7.6. For each edge $vw$ in $G$, we take $\mathcal{B}_{vw}$ to be $v$ and $w$ colored 1,1. The weight of each such subgraph is 0. Note, for each $v \in V(G)$, that $N_0(v) \leqslant \Delta$. Further, for each vertex $v \in V(G)$ and each $(t+1)$-element subset $S \subseteq N_G(v)$, we take $\mathcal{B}_S$ to be $S$ all colored 1. Every such $\mathcal{B}_S$ is determined by each vertex in $S$, so $S$ has weight exactly $t-1$. To construct such a set $S$ containing a given vertex $v$, we first pick a neighbor $w$ of $v$, and then pick $t$ other neighbors of $w$. Thus the number of such sets containing a vertex $v$ is at most $\Delta\binom{\Delta-1}{t}$. That is, $N_{t-1}(v) \leqslant \Delta\binom{\Delta-1}{t}$. Hence, to apply Theorem 7.6, we need

$$k \geqslant \beta + \Delta + \beta^{-(t-1)}\Delta\binom{\Delta-1}{t}.$$

To minimize the right side, let $\beta := \left((t-1)\Delta\binom{\Delta-1}{t}\right)^{1/t}$ and $k := \Delta + \left\lceil \frac{\beta t}{t-1} \right\rceil$. This suffices because $\beta + \Delta + \beta^{-(t-1)}\Delta\binom{\Delta-1}{t} = \Delta + \beta + \frac{\beta\Delta\binom{\Delta-1}{t}}{\beta^t} = \Delta + \beta + \frac{\beta\Delta\binom{\Delta-1}{t}}{(t-1)\Delta\binom{\Delta-1}{t}} = \Delta + \beta + \frac{\beta}{t-1} = \Delta + \frac{\beta t}{t-1} \leqslant k$. □

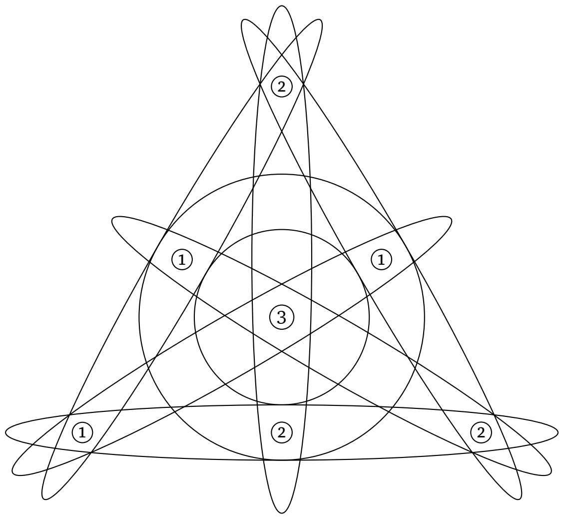



Figure 7.3: The Fano plane is a 3-uniform hypergraph with each vertex in 3 edges. It is not properly 2-colorable, but is properly 3-colorable (as shown).

The case $t = 1$ is similar. Now we must have $k \geqslant \beta + \Delta + \beta^0\Delta(\Delta - 1) = \beta + \Delta^2$. Since we require $\beta \geqslant 1$, we need $k \geqslant \Delta^2 + 1$. This can be proved more directly by greedily coloring $G^2$, since its maximum degree is at most $\Delta^2$.

### 7.3.5 r-Uniform Hypergraph Coloring

A *hypergraph* $H$ consists of a vertex set $V(H)$ and an edge set $E(H)$, where each $e \in E(H)$ is a subset of $V(H)$. Fix an integer $r \geqslant 2$. If $|V(e)| = r$ for all $e \in E(H)$, then $H$ is $r$-*uniform*. (Note that 2-uniform hypergraphs are simply graphs.) The maximum degree of a hypergraph is the maximum number of edges containing a common vertex. A hypergraph is *properly colored* if no edge is monochromatic.

hypergraph
r-uniform
properly colored

**Theorem 7.13.** *Fix integers $r \geqslant 3$ and $\Delta \geqslant 1$ and let $H$ be an $r$-uniform hypergraph with maximum degree $\Delta$. Let $k := \lceil(\frac{r-1}{r-2})((r-2)\Delta)^{1/(r-1)}\rceil$. For every $k$-assignment $L$, the number of proper $L$-colorings of $H$ is at least $((r-2)\Delta)^{|V(H)|/(r-1)}$.*

*Proof.* Let $G$ be a complete graph with $V(G) := V(H)$. For each $e \in E(H)$, let $G_e$ denote the subgraph of $G$ induced by $V(e)$. Let $\mathcal{B}_e$ consist of (a single coloring with) $V(e)$ all colored 1. So each edge of $H$ corresponds to an $r$-element set of $V(G)$, which has weight $r - 2$. Since $H$ has maximum degree $\Delta$, we have $N_{r-2}(v) \leqslant \Delta$ and $N_i(v) = 0$ for all other $i$. So it suffices to have $k \geqslant \beta + \beta^{-(r-2)}\Delta \geqslant \beta + \sum_{\ell \geqslant 0} \beta^{-\ell}N_\ell(v)$. This expression is minimized when $\beta := ((r-2)\Delta)^{1/(r-1)}$. Now we take $k := \lceil(\frac{r-1}{r-2}((r-2)\Delta)^{1/(r-1)}\rceil$, since $\beta + \beta^{-(r-2)}\Delta = (\beta^{r-1} + \Delta)/\beta^{r-2} = ((r-2)\Delta + \Delta)/\beta^{r-2} = \Delta(r-1)/\beta^{r-2} = \beta(r-1)/(r-2) \leqslant k$. $\square$

### 7.3.6 A Slightly Harder Example: Acyclic Coloring

Recall that a proper coloring $\varphi$ is *acyclic* if the subgraph induced by every two color classes is acyclic. Equivalently, $\varphi$ gives at least 3 colors to the vertices of every cycle. The acyclic chromatic number, $\chi_a(G)$, of a graph $G$ is the minimum number of colors that admits an acyclic coloring. It is easy to use our standard approach to bound $\chi_a(G)$.

**Proposition 7.14.** *Every graph* $G$ *with maximum degree* $\Delta$ *satisfies* $\chi_a(G) \leqslant \frac{3}{2}\Delta^{3/2}(1+o(1))$.

*Proof Sketch.* We create an instance $(G, \mathcal{B})$ with templates for 3 types of subgraphs. Each edge in $G$ has a single template, colored 1,1. The weight of each edge is 0. Note that $N_0(v) \leqslant \Delta$. Each 4-cycle in $G$ has a single template, colored 1,2,1,2. Each 4-cycle has weight $4-2-1=1$. Note that $N_1(v) \leqslant \Delta(\Delta-1)^2/2$. Finally, each $P_5$ in $G$, has a single template 1,2,1,2,1; each $P_5$ has weight $5-2-1=2$. Note that $N_2(v) \leqslant 3\Delta(\Delta-1)^3$. Now it is easy to check that (7.1) holds with $\beta := \Delta^{3/2}$ and $k := \lceil \frac{3}{2}\Delta^{3/2} + 4\Delta \rceil$. □

The reason that we use a template for $P_5$, rather than for every even length cycle, is simply to limit ourselves to only three templates, which allows a simpler proof. The reader can check that the leading term of our upper bound comes from the 4-cycle template, so improving the part arising from the $P_5$ does not improve the order of magnitude.

As usual, the same proof works for list coloring. And the number of colorings from each list-assignment is exponential in $|V(G)|$. For the instance $(G, \mathcal{B})$ above, we cannot improve the order of magnitude of $k$. If we increase the magnitude of $\beta$, then $k$ increases in magnitude. But if we decrease the magnitude of $\beta$ then we weaken the bound $N_1(v)\beta^{-1} \leqslant \frac{1}{2}\Delta(\Delta-1)^2\beta^{-1}$, so $k$ again increases. Nonetheless, we can indeed improve this upper bound.

**Theorem 7.15.** *Let* $G$ *be a graph with maximum degree* $\Delta$*. Let* $\beta := 2\Delta^{4/3}$ *and* $k := \lceil 4\Delta^{4/3} + \Delta \rceil$. *If* $L$ *is a* $k$*-assignment for* $G$*, then* $G$ *has at least* $\beta^{|V(G)|}$ *acyclic* $L$*-colorings.*

*Proof.* Again, we will build an instance $(G, \mathcal{B})$ such that every coloring of $(G, \mathcal{B})$ is an acyclic coloring of $G$. As we noted above, when trying to improve our upper bound the "problem area" is our subgraphs of weight 1, namely, 4-cycles. Specifically, we only have that $N_1(v) \leqslant \frac{1}{2}\Delta(\Delta-1)^2$. So we must handle the 4-cycles in $G$ more efficiently.

dangerous pair

Two vertices in $G$ form a *dangerous pair* if they share at least $\Delta^{2/3}$ common neighbors. For each dangerous pair $\{v, w\}$ in $G$, let $\mathcal{B}_{vw}$ consist of a single template with $v, w$ colored 1,1. Such pairs have weight $2-1-1=0$. Further, for each 4-cycle $vwxy$ in $G$, add a template if and only if neither $\{v, x\}$ nor $\{w, y\}$ is a dangerous pair. For each such 4-cycle $vwxy$, let $\mathcal{B}_{vwxy}$ consist of $v, w, x, y$ colored 1,2,1,2. It is easy to check that colorings of the instance $(G, \mathcal{B})$ are in bijection with acyclic colorings of $G$. So it suffices to check that (7.1) holds with $\beta := 2\Delta^{4/3}$ and $k := \lceil 4\Delta^{4/3}(1+o(1)) \rceil$.

The number of walks of length 2 from each vertex $v$ is at most $\Delta^2$. And each dangerous pair containing $v$ is counted by at least $\Delta^{2/3}$ of these. So the number of dangerous pairs containing $v$ is at most $\Delta^2/\Delta^{2/3} = \Delta^{4/3}$. Thus, $N_0(v) \leqslant \Delta + \Delta^{4/3}$. Now the number of 4-cycles containing $v$ but containing no dangerous pair is at most $\Delta(\Delta-1)\Delta^{2/3}/2$, since after picking 3 vertices of the 4-cycle, we have at most $\Delta^{2/3}$ choices for the final vertex (and each 4-cycle is counted in two directions). Thus, $N_1(v) \leqslant \Delta^{8/3}/2$. Finally, for each copy $H$ of $P_5$ in $G$, let $\mathcal{B}_H$ consists of $H$ colored 1,2,1,2,1. Each $P_5$ has weight $5-2-1=2$, and $N_2(v) \leqslant 3\Delta(\Delta-1)^3$. Thus, we need $c \geqslant \beta + \Delta + \Delta^{4/3} + \Delta^{8/3}\beta^{-1}/2 + 3\Delta^4\beta^{-2}$. Substituting $\beta := a\Delta^{4/3}$ gives $c \geqslant \Delta^{4/3}(a + \Delta^{-1/3} + 1 + \frac{1}{2a} + \frac{3}{a^2})$. This expression is approximately minimized when $a := 2$, where it takes the value $4\Delta^{4/3} + \Delta$. $\square$

Why do we choose $\Delta^{2/3}$ when defining a dangerous pair? Suppose, more generally, that we use $\Delta^{\varepsilon}$ for some $\varepsilon \in [0,1]$. We want to minimize $\beta + \Delta + \Delta^{2-\varepsilon} + \Delta^{2+\varepsilon}\beta^{-1}/2 + \Delta^4\beta^{-2}$. Define $b$ such that $\beta = \Theta(\Delta^b)$. Considering exponents, we want to minimize $\max\{b, 1, 2-\varepsilon, 2+\varepsilon-b, 4-2b\}$. Note that $2-\varepsilon \geqslant 1$, so we choose $b$ and $\varepsilon$ such that $b = 2-\varepsilon = 2+\varepsilon-b$. This gives $b = 4/3$ and $\varepsilon = 2/3$. It may seem lucky that also $4-2b = 4/3$. In fact, we could have improved this term by considering separately all 6-cycles and all paths of order 7.

## 7.4 Centered Coloring

A coloring $\varphi$ is $p$-*centered* if, for every connected subgraph $H$ of $G$, either $\varphi$ uses more than $p$ colors on $H$ or there is a color that $\varphi$ uses exactly once on $H$. The $p$-*centered chromatic number* of $G$ is the minimum number of colors allowing a $p$-centered coloring.

p-centered chromatic number

**Example 7.16.** The $p$-centered chromatic number of the path $P_n$ on $n$ vertices is precisely equal to $\min\{p+1, \lceil \lg_2(n+1) \rceil\}$. We denote the vertices of $P_n$ by $v_1, \ldots, v_n$. To achieve this upper bound we either let $\varphi(v_i) \equiv i \pmod{p+1}$ or, when $i$ is written in binary, let $\varphi(v_i)$ be the position of the least significant bit that is 1; see Figure 7.4.

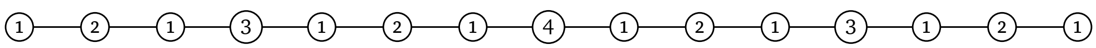



Figure 7.4: A $p$-centered coloring of $P_{15}$ for all positive integers $p$; it is optimal for all $p \geqslant 3$, and uniquely optimal (up to permuting color classes) for all $p \geqslant 4$.

For the lower bound, assume that we use fewer than $p+1$ colors. By induction on $n$ we show that $\lceil \log_2(n+1) \rceil$ colors are needed. Since $\varphi$ gives a $p$-centered coloring of $P_n$, some color appears exactly once on $P_n$, say on vertex $v$. Some component of $P_n - v$ has order at least $(n-1)/2$, so (by induction) needs at least $\lceil \log_2(\frac{n-1}{2}+1) \rceil$ colors. Thus, the number of colors used on $P_n$ is at least $1 + \lceil \log_2(\frac{n-1}{2}+1) \rceil = \lceil \log_2 2(\frac{n+1}{2}) \rceil = \lceil \log_2(n+1) \rceil$. $\diamondsuit$

For 1-centered coloring, the endpoints of every edge must either (a) use more than 1 color in total or else (b) have a color used exactly once. Thus, the endpoints of every edge get distinct colors. So the 1-centered chromatic number is simply the chromatic number. Every 2-centered coloring is also a 1-centered coloring, so it must be proper. On a 4-vertex path, every 2-centered coloring must either (a) use more than 2 colors or else (b) use some color exactly once. Thus, every 2-centered coloring has no 2-colored 4-vertex path. So it is easy to check that the 2-centered chromatic number is simply the star chromatic number.

**Theorem 7.17.** *Let $G$ be a graph with maximum degree $\Delta$. Fix a positive integer $p$, let $\beta := 2^{10}p\Delta^2$, and let $k := \lceil \frac{4}{3}\beta \rceil$. If $L$ is a $k$-assignment for $G$, then $G$ has a $p$-centered $L$-coloring. In fact, the number of $p$-centered $L$-colorings of $G$ is at least $\beta^{|V(G)|}$.*

*Proof.* Here we cannot quite apply Theorem 7.6 directly as stated. So instead we will state a more general version of this theorem, explain how to modify the proof to prove this generalization, and then apply this generalization.

The problem is that now certain subgraphs $H$ will have many templates, so no set of vertices will determine $\mathcal{B}_H$. In the proof of Theorem 7.6, we considered a subgraph $J$ of $G$ and a subgraph $H$ of $J$ with weight $i$. We showed that if $\varphi$ was a coloring of $J$ that was $\mathcal{B}$-good for $J - v$ but $\mathcal{B}_H$-bad for $J$, then we could map $\varphi$ injectively onto a $\mathcal{B}$-good coloring of $J - T$ for some $T \subseteq V(H)$ with $v \in T$ and $|T| = i + 1$. By induction, we thus concluded that $P(J - T, \mathcal{B}, L) \leqslant P(J - v, \mathcal{B}, L)\beta^{-i}$.

Now, rather than looking for a set to determine $\mathcal{B}_H$, we will consider each forbidden template $\tilde{H}_j$ for $H$ individually. If a single template $\tilde{H}_j$ has weight $i$, then the number of $L$-colorings of $G$ that are $\mathcal{B}$-good for $G - v$ but violate $\tilde{H}_j$ on $G$ is still at most $P(G - v, \mathcal{B}, L)\beta^{-i}$. To offset our considering the templates individually, each template of weight $i$, for a subgraph $H$ containing $v$, will count toward $N_i(v)$. The rest of the proof is identical. So now we apply this generalized version.

A coloring $\varphi$ of $G$ fails to be $p$-centered if and only if there exists a connected subgraph $H$ of $G$ and a number of colors $i$, with $i \leqslant p$, such that $\varphi$ uses $i$ colors on $H$ and $\varphi$ uses each color on $H$ at least twice. We count the number of pairs of a connected subgraph $H$ containing a vertex $v$ and a partition of $V(H)$ into $i$ color classes, each of size at least 2. For each such pair, we will add a forbidden template $\tilde{H}_j$.

Since $H$ is connected, it has a spanning tree rooted at $v$. For each vertex $w$, fix an arbitrary ordering of its neighbors. Now consider an arbitrary spanning tree $T$ of $H$ rooted at $v$. We take a walk around $T$, starting and ending at $v$, and traversing each edge twice, visiting the vertices in the spanning tree in depth-first order. Each time that we traverse an edge $e$ and reach a new vertex, we call $e$ a "forward" edge; each other time that we traverse an edge $e$, we call $e$ a "backward" edge. So, in our depth-first walk on $T$, every edge is traversed once (first) as a forward edge and once (second) as a backward edge.

The tree $T$ has $|H| - 1$ edges. To specify $T$, we must know which of the $2|H| - 2$ edges in our walk are forward edges and which of these edges are backward edges. For each forward edge, we must specify which new vertex it reaches. The number of possible orderings for forward and backward edges is at most[2] $\binom{2|H|-2}{|H|-1}$, and the total number of possibilities for the new vertices reached by forward edges is at most $\Delta^{|H|-1}$. So the number of spanning trees is at most $\binom{2|H|-2}{|H|-1}\Delta^{|H|-1} \leqslant (4\Delta)^{|H|}$.

[2] In fact, every prefix of our depth-first walk around $T$ contains at least as many forward edges as backward edges. So this number is at most the $(|H| - 1)$th Catalan number, which improves the bound by a (multiplicative) factor of $|H|$. But we will not need this.

To specify a partition of $V(H)$ into $i$ color classes, it suffices to pick $i$ vertices that will get distinct colors and to assign each other vertex to one of these $i$, with which it will share a color. This can be done in at most $\binom{|H|}{i} i^{|H|-i}$ ways. So, for fixed $|H|$ and $i$, the number of "bad" subgraph/color class partition pairs, i.e., bad templates, containing a given vertex $v$ is at most $\binom{2|H|-2}{|H|-1} \Delta^{|H|-1} \binom{|H|}{i} i^{|H|-i} \leqslant (8\Delta)^{|H|} i^{|H|-i}$. Since each color used on $H$ is used there at least twice, each subgraph/color class partition pair is determined by a set avoiding $v$. Furthermore, the size of a minimum determining set is $i$. So the weight of this template is $|H| - i - 1$. Thus, we need a positive integer $k$ and a real $\beta$ with $\beta \geqslant 1$ such that

$$k \geqslant \beta + \sum_{i=1}^{\min\{p, \lfloor |H|/2 \rfloor\}} \sum_{|H| \geqslant 1} (8\Delta)^{|H|} i^{|H|-i} \beta^{-(|H|-i-1)}.$$

We will show that this inequality holds with $\beta := 2^{10} p \Delta^2$ and $c = \lceil \frac{4}{3} \beta \rceil$. Before evaluating this double sum, we find a convenient upper bound on the summand.

$$\begin{aligned}
& (8\Delta)^{|H|} i^{|H|-i} \beta^{-(|H|-i-1)} \\
\leqslant\; & (8\Delta)^{|H|} p^{|H|-i} \beta^{-(|H|-i-1)} \\
=\; & (8\Delta p \beta^{-1})^{|H|} \beta (\beta p^{-1})^{i} \\
=\; & (2^7 \Delta)^{-|H|} \beta (2^{10} \Delta^2)^{i} \\
=\; & 2^{-|H|} (2^6 \Delta)^{-|H|} \beta (2^{10} \Delta^2)^{i} \\
\leqslant\; & 2^{-|H|} (2^6 \Delta)^{-2i} \beta (2^{10} \Delta^2)^{i} \\
=\; & 2^{-|H|} 4^{-i} \beta.
\end{aligned}$$

Now $\sum_{i=1}^{\min\{p, \lfloor |H|/2 \rfloor\}} \sum_{|H| \geqslant 1} 2^{-|H|} 4^{-i} \beta \leqslant \sum_{i=1}^{\min\{p, \lfloor |H|/2 \rfloor\}} 4^{-i} \beta \leqslant \beta/3$. □

## 7.5 Coloring with Small Connected 2-Colored Subgraphs

In this section we seek a proper coloring of a graph $G$ such that every connected 2-colored subgraph has at most $m$ edges, for some constant $m$. This general result has a number of interesting consequences, which we list below. The following is our main theorem.

**Theorem 7.18.** *For every positive integer $m$, there exists a constant $A_m$ such that every graph with maximum degree $\Delta$ has a proper coloring $\varphi$ with at most $A_m \Delta^{\frac{m+1}{m}}$ colors such that every connected 2-colored subgraph under $\varphi$ has at most $m$ edges.*

Before giving the proof, we state some of the consequences of Theorem 7.18.

**Corollary 7.19.** *Every graph* $G$ *of maximum degree* $\Delta$ *has*

*(a) a star coloring with* $O(\Delta^{3/2})$ *colors;*

*(b) an acyclic coloring with* $O(\Delta^{4/3})$ *colors;*

*(c) a coloring with* $O(\Delta^{8/7})$ *colors in which every bicolored subgraph has treewidth at most 2;*

*(d) a coloring with* $O(\Delta^{9/8})$ *colors in which every bicolored subgraph is planar; and*

*(e) a coloring with* $O(\Delta^{13/12})$ *colors in which every bicolored subgraph has treewidth at most 3.*

*Proof.* (a) Letting $m := 2$, means that $G$ has no 2-colored path on 3 edges. (b) Letting $m := 3$ means that $G$ has no 2-colored cycle. (c) Now we let $m := 7$. A graph $G$ has treewidth at most 2 if and only if $G$ has no $K_4$ minor [42]. It is easy to check (see Exercise 3) that every bipartite graph with a $K_4$ minor has at least 8 edges.

(d) Now we let $m := 8$. The fewest edges in a non-planar graph is 9, in $K_{3,3}$. To see this, note that it suffices to consider graphs with minimum degree at least 3 (why?), and $K_4$ is planar. Now we are done, since $K_5 - e$ is planar, $|E(K_5)| \geqslant 9$, and each graph with at least 6 vertices and minimum degree at least 3 has at least 9 edges.

(e) Now we let $m := 12$. It was shown in [28] that a graph has treewidth at most 3 if and only if it has no minor of any of $K_5$, the Möbius 8-ladder (denoted $M_8$), the octahedron $K_6 - 3K_2$, and the pentagonal prism $C_5 \square K_2$. It is straightforward (see Exercise 3) to show that every bipartite graph with at most 12 edges has none of these graphs as a minor. $\square$

*Proof of Theorem 7.18.* We create an instance $(G, B)$ such that for each connected bipartite subgraph $H$ of $G$ with at least $m + 1$ edges, we define $\mathcal{B}_e$ such that every proper 2-coloring of $H$ in $G$ is $\mathcal{B}$-bad for the instance $(G, \mathcal{B})$. However, sometimes it will be more efficient to create a template in $\mathcal{B}$ with vertices in only one part of $H$. Clearly, if any such template is nonmonochromatic, then $H$ cannot be properly 2-colored.

special $t$-tuple

For each $t \in \{2, \ldots, m\}$ a *special* $t$*-tuple* is a set $\{v_1, \ldots, v_t\} \subseteq V(G)$ such that $|N(v_1) \cap \cdots \cap N(v_t)| \geqslant \Delta^{\frac{m-t+1}{m}}$. The notion of special $t$-tuple is motivated by seeking the threshold where it becomes more efficient to create a template for just some subset of vertices in $H$ (rather than all of $H$). We define the following 4 types of subgraphs.

1. For each edge $xy \in E(G)$, we let $\mathcal{B}_{xy}$ be a coloring with $x$ and $y$ colored 1.

2. For each $v \in V(G)$ and $Q \subseteq N_G(v)$ with $|Q| = m + 1$, we let $\mathcal{B}_Q$ be the single coloring with all vertices of $Q$ colored 1.

3. For each $t \in \{2, \ldots, m\}$ and each special t-tuple R, we let $\mathcal{B}_R$ be the single coloring with all vertices of R colored 1.

4. For each $s \in \{3, \ldots, m+2\}$ and each $S \subseteq V(G)$ such that $|S| = s$ and $G[S]$ is connected and induces at least $m+1$ edges, we consider S provided that neither (a) S contains the vertices of some special t-tuple in one of the parts of $G[S]$ nor (b) $G[S]$ is the star $K_{1,m+1}$. We let $\mathcal{B}_S$ consist of all proper 2-colorings of $G[S]$ (with some specified vertex colored 1, to avoid including templates that differ only by permuting colors).

It is clear that $(G, \mathcal{B})$ is an instance and that every $\mathcal{B}$-good coloring of G has the desired property. We need only to check that each pair $(H, v)$ has $\mathcal{B}_H$ determined by some subset of $V(H) \setminus \{v\}$. (This is why we added the edges of type 2 above, and added condition (b) for the edges of type 4.)

The weights of subgraphs of types (1)–(4) are 0, $m-1$, $t-2$, and $s-3$. So we need

$$k \geqslant \beta + \Delta + \beta^{-(m-1)} N_{m-1}(v) + \sum_{t=2}^{m} \beta^{-(t-2)} N_{t-2}(v) + \sum_{s=3}^{m+2} \beta^{-(s-3)} N'_{s-3}(v), \qquad (7.2)$$

where $N'_i(v)$ is the number of type 4 subgraphs of weight i containing v and $N_i(v)$ is the number of all other subgraphs of weight i containing v. So now we must bound the number of subgraphs of each type that can contain a vertex v.

Clearly, $N_{m-1}(v) \leqslant \Delta\binom{\Delta}{m}$. Each special t-tuple has all vertices forming the leaves of some star $K_{1,t}$ in G. The number of such stars with a given vertex v as a leaf is at most $\Delta\binom{\Delta-1}{t-1}$. For every special t-tuple $\{v_1, \ldots, v_t\}$, by definition $|N(v_1) \cap \cdots \cap N(v_t)| \geqslant \Delta^{\frac{m-t+1}{m}}$. So each special t-tuple appears as the leaves of at least $\Delta^{\frac{m-t+1}{m}}$ stars. Thus, the number of special t-tuples containing a vertex v is at most $\Delta\binom{\Delta-1}{t-1}/\Delta^{\frac{m-t+1}{m}} \leqslant \Delta^{t-1+\frac{t-1}{m}}$.

Finally, we must bound $N'_{s-3}(v)$ for all $v \in V(G)$. Pick a connected bipartite graph H on s vertices with (at least) $m+1$ edges, pick $v_1$ to be an arbitrary vertex of H. Since H need not be induced, we assume that H has exactly $m+1$ edges. Let T be a depth-first search tree of H from $v_1$. Let $v_1, \ldots, v_s$ denote the order that the vertices of H are encountered in T. In particular, note that $H[\{v_1, \ldots, v_i\}]$ is connected for all $i \in [s]$. Let $w_1 := v$. Finally, for each $i \in \{2, \ldots, s\}$, choose $w_i \in V(G)$ such that $w_i w_j \in E(G)$ whenever $v_i v_j \in E(H)$. Note that every set $S \subseteq V(G)$ (containing v) that gives rise to a type 4 subgraph in H can be formed in this way. Now we must bound the number of such sets S.

Fix H and T. For each $i \in \{2, \ldots, s\}$, let $d_i + 1$ be the number of vertices in $\{v_1, \ldots, v_{i-1}\}$ adjacent in H to $v_i$. (Note that $\sum_{i=2}^{s}(d_i+1) = m+1$, so $\sum_{i=2}^{s} d_i = m-s+2$.) By our criteria for type (4) subgraphs, we assume these $d_i + 1$ vertices, for each i, do not form a special $(d_i+1)$-tuple. As a result, our number of choices for $w_i$ is at most $\Delta^{\frac{m-(d_i+1)+1}{m}} = \Delta^{1-\frac{d_i}{m}}$. Thus, the total number of choices for such sets is at most $\prod_{i=2}^{s} \Delta^{1-\frac{d_i}{m}} = \Delta^{s-1-\frac{d_2+\cdots+d_s}{m}} = \Delta^{s-1-\frac{m-s+2}{m}} = \Delta^{(s-2)(\frac{m+1}{m})}$.

Let $A_m$ denote the maximum, over all $s \leqslant m+2$, of the number of connected bipartite graphs with $s$ vertices and at least $m+1$ edges. So (7.1) is implied by the following.

$$k \geqslant \beta + \Delta + \beta^{-(m-1)}\Delta\binom{\Delta}{m} + \sum_{t=2}^{m} \beta^{-(t-2)}\Delta^{(t-1)\frac{m+1}{m}} + \sum_{s=3}^{m+2} A_m \Delta^{(s-2)(\frac{m+1}{m})}\beta^{-(s-3)}.$$

Let $\beta := 2\Delta^{\frac{m+1}{m}}$. Now

$$\begin{aligned}
&\beta + \Delta + \beta^{-(m-1)}\Delta\binom{\Delta}{m} + \sum_{t=2}^{m} \beta^{-(t-2)}\Delta^{(t-1)\frac{m+1}{m}} + \sum_{s=3}^{m+2} A_m \Delta^{(s-2)(\frac{m+1}{m})}\beta^{-(s-3)}\\
&\leqslant \beta + \Delta + \Delta^{\frac{m+1}{m}}\left([\Delta^{\frac{m+1}{m}}\beta^{-1}]^{(m-1)} + \sum_{t=2}^{m}[\Delta^{\frac{m+1}{m}}\beta^{-1}]^{t-2} + \sum_{s=3}^{m+2}[\Delta^{\frac{m+1}{m}}\beta^{-1}]^{s-3}A_m\right)\\
&\leqslant 2\Delta^{\frac{m+1}{m}} + \Delta + \Delta^{\frac{m+1}{m}}\left(\sum_{t=2}^{m+1} 2^{-(t-2)} + \sum_{s=3}^{m+2} 2^{-(s-3)}A_m\right)\\
&\leqslant 2\Delta^{\frac{m+1}{m}} + \Delta^{\frac{m+1}{m}}(3+2A_m)\\
&= \Delta^{\frac{m+1}{m}}(5+2A_m).
\end{aligned}$$
□

## 7.6 Coloring Triangle-free Graphs

In this section we prove that a triangle-free graph $G$ with maximum degree $\Delta$ has $\chi(G) = O(\Delta/\ln\Delta)$. This is a well-known result, first proved in 1996 (more details are in the Notes). However, earlier proofs were difficult. In contrast, as we shall now see, Rosenfeld Counting facilitates a proof that is much easier.

Our proof in this section looks a little different from those we have seen previously. When it applies, Theorem 7.6 shows that, for a graph $G$ and a vertex $v$, *every* coloring of $G - v$ extends to at least $\beta$ colorings of $G$. In contrast, here we will prove that *most* colorings of $G - v$ extend to (far) more than $\beta$ colorings of $G$. This distinction allows for a small number of colorings of $G - v$ that have few extensions. This new approach is essential, since the number of colors allowed for each vertex $v$ is much smaller than its degree; thus, some coloring of $G - v$ might not admit any extensions to $G$.

**Theorem 7.20 (Johansson's Theorem).** *Fix $\varepsilon > 0$. There exists $\Delta_\varepsilon$ such that if $G$ is a triangle-free graph with maximum degree $\Delta$ and $\Delta \geqslant \Delta_\varepsilon$, then $\chi(G) \leqslant \lceil(1+\varepsilon)\Delta/\ln\Delta\rceil$.*

$k, \ell, \mathcal{C}(G)$

Let $k := \lceil(1+\varepsilon)\Delta/(\ln\Delta)\rceil$ and let $\ell := \ln^2\Delta$. Let $\mathcal{C}(G)$ be the set of proper $k$-colorings of $G$. Theorem 7.20 follows immediately from Lemma 7.21, by induction on $|G|$. As usual, the proof guarantees exponentially many colorings, which is essential to the inductive argument. It also works for list-coloring (and correspondence coloring), but we do not emphasize this point.

**Lemma 7.21.** *If $\Delta$ is sufficiently large and $G$ is any triangle-free graph with maximum degree at most $\Delta$, then $|\mathcal{C}(G)|/|\mathcal{C}(G-v)| \geqslant \ell$ for every $v \in V(G)$.*

*Proof.* For any partial proper coloring $\varphi$ of $G$, let $L_\varphi(w)$ denote the set of colors available for $w$ (those not used on any neighbor of $w$); this definition is independent of whether or not $w$ is already colored. We want to show that $|\mathcal{C}(G)|/|\mathcal{C}(G-v)| \geqslant \ell$. We have $L_\varphi$

$$\frac{|\mathcal{C}(G)|}{|\mathcal{C}(G-v)|} = \frac{\sum_{\varphi \in \mathcal{C}(G-v)} |L_\varphi(v)|}{|\mathcal{C}(G-v)|} = \mathbb{E}[|L_\varphi(v)|].$$

We will find it more convenient to work with $\mathbb{E}[|L_\varphi(v)|]$. So we will show that, for $\Delta$ sufficiently large, we have $\mathbb{E}[|L_\varphi(v)|] \geqslant \ell$.

When choosing a coloring of $G-v$, the key idea is to condition on a coloring of $G-N[v]$. Because $G$ is triangle-free, $N(v)$ is an independent set. So, to extend a coloring $\varphi$ of $G-N[v]$ to a coloring of $G-v$, we simply choose a color in $L_\varphi(w)$ for each $w \in N(v)$. Furthermore, all such extensions are equally likely. So it will suffice to show that for an extension $\varphi'$ of $\varphi$ to $G-v$ we have $\mathbb{E}[|L_{\varphi'}(v)|] \gg \ell$. (For $f$ and $g$, both functions of $\Delta$, we write $f \gg g$ to denote that $\lim_{\Delta\to\infty} g/f = 0$. Similarly, $f \ll g$ precisely when $g \gg f$.) We want to show, for each $w \in N(v)$, that $|L_\varphi(w)|$ is typically large. If $v$ has many neighbors $w$ with $|L_\varphi(w)|$ large, then many of them will likely use repeated colors, so $\mathbb{E}[|L_{\varphi'}(v)|]$ will be large.

Fix a positive integer $t$, which we specify later, and fix $w \in N(v)$. We want to bound the probability that $|L_\varphi(w)| \leqslant t$. Note that each coloring $\varphi$ of $G-v$ with $|L_\varphi(w)| \leqslant t$ can be formed from a coloring $\varphi'$ of $G-v-w$ by giving $w$ one of at most $t$ colors. By induction,

$$\Pr[|L_\varphi(w)| \leqslant t] \leqslant \frac{t|\mathcal{C}(G-v-w)|}{|\mathcal{C}(G-v)|} \leqslant \frac{t}{\ell}.$$

Thus, the expected number of such vertices in $N(v)$ is at most $\Delta t/\ell$. Recall that $\ell = \ln^2\Delta$. To ensure that, with high probability, $o(\Delta/\ln\Delta)$ neighbors of $v$ have at most $t$ available colors, we require $1 \ll t \ll \ell/\ln\Delta = \ln\Delta$. For concreteness, let $t := \ln^{1/2}\Delta$. Markov's Inequality states: If $Y$ is a nonnegative random variable and $a > 0$, then $\Pr[Y \geqslant a\,\mathbb{E}[Y]] \leqslant 1/a$. Let $Y$ denote the number of neighbors $w$ of $v$ such that $|L_\varphi(w)| \leqslant t$. Let $a := \sqrt{\ell/(t\ln\Delta)} = \ln^{1/4}\Delta$. Now $t$ $a$

$$\Pr[Y \geqslant \Delta/\ln^{5/4}\Delta] = \Pr[Y \geqslant a\frac{t\Delta}{\ell}] \leqslant 1/a.$$

In particular, since $a \to \infty$ as $k \to \infty$, we get $\Pr[Y = o(k)] \to 1$ as $k \to \infty$.

Each neighbor $w$ of $v$ with $|L_\varphi(w)| \leqslant t$ removes at most one color from $L_\varphi(v)$. So we must show that the remaining neighbors of $v$, those with more than $t$ available colors, do not remove too many more colors from $L_\varphi(v)$.

Fix a coloring $\varphi'$ of $G-N[v]$. Let $B$ be the set of neighbors of $v$ with at most $t$ available colors, i.e., $x$ such that $|L_{\varphi'}(x)| \leqslant t$. Color each $x \in B$ uniformly at random from $L_{\varphi'}(x)$. By symmetry, assume each $x \in B$ is colored from $\{k-|B|+1,\ldots,k\}$. Now color each remaining $\varphi', B$

$X, k'$ neighbor $w$ of $v$ from $L_{\varphi'}(w)$. Call the resulting coloring $\varphi$. Let $X := |L_\varphi(v)|$. Let $k' := k - |B|$.
$X_i, \Delta'$ For each $i \in [k']$, let $X_i := 1$ if $i \in L_\varphi(v)$ and otherwise $X_i := 0$. Let $\Delta' := |N(v)| - |B|$. We
$w_j, L_j$ denote the vertices of $N(v) \setminus B$ by $w_1, \ldots, w_{\Delta'}$, and we denote $L_{\varphi'}(w_j)$ by $L_j$.

$$\begin{aligned}
|L_\varphi(v)| &\geqslant \sum_{i=1}^{k'} X_i \geqslant \sum_{i=1}^{k'} \prod_{\substack{j \in \Delta' \\ L_j \ni i}} \left(1 - \frac{1}{|L_j|}\right) \\
&\geqslant k' \left[ \prod_{i=1}^{k'} \prod_{\substack{j \in \Delta' \\ L_j \ni i}} \left(1 - \frac{1}{|L_j|}\right) \right]^{1/k'} \\
&= k' \left[ \prod_{j=1}^{\Delta'} \prod_{i \in L_j} \left(1 - \frac{1}{|L_j|}\right) \right]^{1/k'}.
\end{aligned}$$

The final inequality is by the Arithmetic Mean–Geometric Mean inequality. To lower bound the double product on the second line, we reverse the order of the products. Using the fact that $\{(1 - 1/x)^x)\}_{x=1}^{\infty}$ is an increasing sequence, we get the following.

$$\prod_{j=1}^{\Delta'} \prod_{i \in L_j} \left(1 - \frac{1}{|L_j|}\right) \geqslant ((1 - 1/t)^t)^{\Delta'} \geqslant ((1 - 1/t)^t)^{\Delta} = e^{-(1+O(1/t))\Delta}.$$

Thus, as desired,

$$|L_\varphi(v)| \geqslant k' \exp(-\frac{(1 + o(1))\Delta}{k'}) = \Theta(\frac{\Delta}{\ln \Delta}) \exp(-\frac{1 + o(1)}{1 + \varepsilon} \ln \Delta) \geqslant \Delta^{\varepsilon/(1+\varepsilon)-o(1)}.$$

We showed that, with probability going to 1 as $k \to \infty$, we have $|L_\varphi(v)| \geqslant \Delta^{\varepsilon/(1+\varepsilon)-o(1)} \gg \ell$. This implies also that $\mathbb{E}[|L_\varphi(v)|] \gg \ell$, which is what we aimed to prove. □

## Notes

In this chapter we have considered a wide variety of coloring problems. We often believe intuitively that a "random coloring" should have the properties we desire. This notion is imprecise, so we consider ways to make it more formal. With the Local Lemma, we create a "bad event" (similar to our forbidden templates) for every way in which an arbitrary coloring could fail to have the properties we desire. If each bad event is sufficiently unlikely in a random coloring, and is independent of all but a few other bad events, then the Local Lemma guarantees that our desired coloring exists.

The Local Lemma has been used widely throughout extremal combinatorics, and particularly graph coloring [307]. One of its chief drawbacks was the nonconstructive nature of its proof [307, Chapter 19]. This led to significant efforts toward proving an "algorithmic" Local Lemma. The first breakthrough in this direction was due to Beck [34], who gave an algorithm under more restrictive hypotheses. Beck's work sparked a long series of improvements (see the introduction of [313] and references therein), culminating with work of Moser [312] and Moser and Tardos [313], which gave an efficient algorithm that applies to nearly every example where the Local Lemma does.

Entropy Compression is the name given to the algorithms in [312, 313]. More generally, it is a paradigm to prove that a randomized coloring algorithm will eventually (typically, in expected polynomial time) produce a desirable coloring with high probability.[3] Now we color randomly, but when we create a defect in our coloring, we simply uncolor the defective part and try again. The idea is to store each (failed) run of the randomized algorithm in a compressed form, called a log, so that the entire run of the algorithm can be recovered from the log. That is, failed runs map injectively to logs. If the total number of logs, for failed runs of a given length, is smaller than the total number of random bit strings used by runs of that length, then not all runs can map to logs. Thus, some run must succeed. The name Entropy Compression originated in a blog post of Terry Tao [371]. It owes to the fact that we map the entropy (represented by the random bit string required for each run of the algorithm) into a compressed form (the log).

Rosenfeld Counting was introduced by Matthieu Rosenfeld [348] to prove that every graph with maximum degree $\Delta$ has a nonrepetitive coloring with at most $\Delta^2(1 + o(1))$ colors, Theorem 7.11. Earlier, this result was proved by Dujmović, Joret, Kozik, and Wood [123], using Entropy Compression, but their proof was far more complicated. Rosenfeld also used his technique to prove upper bounds for other related types of coloring [348], called "total Thue coloring" and "weak total Thue coloring", as well as game versions of further coloring problems [349]. More importantly, he noted that this technique could be applied much more broadly and outlined such an approach. Rosenfeld wrote:

> This technique [Rosenfeld Counting] is strongly related to the entropy-compression technique. In fact, in the particular context of colorings of graphs of bounded degree, it is equivalent of [179, Theorem 12]. Using our technique one can in fact provide a simpler proof of their Theorem 12... but it does not seem to be worth the trouble of introducing all the necessary formalism only to provide an alternative proof of the exact same result. However, even if we can simplify the proof and match the bound of their theorem, we cannot easily improve the bound.

The proof we present of Theorem 7.3, specifically Lemma 7.4, is due to Rosenfeld [348, Section 2.1], although the same result was proved previously using the Local Lemma [190] and Entropy Compression [189]. Theorem 7.6 and Lemma 7.7 are due to Wanless and Wood [410]. Although we use slightly different terminology[4], our presentation follows theirs closely. This

[3] In fact, both the Local Lemma and Entropy Compression have been used more widely than just in graph coloring. However, in these Notes we restrict ourselves to those applications.

[4] They phrase their results in terms of hypergraph coloring. We avoid this notion, to keep our presentation of the easier examples as simple as possible.

chapter was inspired by [410], which also provided many of our examples.

In the rest of these Notes, for brevity we typically omit mention of the list-coloring variant of each problem. Proofs relying on the Local Lemma, on Entropy Compression, and on Rosenfeld Counting all generally work equally well for list-coloring variants. Similarly, we generally omit mention of the fact when we get exponentially many of the desired coloring. This comes for free with Rosenfeld Counting, and it can usually be deduced with a small amount of extra effort when using the Local Lemma or Entropy Compression.

Early work on star chromatic number, denoted $\chi_s(G)$, was done by Fertin, Raspaud, and Reed [161], who used the Local Lemma to prove that every graph G with maximum degree $\Delta$ has $\chi_s(G) \leqslant 20\Delta^{3/2}$. They also constructed graphs G with maximum degree $\Delta$ such that $\chi_s(G) \geqslant C\Delta^{3/2}/(\log\Delta)^{1/2}$ for some constant C. This upper bound was improved to $\chi_s \leqslant 4.34\Delta^{3/2} + 1.5\Delta$ by Ndreca, Procacci, and Scoppola [320] and subsequently to $\chi_s(G) \leqslant \sqrt{8}\Delta^{3/2} + \Delta$ by Esperet and Parreau [155, Corollary 2] in their excellent tutorial on Entropy Compression. This is Theorem 7.9, but our presentation more closely follows [410].

The nonrepetitive chromatic number, denoted $\pi(G)$, was introduced by Alon, Grytczuk, Hatuszczak, and Riordan, who proved [16, Section 4.3] that $\pi(G) = O(\Delta^2)$ for every graph G with maximum degree $\Delta$. (They also found an absolute constant C such that, for each positive integer $\Delta$, there exist graphs with maximum degree $\Delta$ and $\pi(G) \geqslant C\Delta^2/\log\Delta$.) This upper bound was improved to $36\Delta^2$ by Grytczuk [188], to $16\Delta^2$ by Grytczuk [187], to $(12.2+o(1))\Delta^2$ by Harant and Jendrol' [203], and to $10.4\Delta^2$ by Kolipaka, Szegedy, and Xu [266]. Each of these proofs used the Local Lemma. Ultimately, Dujmović, Joret, Kozik, and Wood [123] used Entropy Compression to show that $\pi(G) \leqslant \Delta^2 + O(\Delta^{5/3})$. This same upper bound was later reproved by Bernshteyn using his Local Cut Lemma [35] and independently using cluster-expansion [25, 41]. More recently, Rosenfeld [348] reproved it using the inductive approach of Theorem 7.6. We present his proof in Theorem 7.11.

The minimum number of colors in an acyclic edge-coloring of a graph G is denoted $\chi'_a(G)$. Alon, McDiarmid, and Reed [18] showed that $\chi'_a(G) \leqslant 64\Delta$. This was improved to $\chi'_a(G) \leqslant 16\Delta$ by Molloy and Reed [306, Theorem 2.2], to $\chi'_a(G) \leqslant \lceil 9.62(\Delta - 1)\rceil$ by Ndreca, Procacci, and Scoppola [320], and to $\chi'_a(G) \leqslant 9\Delta$ by Molloy and Reed [307, Theorem 19.4]. All four of these proofs use the Local Lemma.

Esperet and Parreau [155] strengthened the upper bound to $\chi'_a(G) \leqslant 4(\Delta - 1)$. This was improved to $\chi'_a(G) \leqslant \lceil 3.74(\Delta-1)\rceil + 1$ by Giotis, Kirousis, Psaromiligkos, and Thilikos [176], to $\chi'_a(G) \leqslant \lceil 3.569(\Delta - 1)\rceil + 1$ by Fialho, de Lima, and Procacci, [162], and eventually to $\chi'_a(G) \leqslant 2\Delta - 1$ by Kirousis and Livieratos [260]. All four of these latter proofs use Entropy Compression, and they are at least somewhat involved. In Theorem 7.10, we proved the weaker bound $\chi'_a(G) \leqslant \lceil 4.6(\Delta - 1)\rceil$, since it admits a much simpler proof.

Recall that a proper coloring is t-*frugal* if each color appears at most t times on the neighborhood of each vertex. Hind, Molloy, and Reed [218] proved, for each integer $t \geqslant 1$, that every graph with sufficiently large maximum degree $\Delta$ has a t-frugal coloring with $\max\{(t+1)\Delta, \frac{e^3}{t}\Delta^{1+1/t}\}$ colors. Alon [218] constructed graphs with maximum degree $\Delta$ and no t-frugal coloring with $\frac{1}{2t}\Delta^{1+1/t}$ colors. Thus, the bound of Hind et al. [218] is sharp up to a constant factor. Wanless and Wood [410] strengthened the upper bound to slightly better than $\frac{4}{t}\Delta^{1+1/t} + 2\Delta$; when $\Delta > t$ and $t \to \infty$, they improved this to $(e + o(1))\Delta^{1+1/t}/t$. This result is Theorem 7.12.

Coloring r-uniform hypergraphs is the standard example used to illustrate the Local Lemma. Erdős and Lovász [151] proved, for an r-uniform hypergraph G with maximum degree $\Delta$, that $\chi(G) \leqslant \lceil (4r\Delta)^{1/(r-1)} \rceil$. Using a stronger version of the Local Lemma, Spencer [365] improved this to $\chi(G) \leqslant \lceil (e(r(\Delta-1)+1))^{1+1/(r-1)} \rceil$. Wanless and Wood [410] improved it further to $\chi(G) \leqslant \lceil \frac{r-1}{r-2}((r-2)\Delta)^{1+1/(r-1)} \rceil$, which is Theorem 7.13.

For the acyclic chromatic number, denoted $\chi_a(G)$, Alon, McDiarmid, and Reed [18] proved $\chi_a(G) \leqslant \lceil 50\Delta^{4/3} \rceil$ for every graph G with maximum degree $\Delta$. They also constructed an infinite family of graphs needing $\Omega(\Delta^{4/3}/(\log \Delta)^{1/3})$ colors. The upper bound was improved by Ndreca, Procacci, and Scoppola [320] to $\chi_a(G) \leqslant \lceil 6.59\Delta^{4/3} + 3.3\Delta \rceil$, by Sereni and Volec [358] and to $\chi_a(G) \leqslant 2.835\Delta^{4/3} + \Delta$, by Gonçalves, Montassier, and Pinlou [179] to $\chi_a(G) \leqslant \Delta^{4/3}(3/2 + o(1))$. Finally, Kirousis and Livieratos [260] proved the currently best known result: $\chi_a(G) \leqslant \lceil (2^{-1/3} + \varepsilon)\Delta^{4/3} \rceil + \Delta + 1$ for each positive $\varepsilon$ and $\Delta$ sufficiently large, as a function of $\varepsilon$. Theorem 7.15 shows that $\chi_a(G) \leqslant \lceil 4\Delta^{4/3} + \Delta \rceil$. Our proof is that of Alon, McDiarmid, and Reed, rephrased in the language of Rosenfeld Counting. In particular, the notion of "dangerous pairs" (which they called "special pairs") comes from their proof.

Recall that a coloring $\varphi$ is p-*centered* if each connected subgraph H either (a) receives more than p colors under $\varphi$ or (b) has a vertex that receives a color used nowhere else on H. The p-centered chromatic number, $\chi_p(G)$, plays a key role in the theory of sparse graph classes, pioneered by Nešetřil and Ossona de Mendez [322]. A class $\mathcal{G}$ of graphs has *bounded expansion* if and only if there exist a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $\chi_p(G) \leqslant f(p)$ for every graph G in $\mathcal{G}$. Nešetřil and Ossona de Mendez originally defined this notion in terms of the maximum density of shallow minors, but later proved [321] that the two definitions are equivalent.

It was known that every graph class with bounded degree has bounded p-centered chromatic number. However, it was conjectured that this dependency on p was exponential, i.e., that there existed graphs with bounded maximum degree and p-centered chromatic number $\Omega(a^p)$ for some $a > 1$. This conjecture was refuted by Dębski, Felsner, Micek, and Schröder [120], who proved that $\chi_p(G) = O(\Delta^{2-1/p})$. This is nearly optimal, since there exist [122] graphs G with maximum degree $\Delta$ and $\chi_p(G) = \Omega(\Delta^{2-1/p} p \ln^{-1/p} \Delta)$. This upper bound is proved using Entropy Compression. In Theorem 7.17, we used Rosenfeld Counting to prove that $\chi_p(G) = O(\Delta^2 p)$. The key ideas in the proof all come from the proof of Dębski at al. We suspect that with more effort we could recover the bound $\chi_p(G) = O(\Delta^{2-1/p} p)$, using Rosenfeld Counting. But we prefer the proof given, since it illustrates the approach well, while minimizing technical details.

Now we consider proper colorings where 2-colored connected subgraphs must have at most m edges (for some fixed m). Theorem 7.18, which bounds the number of colors needed, is due to Aravind and Subramanian [27]. We are not aware of previous work in the direction of this generalized result. As we noted in Corollary 7.19, the cases $m = 2$ and $m = 3$ correspond to star coloring and acyclic coloring, which are discussed above. Corollary 7.19(d), which requires every 2-colored subgraph to be planar, improves a previous result of the same authors [26], strengthening their bound of $O(\Delta^{8/7})$ to $O(\Delta^{9/8})$.

In the same paper [27], Aravind and Subramanian also proved the following (nearly matching) lower bound: "If $H$ is a bipartite graph with $m+1 \geqslant 2$ edges, then there exists an infinite family $\mathcal{G}$ of graphs with increasing maximum degree such that for each $G \in \mathcal{G}$ of maximum degree $\Delta$, a bicolored copy of $H$ appears in every proper coloring of $G$ with $k$ colors, where $k$ is a function satisfying $k = \Omega(\Delta^{(m+1)/m} \log^{-1/m} \Delta)$." Thus, Theorem 7.18 is best possible, up to a factor of at most $\log^{-1/m} \Delta$. In [27], Theorem 7.18 is proved using the Local Lemma, and that proof can also be phrased using Entropy Compression. The proof we presented largely follows the same lines, although ours avoids needing to verify some tedious inequalities.

Theorem 7.20 was first proved by Johansson [231] in 1996 (with a worse multiplicative constant), but the proof was not published. The first published proof appeared in [307], but the proof was still fairly hard. About 20 years later, Molloy [305] used entropy compression to give a simpler proof, which also improved the multiplicative constant to 1. Following this, Bernshteyn [36] gave a proof using the Lopsided Local Lemma in place of entropy compression; Bernshteyn, Brazelton, Cao, and Kang [37] used Rosenfeld Counting to give an asymptotically sharp lower bound on the number of $\Delta/\log \Delta$-colorings; and Hurley and Pirot [222] proved a more general version (also using Rosenfeld Counting) that allows the graph to have a moderate number of triangles. Finally, Martinsson [294] simplified the proof of Hurley and Pirot, in the case that $G$ is triangle-free. It is this proof of Martinsson that we presented in Section 7.6.

## Exercises

7.1. Consider a string on the alphabet $\{1,2,3\}$. Show that the substitution $1 \to 12312$, $2 \to 131232$, $3 \to 1323132$ preserves squarefreeness. For example, 1232 becomes 12312,131232, 1323132,131232 (the commas are included only to improve clarity). By applying this substitution repeatedly, starting with 1, we converge to an infinite squarefree word. This proves that every path has nonrepetitive chromatic number at most 3. [386]

7.2. (a) Find an improved function $f$ such that $\chi'_a(G) \leqslant \lceil (2+f(g))(\Delta-1) \rceil$, where $g$ is the girth of $G$ and $\lim_{g\to\infty} f(g) = 1$. (b) Find an improved function $h$ such that $\chi_a(G) = O(\Delta^{h(t)})$, when $G$ has maximum degree at most $\Delta$ but has no subgraph $K_{2,t}$ (not necessarily induced); here $t$ could be a function of $\Delta$. [18]

7.3. (a) Verify that every bipartite graph with a $K_4$ minor has at least 8 edges. (b) Verify that if $G$ is bipartite with at most 12 edges, then $G$ has none of the following as a minor: $K_5$, the Wagner graph (Möbius 8-ladder), the octahedron $K_6 - 3K_2$, and the prism $C_5 \square K_2$. [72]

# Chapter 8

# The Combinatorial Nullstellensatz

> The apex of mathematical achievement occurs when two or more fields which were thought to be entirely unrelated turn out to be closely intertwined. Mathematicians have never decided whether they should feel excited or upset by such events.
>
> —Gian-Carlo Rota

Here we use properties of polynomials to prove upper bounds on chromatic number, choice number, and paint number. By the Fundamental Theorem of Algebra, if a real one-variable polynomial $f$ has degree at most $n$, and is not identically 0, then it has at most $n$ real roots. So if we take $n+1$ distinct points on the real line, $x_1, \ldots, x_{n+1}$, then there exists some $i \in [n+1]$ such that $f(x_i) \neq 0$. The Combinatorial Nullstellensatz, Theorem 8.1 below, is an analogous statement for polynomials with multiple variables.

But what does this have to do with coloring? Given a graph $G$, we construct a polynomial $f_G$ such that, for every assignment $\varphi$ of colors to its vertices, $f_G(\varphi) \neq 0$ if and only if $\varphi$ is a proper coloring of $G$. Much like the single-variable case, we need the size of the list for each vertex $v$ to exceed the maximum degree in the polynomial of the variable corresponding to $v$.

## 8.1 The Alon–Tarsi Theorem

In this chapter, we present numerous applications of our next theorem, the Combinatorial Nullstellensatz, to prove coloring results. Understanding its proof, which is essentially polynomial long division, is not really necessary to grasp these applications. (So the impatient reader should feel free to skip ahead to the applications, and perhaps return to the proof later.) But the proof is short, so we include it for completeness.

We need to recall the following definition. The degree, $\deg(f)$, of a polynomial $f$ in a ring $\mathbb{F}[x_1, \ldots, x_n]$ is the maximum sum of exponents, over all monomials in its expansion. Typically, our choice of $\mathbb{F}$ is unimportant; so we usually let $\mathbb{F} := \mathbb{R}$ or $\mathbb{F} := \mathbb{Z}$. $\deg(f)$

**Theorem 8.1 (Combinatorial Nullstellensatz).** *Let $\mathbb{F}$ be a field and fix $f \in \mathbb{F}[x_1, \dots, x_n]$. Let $d_1, \dots, d_n$ be nonnegative integers such that $\sum_{i=1}^n d_i = \deg(f)$, and suppose the coefficient of $x_1^{d_1} \cdots x_n^{d_n}$ in $f$ is nonzero. Now for any subsets $L_1, \dots, L_n$ of $\mathbb{F}$ such that $|L_i| > d_i$ for each $i$, there exist elements $\alpha_i \in L_i$ such that $f(\alpha_1, \cdots, \alpha_n) \neq 0$.*

$f$, $L_i$ *Proof.* Suppose the theorem is false, and choose a counterexample $f$ and $L_1, \dots, L_n$ to minimize $\deg(f)$. First, suppose that $\deg(f) = 1$. By symmetry, we assume that $d_1 = 1$. Choose $\alpha_i \in L_i$ arbitrarily, for each $i \geqslant 2$. Now $f(x_1, \alpha_2, \dots, \alpha_n)$ is a linear function in one variable. Since $|L_1| > 1$, we can choose $\alpha_1 \in L_1$ such that $f(\alpha_1, \dots, \alpha_n) \neq 0$. Thus, $f$ is not a counterexample.

Instead assume $\deg(f) > 1$. We use polynomial division to find a counterexample $q$ with
$\alpha_1$ $\deg(q) < \deg(f)$, contradicting our choice of $f$. Choose $\alpha_1 \in L_1$ arbitrarily, and use long
$q$, $r$ division to find polynomials $q$ and $r$ (for quotient and remainder) such that

$$f(x_1, \dots, x_n) = (x_1 - \alpha_1)q(x_1, \dots, x_n) + r(x_2, \dots, x_n). \tag{8.1}$$

Note that the degree of $x_1$ in $r$ is less than in $x_1 - \alpha_1$. That is, $x_1$ does not appear in $r(x_2, \dots, x_n)$. By assumption, $f(x_1, \dots, x_n) = 0$ whenever $x_i \in L_i$ for all $i$. So, when $x_i \in L_i$ for all $i \geqslant 2$, we have $r(x_2, \dots, x_n) = f(\alpha_1, x_2, \dots, x_n) = 0$. Thus, when $x_i \in L_i$ for all $i$, we have $f(x_1, \dots, x_n) = (x_1 - \alpha_1)q(x_1, \dots, x_n) + r(x_2, \dots, x_n) = (x_1 - \alpha_1)q(x_1, \dots, x_n)$. By assumption, this value is always 0. So, when $x_1 \in L_1 \setminus \{\alpha_1\}$ and $x_i \in L_i$ for all $i \geqslant 2$, we have $q(x_1, \dots, x_n) = 0$. By hypothesis, the coefficient in $f$ of $x_1^{d_1} \cdots x_n^{d_n}$ is nonzero. Thus, the coefficient in $q$ of $x_1^{d_1 - 1} x_2^{d_2} \cdots x_n^{d_n}$ is also nonzero (since $\deg(q) < \deg(f) = \sum_{i=1}^n d_i$). However, now $q$ is a counterexample to the theorem, with subsets $L_1 \setminus \{\alpha_1\}, L_2, \dots, L_n$. Since $\deg(q) < \deg(f)$, this contradicts the minimality of $f$. $\square$

Eulerian **Definition 8.2.** Recall that a digraph $H$ is *Eulerian* if $d_H^+(v) = d_H^-(v)$ for all $v \in V(G)$. For
$EE(D)$, $OE(D)$ an orientation $D$ of a graph $G$, let $EE(D)$ and $OE(D)$ denote the sets of spanning Eulerian subgraphs $H$ of $D$ such that $\|H\|$ is even and odd, respectively. For an orientation $D$, let
$f_D$ $f_D := \prod_{v_i v_j \in E(D)} (x_i - x_j)$. For any two orientations $D'$ and $D''$ of $G$, note that $f_{D'} = \pm f_{D''}$.
$f_{d_1,\dots,d_n}$ For a field $\mathbb{F}$ and polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$, let $f_{d_1,\dots,d_n}$ denote the coefficient in $f$ of
$f_G$ $x_1^{d_1} \cdots x_n^{d_n}$. Typically, we mainly care about whether or not $f_{d_1,\dots,d_n} \neq 0$ (for some choice
graph polynomial $d_1, \dots, d_n$). So we often write $f_G$, or *graph polynomial* (when $G$ is clear from context), to denote $f_D$ for some arbitrary orientation $D$ of $G$; this determines $f_G$ up to a factor of $\pm 1$.

We will use the Combinatorial Nullstellensatz to strengthen results for list-coloring. Thus, we consider the graph polynomial. Given a list assignment $L_1, \dots, L_n$, the proper $L$-colorings correspond precisely to the choices $\alpha_i \in L_i$ such that $f(\alpha_1, \dots, \alpha_n) \neq 0$. So, to apply the Combinatorial Nullstellensatz, we must find $d_1, \dots, d_n$ such that $f_{d_1,\dots,d_n} \neq 0$. (This approach strengthens results on list-coloring, since having this nonzero coefficient is a sufficient—but not necessary—condition for being $L$-colorable.) Alon and Tarsi showed how to interpret the coefficient of $x_1^{d_1} \cdots x_n^{d_n}$ in terms of orientations of $G$ in which $d^+(v_i) = d_i$.

**Theorem 8.3 (Alon–Tarsi Theorem).** *Let* $D$ *be an orientation of a graph* $G$ *such that* $|EE(D)| - |OE(D)| \neq 0$*. If* $|L(v_i)| > d^+_D(v_i)$ *for all* $i$*, then* $G$ *is* $L$*-colorable.*

*Proof.* Let $n := |G|$. When we expand the graph polynomial $\prod_{v_iv_j \in E(G), i<j}(x_i - x_j)$, for each factor we select either $x_i$ or $-x_j$. We can interpret this selection as orienting the edge $v_iv_j$. When we select $x_i$, we orient away from $v_i$, and call this edge *increasing*. And when we select $-x_j$, we orient away from $v_j$, and call this edge *decreasing*.

For each term $x_1^{d_1} \cdots x_n^{d_n}$ in the expansion, note that the sequence of exponents is the sequence of outdegrees. Further, a term has coefficient 1 precisely when the number of decreasing edges in its corresponding orientation is even, and has coefficient $-1$ when this number is odd. Thus, after collecting like terms in the expansion, the coefficient on $x_1^{d_1} \cdots x_n^{d_n}$ equals the difference in the numbers of orientations (with this outdegree sequence) with even and odd numbers of decreasing edges. So this gives a necessary and sufficient condition for applying the Combinatorial Nullstellensatz.

To complete the proof, we extend this bijection a bit further, showing that this difference equals $\big|\, |EE(D)| - |OE(D)| \,\big|$ for a suitable orientation $D$. Given a graph $G$ and a sequence of outdegrees $d_1, \cdots, d_n$, choose an arbitrary orientation $D$ of $G$ with $d^+_D(v_i) = d_i$ for all $i$. Consider another orientation $\widehat{D}$ with $d^+_{\widehat{D}}(v_i) = d^+_D(v_i)$ for all $i$. Note that the subgraph induced by edges oriented oppositely in $D$ and $\widehat{D}$ is Eulerian. Further, the set of orientations with degree sequence equal to that of $D$ is in bijection with $EE(D) \cup OE(D)$. In fact, if an Eulerian subgraph $D_1 \in EE(D)$, then the numbers of decreasing edges in $D$ and $D \oplus D_1$ have the same parity. But if $D_1 \in OE(D)$, then the numbers of decreasing edges in $D$ and $D \oplus D_1$ have opposite parities. Thus, the absolute value of the coefficient of $x_1^{d_1} \cdots x_n^{d_n}$ in $f_G$ is exactly $\big|\, |EE(D)| - |OE(D)| \,\big|$. □

The previous theorem motivates our next definition.

**Definition 8.4.** The *Alon–Tarsi number, AT*$(G)$, is the minimum $k$ such that there exist integers $d_1, \ldots, d_n$ with $0 \leqslant d_i < k$ and $f_{d_1,\ldots,d_n} \neq 0$, where $f$ is the graph polynomial of $G$. The first paragraph of the previous proof shows that $AT(G)$ can be defined equivalently as the minimum integer $k$ such that $G$ has an orientation $D$ with $d^+_D(v) < k$ for all $v \in V(G)$ and $|EE(D)| - |OE(D)| \neq 0$. If $AT(G) \leqslant k$, then we say that $G$ is $k$-*AT*. More generally, for a function $g : V(G) \to \mathbb{Z}^+$, the graph $G$ is $g$-*AT* if there exist $d_1, \ldots, d_n$ such that $d_i < g(v_i)$ for all $i$ and $f_{d_1,\ldots,d_n} \neq 0$. (So $k$-AT is the special case when $g(v) := k$ for all $v \in V(G)$.) Similarly, $g$-AT can be defined in terms of an orientation $D$ and $|EE(D)| - |OE(D)|$.

Alon–Tarsi number, $AT(G)$

$k$-AT

$g$-AT

Theorem 8.3 gives $\chi_\ell(G) \leqslant AT(G)$ for every graph $G$. As we will see in Section 8.8, the Alon–Tarsi number also bounds the paint number; that is, $\chi_\ell(G) \leqslant \chi_p(G) \leqslant AT(G)$.

### 8.1.1 First Examples and Easy Lemmas

We begin with some examples and easy lemmas, to help build the reader's intuition.

**Example 8.5.** Let $G := C_n$. Each $x_i$ appears in exactly two factors of $f_G$; thus, exactly two terms in the expansion of $f_G$ contribute to the coefficient of $\prod_{i=1}^n x_i$. When $n$ is even, both terms are positive, so $f_{1,\ldots,1} = 2 \neq 0$. And when $n$ is odd, one is positive and one is negative, so $f_{1,\ldots,1} = 0$. Equivalently, consider an orientation $D$ with $d_D^+(v) = d_D^-(v) = 1$ for all vertices $v$; see Figure 8.1. The only two spanning Eulerian subgraphs of $D$ contain all and none of its edges. Thus, when $n$ is even $|EE(D)| = 2 \neq 0 = |OE(D)|$, so $AT(C_n) = 2$. When $n$ is odd $|EE(D)| = 1 = |OE(D)|$, so $AT(C_n) > 2$. (How do we verify that $AT(C_n) = 3$?)

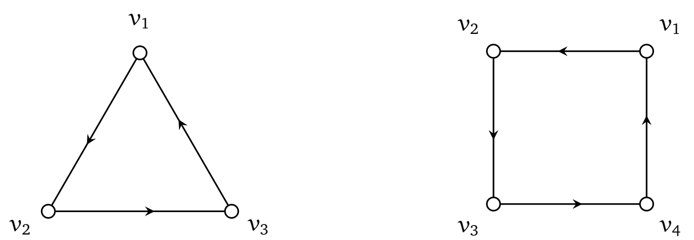



Figure 8.1: Every directed cycle $D_n$ has 2 spanning Eulerian subgraphs: the edgeless graph and all of $D_n$. When $n$ is even, $|EE(D_n)| = 2$ and $|OE(D_n)| = 0$, so $AT(C_n) = 2$. When $n$ is odd, $|EE(D_n)| = |OE(D_n)| = 1$, so $AT(C_n) > 2$. See Example 8.5.

Another simple application of the Alon–Tarsi Theorem is to d-degenerate graphs. Suppose $G$ is $d$-degenerate, and let $v_1, \ldots, v_n$ be an order where each $v_i$ has at most $d$ neighbors later in the order. We orient each edge as $v_i \to v_j$, where $i < j$. This gives an acyclic orientation $D$, with maximum outdegree at most $d$. Since $D$ is acyclic, its only Eulerian subgraph is the edgeless graph. Thus, $EE(D) = 1 \neq 0 = OE(D)$. So $AT(G) \leqslant d+1$.

In Chapter 5 we used the Kernel Method to show that a graph $G$ is $(d+1)$-choosable whenever $G$ is bipartite and $\text{mad}(G) \leqslant 2d$. In particular, planar bipartite graphs are 3-choosable. The idea was to find an orientation $D$ of $G$ with outdegree at most $d$. In fact, we can prove the same result by applying the Alon–Tarsi Theorem to $D$. (Indeed, this is how it was first proved.) For any Eulerian subgraph $H$ of $D$, the edges of $H$ can be partitioned into directed cycles. Since $G$ is bipartite, each directed cycle of $H$ must be even. Hence $\|H\|$ is also even. Since $G$ always has the edgeless graph as an Eulerian subgraph, we have $|EE(D)| \geqslant 1 \neq 0 = |OE(D)|$. So, again $AT(G) \leqslant d+1$. ♢

Motivated by our intuition from coloring and list-coloring, we naturally guess that the Alon–Tarsi number of a graph is the maximum of the Alon–Tarsi numbers of its components. Indeed, this is true. More generally, the union of "good" orientations for all components of G is a good orientation for G. This is a special case of Lemma 1.40(ii). But as a warmup, we reprove it.

**Lemma 8.6.** *Fix a graph* $G$ *and vertex disjoint subgraphs* $G_1$ *and* $G_2$ *with* $V(G) = V(G_1) \cup V(G_2)$ *and* $E(G) = E(G_1) \cup E(G_2)$. *Fix* $g : V(G) \to \mathbb{Z}^+$. *Let* $g_1$ *and* $g_2$ *be the restrictions of* $g$ *to* $V(G_1)$ *and* $V(G_2)$. *If* $G_1$ *is* $g_1$*-AT and* $G_2$ *is* $g_2$*-AT, then also* $G$ *is* $g$*-AT.*

*Proof.* By hypothesis $G_i$ is $g_i$-AT, for each $i \in \{1, 2\}$. So there exist orientations $D_i$ such that $d^+_{D_i}(v) < g_i(v)$ for all $v \in V(G_i)$ and $|EE(D_i)| - |OE(D_i)| \neq 0$. For brevity, we denote $|EE(D_i)|$ and $|OE(D_i)|$ by $EE_i$ and $OE_i$. Let $D := D_1 \cup D_2$. Now $|EE(D)| - |OE(D)| = (EE_1EE_2 + OE_1OE_2) - (EE_1OE_2 + EE_2OE_1) = (EE_1 - OE_1)(EE_2 - OE_2) \neq 0$. □

For any subgraph H of a graph G, clearly $\chi(H) \leqslant \chi(G)$ and $\chi_\ell(H) \leqslant \chi_\ell(G)$. This observation motivates the following analogous lemma for Alon–Tarsi number.

**Lemma 8.7.** *If* $H$ *is a subgraph of a graph* $G$*, then* $AT(H) \leqslant AT(G)$*. More generally, fix* $g : V(G) \to \mathbb{Z}^+$*, and let* $\widehat{g}$ *be the restriction of* $g$ *to* $V(H)$*. If* $G$ *is* $g$*-AT, then* $H$ *is* $\widehat{g}$*-AT.*

*Proof.* It suffices to show that if $e \in E(G)$, and G is g-AT, then also $G - e$ is g-AT. From this the result follows by induction on $\|G\| - \|H\|$. If $|V(H)| < |V(G)|$, then for each vertex in $V(G) \setminus V(H)$, we first delete all of its incident edges. Now removing an isolated vertex has no effect on the graph polynomial.

Suppose the lemma is false. Fix a graph G, a function g, and an edge e such that G is g-AT, but $G - e$ is not g-AT. Let $v_1$ and $v_2$ denote the endpoints of e. Since $G - e$ is not g-AT, for every term $x_1^{c_1} \cdots x_n^{c_n}$ of $f_{G-e}$ with nonzero coefficient, there exists i such that $g(v_i) \leqslant c_i$. Now $f_G = (x_1 - x_2)f_{G-e}$. Terms may cancel, but exponents never decrease. Thus, for any term $x_1^{d_1} \cdots x_n^{d_n}$ with nonzero coefficient in $f_G$, there exists i such that $g(v_i) \leqslant d_i$; in particular, $g(v_i) \leqslant c_i \leqslant d_i$, where i is chosen to show that $G - e$ is not g-AT. Hence, G is not g-AT, contradicting the hypothesis. □

Next we prove a version of Brooks' Theorem for Alon–Tarsi number.

**Theorem 8.8.** *If* $\Delta(G) \geqslant 3$ *and* $G$ *contains no copy of* $K_{\Delta+1}$*, then* $AT(G) \leqslant \Delta$*.*

*Proof.* By Lemma 8.6, we assume G is connected.

**Case 1: There is $v \in V(G)$ such that $d(v) < \Delta(G)$.** We order $V(G)$ by weakly increasing distance from v, and orient each edge toward its endpoint later in the order. The resulting digraph D is acyclic, so $|EE(D)| = 1 \neq 0 = |OE(D)|$. Now we are done, since each vertex w other than v has an inneighbor on the $v, w$-path in D, and thus $\Delta^+(D) < \Delta(G)$.

**Case 2: G is regular.** If G is 2-connected, then it contains as an induced subgraph $\mathcal{C}$ an even cycle with at most one chord, by Rubin's Block Lemma (Lemma 1.38). If G has a cut-vertex, then we find such a $\mathcal{C}$ within a leaf block, which must not be regular, since G is regular. We again orient all edges outside $\mathcal{C}$ away from $\mathcal{C}$, as in Case 1. We also orient the edges of $\mathcal{C}$ as a directed cycle and the chord, if it exists, arbitrarily. Denote by D this orientation and by $\mathcal{C}_D$ its restriction to $\mathcal{C}$. As in Case 1, we have $\Delta^+(D) < \Delta(G)$. Note that every spanning Eulerian subgraph of D has all its edges in $\mathcal{C}_D$ (otherwise some vertex w outside $\mathcal{C}$ has $d^-(w) \geqslant 1 > 0 = d^+(w)$, contradicting that the subgraph is Eulerian). But it is easy to check that $|EE(\mathcal{C}_D)| \geqslant 2 > 1 \geqslant |OE(\mathcal{C}_D)|$. □

### 8.1.2 Squares of Cycles

Note that $\chi(C_n^2) = 3$ if and only if 3 divides $n$. Sufficiency follows from the coloring $1, 2, 3, 1, 2, 3, \ldots$ and necessity follows from the fact that $C_n^2$ has independence number $\lfloor n/3 \rfloor$. (In fact, Brooks' Theorem implies that $\chi(C_n^2) \leqslant 4$, except when $n = 5$, in which case $C_5^2 = K_5$.) Here we prove the analogous statement for $AT(C_n^2)$.

**Theorem 8.9.** *We have $AT(C_n^2) = 3$ if and only if $3|n$.*

As in the case of coloring, necessity follows from the independence number, so we only need to prove sufficiency. We would like to fix an Eulerian orientation $D$ and use recursion to evaluate $|EE(D)| - |OE(D)|$. But $C_n^2$ is not directly amenable to recursion. Instead we work with the squared path $P_n^2$, which *is* amenable to recursion. We then relate the subgraphs of $C_n^2$ to those of $P_n^2$.

$\overrightarrow{P_n}$ weakly Eulerian $EE_i$, $OE_i$ $h_i(n)$

**Definition 8.10.** Form $\overrightarrow{P_n}$ from $P_n^2$ by orienting all edges from left to right. Number the vertices, from left to right, as $v_1, \ldots, v_n$. A spanning subgraph $D$ of $\overrightarrow{P_n}$ is *weakly Eulerian* if each $w \notin \{v_1, v_n\}$ satisfies $d^+(w) = d^-(w)$ and $d^+(v_1) = d^-(v_n) = i$ for some $i \in \{1, 2\}$. Let $EE_i(\overrightarrow{P_n})$ and $OE_i(\overrightarrow{P_n})$ denote, respectively, the sets of even and odd weakly Eulerian subgraphs where $d^+(v_1) = d^-(v_n) = i$. Finally, let $h_i(n) := |EE_i(\overrightarrow{P_n})| - |OE_i(\overrightarrow{P_n})|$.

To prove Theorem 8.9, we need the following lemma about $h_i(n)$.

**Lemma 8.11.** *If $n = 3k + j$ for some positive integer $k$ and $j \in \{-1, 0, 1\}$, then $h_1(n) = j$ and for $n \geqslant 4$ also $h_2(n) = -h_1(n-2)$, with $h_i(n)$ as in Definition 8.10.*

*Proof.* Rather than directly counting weakly Eulerian subgraphs, we use a parity-reversing (partial) bijection. (We will see the same technique in the proof of Theorem 8.17 and in Example 8.24.) That is, we group many of the weakly Eulerian subgraphs into pairs, where

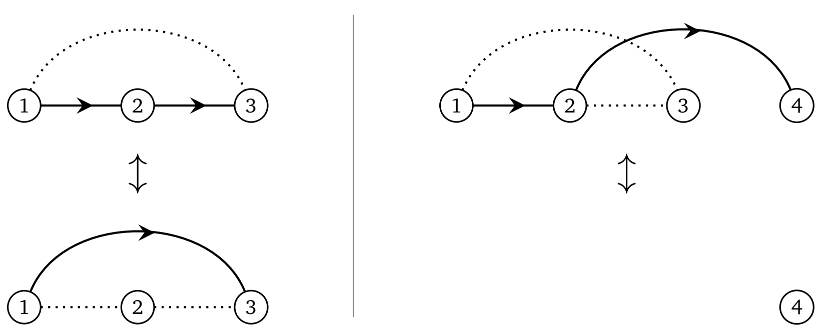



Figure 8.2: The calculation of $h_1(n)$ in Lemma 8.11. Left: Many of the weakly Eulerian subgraphs counted by $h_1(n)$ can be paired up and disregarded, since the total contribution of the pair to $h_1(n)$ is 0. Right: Those subgraphs that cannot be paired up are mapped bijectively to subgraphs counted by $h_1(n-3)$.

the numbers of edges in the two subgraphs in each pair have opposite parities. Thus, the total contribution of the pair to $h_i(n)$ is 0, and the pair can be safely ignored when computing $h_i(n)$; see the left of Figure 8.2.

We first prove that $h_2(n) = -h_1(n-2)$. The complement of each $D \in EE_2(\overrightarrow{P_n}) \cup OE_2(\overrightarrow{P_n})$ D has $d^+(v_2) = d^-(v_{n-1}) = 1$ and $d^+(w) = d^-(w)$ for each $w \notin \{v_1, v_2, v_{n-1}, v_n\}$ (and $d^+(v_1) = d^-(v_n) = d^-(v_2) = d^+(v_{n-1}) = 0$). Since $\overrightarrow{P_n}$ has $2n-3$ edges, each digraph has parity opposite that of its complement; thus, $h_2(n) = -h_1(n-2)$.

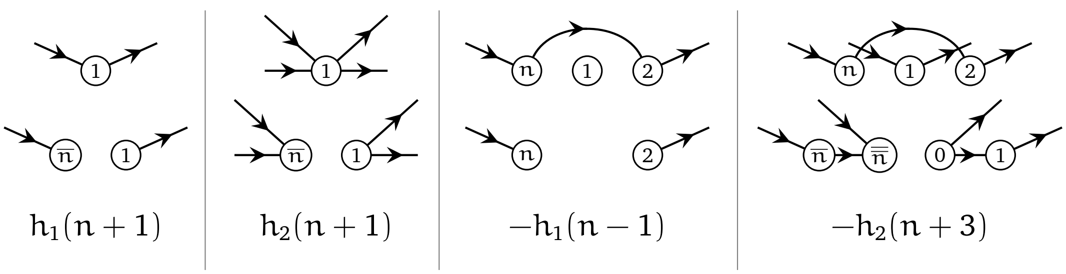
Now we determine $h_1(n)$. Let D be a weakly Eulerian subgraph with $d^+(v_1) = 1$. Consider the directed paths $v_1v_3$ and $v_1v_2, v_2v_3$. If D contains all of one path and none of the other, then we can pair D with its complement, which has opposite parity. If neither of these cases holds, then we must have $v_1v_2, v_2v_4 \in \mathcal{D}$ and $v_1v_3, v_2v_3 \notin D$. This yields $h_1(n) = h_1(n-3)$. It remains only to check that $h_1(2) = -1$, $h_1(3) = 0$, and $h_1(4) = 1$. □

Our next result is a slightly stronger version of Theorem 8.9.

**Lemma 8.12.** *For each integer $n \geqslant 3$, there exists an orientation $\overrightarrow{C_n}$ of $C_n^2$ such that $|EE(\overrightarrow{C_n})| - |OE(\overrightarrow{C_n})| = 6$ if $3|n$. Thus, $AT(C_n^2) = 3$ if $3|n$.*

*Proof.* Let $\overrightarrow{C_n}$ denote the orientation of $C_n^2$ with vertices $v_1, \ldots, v_n$ spaced equally around a $\overrightarrow{C_n}$ circle in clockwise order and each edge oriented clockwise. Fix an Eulerian subgraph D of $\overrightarrow{C_n}$. D We classify D by the value of $d_D^+(v_1)$ and whether or not $\overrightarrow{v_nv_2} \in D$. Since $0 \leqslant d_D^+(v_1) \leqslant 2$, a priori we have 6 cases; however, 2 of these are trivial. If $\overrightarrow{v_nv_2} \notin D$ and $d_D^+(v_1) = 0$, then $E(D) = \emptyset$. And if $\overrightarrow{v_nv_2} \in D$ and $d^+(v_1) = 2$, then $E(D) = E(\overrightarrow{C_n})$. Now we consider the 4 remaining cases. In each case, we map D to a weakly Eulerian subgraph in $EE_i(P_\ell) \cup OE_i(P_\ell)$ for some $\ell$ and some $i \in \{1, 2\}$.

(a) If $d_D^+(v_1) = 1$ and $\overrightarrow{v_nv_2} \notin D$, then we map D to a subgraph of $EE_1(\overrightarrow{P_{n+1}}) \cup OE_1(\overrightarrow{P_{n+1}})$; see Figure 8.3(a). To map subgraphs of $\overrightarrow{C_n}$ to subgraphs of $\overrightarrow{P_{n+1}}$, we split (old) vertex 1 into two vertices: (new) vertex 1 inherits its outedge and (new) vertex $n+1$ inherits its inedge.




(a) (b) (c) (d)

Figure 8.3: The top two rows show the mappings in the proof of Lemma 8.12 from subgraphs of $\overrightarrow{C_n}$ to subgraphs of $\overrightarrow{P_\ell}$, for certain values of $\ell$. The third row shows the contribution of these subgraphs to $|EE(\overrightarrow{C_n})| - |OE(\overrightarrow{C_n})|$. Here we denote $n+1$ by $\overline{n}$ and denote $n+2$ by $\overline{\overline{n}}$. (a) $d_D^+(v_1) = 1$ and $\overrightarrow{v_nv_2} \notin D$. (b) $d_D^+(v_1) = 2$ and $\overrightarrow{v_nv_2} \notin D$. (c) $d_D^+(v_1) = 0$ and $\overrightarrow{v_nv_2} \in D$ (d) $d_D^+(v_1) = 1$ and $\overrightarrow{v_nv_2} \in D$.

(b) If $d^+_D(v_1) = 2$ and $\overrightarrow{v_n v_2} \notin D$, then we map D to a subgraph of $EE_2(\overrightarrow{P_{n+1}}) \cup OE_2(\overrightarrow{P_{n+1}})$; see Figure 8.3(b). The vertex splitting here is similar to in part (a).

(c) If $d^+_D(v_1) = 0$ and $\overrightarrow{v_n v_2} \in D$, then we map D to a subgraph of $EE_1(\overrightarrow{P_{n-1}}) \cup OE_1(\overrightarrow{P_{n-1}})$; see Figure 8.3(c). To map subgraphs of $\overrightarrow{C_n}$ to subgraphs of $\overrightarrow{P_{n-1}}$, we delete vertex 1 and also delete edge $\overrightarrow{v_n v_2}$.

(d) If $d^+_D(v_1) = 1$ and $\overrightarrow{v_n v_2} \in D$, then we map D to a subgraph of $EE_2(\overrightarrow{P_{n+3}}) \cup OE_2(\overrightarrow{P_{n+3}})$; see Figure 8.3(d). This case is a bit more subtle. We delete edge $\overrightarrow{v_n v_2}$ and create three new vertices: $0$, $n+1$, $n+2$. Vertex 0 sends outedges to 1 and 2. Vertex (new) 1 inherits the outedge from vertex (old) 1. Vertex $n+2$ gets inedges from $n$ and (new) $n+1$. And vertex (new) $n+1$ inherits the inedge from (old) 1.

As suggested above, we can now write $|EE(\overrightarrow{C_n})| - |OE(\overrightarrow{C_n})|$ as a sum of terms of the form $h_i(\ell)$ with $i \in \{1, 2\}$. Note that the sign of a term in this sum is negative if the map for the corresponding term in the sum flips the parity of the number of edges in each subgraph. The term 2 below comes from the 2 trivial cases at the start. Thus we get

$$|EE(\overrightarrow{C_n})| - |OE(\overrightarrow{C_n})| = 2 + h_1(n+1) + h_2(n+1) - h_1(n-1) - h_2(n+3).$$

By applying Lemma 8.11(b) twice, this expression simplifies to $2 + 2h_1(n+1) - 2h_1(n-1)$. Now the lemma follows from Lemma 8.11(a) by direct substitution. If $n \bmod 3 \equiv -1$, then we get $2 + 2(0) - 2(1) = 0$. If $n \bmod 3 \equiv 0$, then we get $2 + 2(1) - 2(-1) = 6$. If $n \bmod 3 \equiv 1$, then we get $2 + 2(-1) - 2(0) = 0$. $\square$

### 8.1.3 The Product of a Cycle and a Path

Cartesian product

$G_1 \square G_2$

**Definition 8.13.** The *Cartesian product*, $G_1 \square G_2$, of graphs $G_1$ and $G_2$ has $V(G_1 \square G_2) = \{(x, y) : x \in V(G_1) \text{ and } y \in V(G_2)\}$ and $E(G_1 \square G_2) = \{(x_1, y_1)(x_2, y_2) : x_1 = x_2 \text{ and } y_1 y_2 \in E(G_2) \text{ or } y_1 = y_2 \text{ and } x_1 x_2 \in E(G_1)\}$. For example, $P_k \square P_\ell$ is a $k \times \ell$ grid.

**Theorem 8.14.** *If $G$ is a Cartesian product $C_{2k+1} \square P_\ell$ of an odd cycle and a path, then $AT(G) = 3$.*

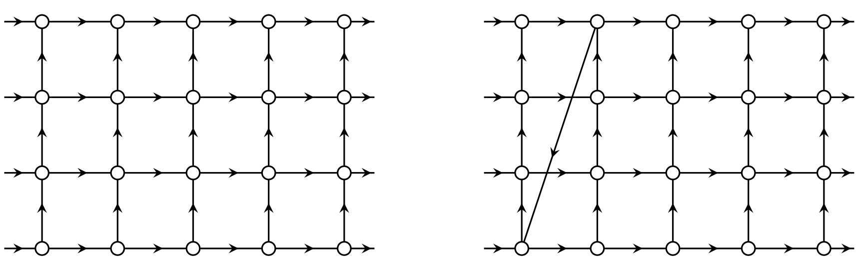


Figure 8.4: $D$ on the left and $D^*$ on the right (when $G = C_5 \square P_4$).

*Proof.* Clearly $\mathrm{AT}(G) \geqslant \mathrm{AT}(C_{2k+1}) \geqslant \chi(C_{2k+1}) = 3$. So we must show that $\mathrm{AT}(G) \leqslant 3$.

Denote $V(C_{2k+1})$ by $\{x_1, \ldots, x_{2k+1}\}$ and $V(P_\ell)$ by $\{y_1, \ldots, y_\ell\}$. Let $V(G) := \{(x_i, y_j) : 1 \leqslant i \leqslant 2k+1 \text{ and } 1 \leqslant j \leqslant \ell\}$, let $X_i := \{(x_i, y_j) : 1 \leqslant j \leqslant \ell\}$, and let $Y_j := \{(x_i, y_j) : 1 \leqslant i \leqslant 2k+1\}$. So each $X_i$ induces $P_\ell$ and each $Y_j$ induces $C_{2k+1}$. Orient each edge on a path upward and each edge on a cycle to the right; here each edge $(x_{2k+1}, y_j), (x_1, y_j)$ "wraps around" with head at $(x_1, y_j)$. See Figure 8.4. Call this orientation $D$. Note that the only directed cycles in $D$ are those induced by each $Y_j$. Form $D^*$ from $D$ by adding an edge from $(x_2, y_\ell)$ to $(x_1, y_1)$; call this edge $e^*$. Let $G^*$ denote the undirected graph underlying $D^*$. We will show that $|\mathrm{EE}(D^*)| \neq |\mathrm{OE}(D^*)|$. Since $D^*$ has maximum outdegree 2, by Lemma 8.7 this implies that $3 \geqslant \mathrm{AT}(G^*) \geqslant \mathrm{AT}(G)$, as desired.

$X_i, Y_j$ $D$ $D^*$ $e^*, G^*$

To show that $|\mathrm{EE}(D^*)| \neq |\mathrm{OE}(D^*)|$, we prove the stronger result that $|\mathrm{EE}(D^*)| + |\mathrm{OE}(D^*)|$ is odd. Let $\mathcal{E} \subseteq \mathrm{EE}(D^*) \cup \mathrm{OE}(D^*)$ denote the set of Eulerian subgraphs $H$ of $D^*$ for which there exists $j$ such that $H$ contains either all or none of the edges induced by $Y_j$. Starting from $H$, for the minimum such $j$, if $H$ contains all edges induced by $Y_j$, then remove all of them; and if $H$ contains no edges induced by $Y_j$, then add all of them. Call the resulting Eulerian digraph $g(H)$. Clearly, $g(g(H)) = H$, and $g$ has no fixed point. Thus, $|\mathcal{E}|$ is even. Let $\overline{\mathcal{E}}$ denote $\mathrm{EE}(D^*) \cup \mathrm{OE}(D^*) \setminus \mathcal{E}$. Since $|\mathcal{E}|$ is even, it suffices to show that $|\overline{\mathcal{E}}|$ is odd. Fix $H \in \overline{\mathcal{E}}$. Recall that every directed cycle in $D$ is induced by some $Y_j$. Since every Eulerian subgraph decomposes

$\mathcal{E}$ $\overline{\mathcal{E}}$

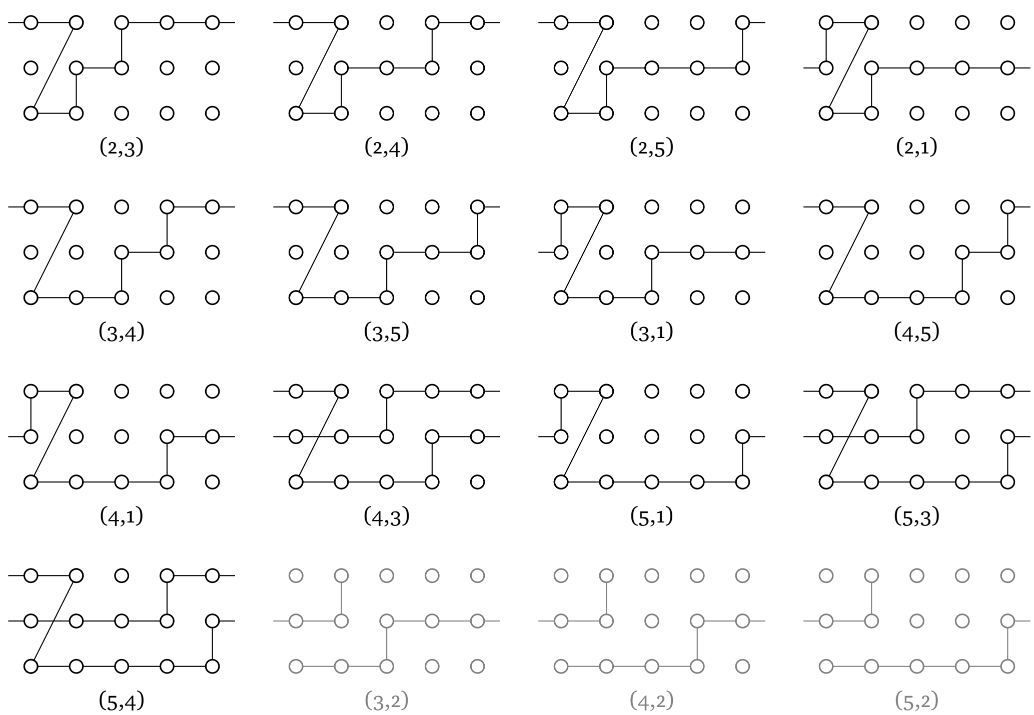



Figure 8.5: The 16 ordered pairs in $B_3$, for $C_5 \square P_3$: the first 13 in $B_3'$, and the last 3 in $B_3''$.

into directed cycles, $H$ is a single directed cycle containing edge $e^*$. Since $e^*$ is the only edge directed downward, $H$ contains between 1 and $2k$ consecutive edges induced by $Y_j$, for each $j \in [\ell]$. We define the following three sets of ordered $(s-1)$-tuples. Clearly, $B_s$ is the disjoint union of $B'_s$ and $B''_s$.

$$B_s := \{(i_1, \ldots, i_{s-1}) : i_h \in [2k+1], i_1 \neq 1, \text{ and } i_{h+1} \neq i_h \text{ for all } h \in [s-2]\}$$
$$B'_s := \{(i_1, \ldots, i_{s-1}) \in B_s : i_{s-1} \neq 2\}$$
$$B''_s := \{(i_1, \ldots, i_{s-1}) \in B_s : i_{s-1} = 2\}$$

Since $H$ is uniquely determined by its vertical edges, $|\overline{\mathcal{E}}| = |B'_s|$. More specifically, $i_h$ is the column of the vertical edge out of $Y_h$. We need that $i_{h+1} \neq i_h$ to ensure that some edge of $Y_{h+1}$ lies on $D$. And we need $i_{s-1} \neq 2$ to ensure that some edge of $Y_s$ lies on $D$, since $e^*$ has its tail in column 2. Figure 8.5 shows the elements of $B_3$ when $G := C_5 \square P_3$.

Observe that $|B_2| = 2k$, so $|B_{s+1}| = (2k)^s$ since, given any element of $B_s$, we can extend it to an element of $B_{s+1}$ by choosing $i_s$ to be any value in $\{1, \ldots, 2k+1\} \setminus \{i_{s-1}\}$. We prove that $|B'_s|$ is odd, by induction on $s$. It is easy to check that $|B'_2| = 2k-1$, and $|B''_2| = 1$. Each element of $B'_s$ can be extended to an element of $B'_{s+1}$ by choosing $i_s$ to be any element of $\{1, \ldots, 2k+1\} \setminus \{2, i_{s-1}\}$. Similarly, each element of $B''_s$ can be extended to an element of $B'_{s+1}$ by choosing $i_s$ to be any element of $\{1, \ldots, 2k+1\} \setminus \{i_{s-1}\}$. Thus, $|B'_{s+1}| = |B'_s|(2k-1) + |B''_s|(2k)$. Since $|B'_s|$ is odd by induction, also $|B'_{s+1}|$ is odd. □

## 8.2 The Cycle-Plus-Triangles Theorem

CPT graph **Definition 8.15.** A *cycle-plus-triangles graph* $G$ (*CPT graph*, for short) is a 4-regular graph formed from a cycle $v_1, \ldots, v_{3n}$ by adding the edges of $n$ vertex-disjoint triangles on the same vertex set; see Figure 8.7. When we speak of a CPT graph $G$, we mean an embedding of $G$ with the cycle $v_1, \ldots, v_{3n}$ on the outer face, with its vertices in convex position, and with each triangle edge drawn as a chord inside this long cycle.

Let $D$ be an arbitrary Eulerian orientation of $G$ (for example formed by orienting cyclically each triangle, as well as the long cycle $v_1, \ldots, v_{3n}$). To prove that $AT(G) = 3$, we need only show that $|EE(D)| - |OE(D)| \neq 0$.

To prove Theorem 8.14, we showed that $|EE(D^*)| + |OE(D^*)|$ is odd, which immediately implied that $|EE(D^*)| \neq |OE(D^*)|$; so $AT(G) \leqslant 1 + \Delta^+(D^*) = 3$. We would like to use the same approach here, but it won't quite work, since this sum is actually even, as we now show.

$g(D')$ Fix an Eulerian orientation $D$ of $G$ and an Eulerian subgraph $D'$ of $D$. Define $g(D')$ to be the spanning subgraph of $D$ containing precisely the edges that are missing from $D'$. That is, $E(D') \cap E(g(D')) = \emptyset$ and $E(D') \cup E(g(D')) = E(D)$. Note also, that $g(g(D')) = D'$. So $g$ pairs up the elements of $EE(D) \cup OE(D)$.

However, since $D$ is 4-regular, $\|G\|$ is even. Thus, if $D' \in EE(D)$, then $g(D') \in EE(D)$; otherwise, $D' \in OE(D)$ and $g(D') \in OE(D)$. So $|EE(D)|$ and $|OE(D)|$ are both even, which means that our strategy to prove Theorem 8.14 cannot work here. However, we can modify it slightly. We will show that $|\mathrm{EE}(D)| + |\mathrm{OE}(D)| \equiv 2 \pmod 4$. Since $|\mathrm{EE}(D)|$ and $|\mathrm{OE}(D)|$ are both even, this implies that $|\mathrm{EE}(D)| \neq |\mathrm{OE}(D)|$, as desired.

We begin with a seemingly unrelated problem: counting the sets $U$ of edges from triangles such that $U$ contains exactly one edge from each triangle, and each edge in $U$ intersects an even number of edges in $U$. We will show that the number, $n_U$, of such sets $U$ is odd, and later show that $\big|\, |\mathrm{EE}(D)| - |\mathrm{OE}(D)| \,\big| = 2n_u \neq 0$. $n_U$

**Lemma 8.16.** *Fix a CPT graph, embedded as in Definition 8.15. Let $W_1, \dots, W_n$ denote the edge sets of the triangles of $G$. Form a graph $J$ with $V(J) := \cup_{i=1}^n W_i$ and let $w_i, w_j \in V(J)$ be adjacent in $J$ if their corresponding edges intersect in the embedding of $G$. Consider subsets $U \subset V(J)$ such that $|U \cap W_i| = 1$ for all $i$ and $J[U]$ is Eulerian. That is, $U$ picks one edge from each triangle and picked edges each intersect an even number of picked edges. The number of such subsets $U$ is odd.* $J$

*Proof.* Consider a sequence $(y_1, \dots, y_n, z_1, \dots, z_n)$ such that (1)–(3) below hold for all $i$:

1. $y_i \in W_i$,
2. $z_i \in \{y_1, \dots, y_n\}$, and
3. either $z_i = y_i$ or else $y_i$ and $z_i$ are adjacent in $J$.

We call this a *special sequence*. Fix a sequence $(y_1, \dots, y_n)$ satisfying (1). Let $U := \{y_1, \dots, y_n\}$, and let $J' := J[U]$. Now $(y_1, \dots, y_n)$ can be extended to exactly $\prod_{y \in U}(1 + d_{J'}(y))$ special sequences. So $U$ gives rise to an odd number of special sequences precisely when $J'$ is Eulerian. Thus, the number of special sequences and the number of sets $U$ (such that $J[U]$ is Eulerian) have the same parity. So, it suffices to show that the number of special sequences is odd. special sequence

For each special sequence $\sigma := (y_1, \dots, y_n, z_1, \dots, z_n)$ we form a digraph $G(\sigma)$ with vertex set $[n]$ as follows. Add edge $i \to j$ if $z_i \in W_j$ and $i \neq j$. Note that $\Delta^+(G(\sigma)) \leqslant 1$ for all $\sigma$. Further, if we fix $y_1, \dots, y_n$, then $G(\sigma)$ defines at most one choice of $z_1, \dots, z_n$. Specifically, if $i \to j$, then $z_i = y_j$ and if $d^+(i) = 0$, then $z_i = y_i$. So we refer to a special sequence $\sigma$ equivalently by its graph $G(\sigma)$. Let $\mathcal{S}$ denote the set of all special sequences. A directed cycle in $G(\sigma)$ with length at least 3 is *long*. Let $\mathcal{S}_{long}$ denote the set of special sequences with at least one long cycle. Let $\mathcal{S}_{short}$ denote the set of special sequences with no long cycle, but at least one edge. And let $\mathcal{S}_\emptyset$ denote the set of special sequences with no edges. So $|\mathcal{S}| = |\mathcal{S}_{long}| + |\mathcal{S}_{short}| + |\mathcal{S}_\emptyset|$. Note that $|\mathcal{S}_\emptyset|$ is simply $\prod_{i=1}^n |W_i| = 3^n$, which is odd. Thus, it suffices to show that both $|\mathcal{S}_{long}|$ and $|\mathcal{S}_{short}|$ are even. $\sigma, G(\sigma)$ $\mathcal{S}$ $\mathcal{S}_{long}$ $\mathcal{S}_{short}$ $\mathcal{S}_\emptyset$

To prove that $\left|\mathcal{S}_{long}\right|$ is even, we pair up its elements. Suppose $\sigma \in \mathcal{S}_{long}$ and among all directed cycles, pick the one, $C$, that visits the smallest numbered vertex. Form $g(\sigma)$ from $\sigma$ by reversing the edges of $C$; see Figure 8.6. Note that $g(g(\sigma)) = \sigma$. Thus, $|\mathcal{S}_{long}|$ is even.

Now we show that $|\mathcal{S}_{short}|$ is also even. Fix $\sigma \in \mathcal{S}_{short}$. Let $N(x) := N^+(x) \cup N^-(x)$ for all $x \in V(G(\sigma))$. Choose the smallest $i$ such that $|N(i)| = 1$, and let $\{j\} := N(i)$. We will form a new special sequence $g(\sigma)$ from $\sigma$ by replacing $y_i$ with some edge $y_i' \in W_i$ such that

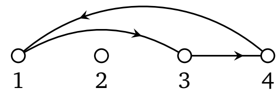
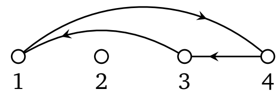
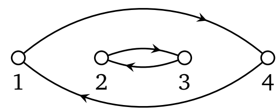
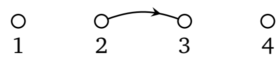
| | $\sigma, g(\sigma)$ | $G(\sigma), G(g(\sigma))$ |
|---|---|---|
| $\mathcal{S}_{long}$ | $(a_1, b_3, c_2, d_2, c_2, b_3, d_2, a_1)$ |  |
| | $(a_1, b_3, c_2, d_2, d_2, b_3, a_1, c_2)$ |  |
| $\mathcal{S}_{short}$ | $(a_1, b_3, c_2, d_2, d_2, c_2, b_3, a_1)$<br>$(a_3, b_3, c_2, d_2, d_2, c_2, b_3, a_3)$ |  |
| $\mathcal{S}_{short}$ | $(a_1, b_3, c_2, d_2, a_1, c_2, c_2, d_2)$<br>$(a_1, b_2, c_2, d_2, a_1, c_2, c_2, d_2)$ |  |

Figure 8.6: Pairs of special sequences $\sigma$ and $g(\sigma)$, for the CPT graph shown on the top left of Figure 8.7.

$G(g(\sigma)) = G(\sigma)$. Further, we want that $g(g(\sigma)) = \sigma$. It is easy to check that each edge $y_j$ that intersects a triangle $W_i$ intersects exactly two of its edges. (Since $G$ is 4-regular, $y_j$ cannot intersect any vertex of $W_i$.) One of the edges of $W_i$ that $y_j$ intersects is $x_i$; denote the other by $x_i'$. Form $g(\sigma)$ from $\sigma$ by replacing each instance of $x_i$ by $x_i'$. Thus, $g$ pairs up elements of $\mathcal{S}_{short}$, which implies that $|\mathcal{S}_{short}|$ is even, as desired. $\square$

**Theorem 8.17.** *Let $G$ be a cycle-plus-triangles graph with vertices $v_1, \ldots, v_{3n}$, and let $D$ be an arbitrary Eulerian orientation of $G$. Now $|EE(D)| - |OE(D)| \neq 0$. Thus, $AT(G) = 3$.*

*Proof.* Since $G$ is 4-regular, and $D$ is Eulerian, $d_D^+(v) = 2$ for all $v \in V(G)$. So the bound $AT(G) \leqslant 3$ follows directly from the Alon–Tarsi Theorem. Rather than count the Eulerian
$OE(G), EE(G)$ subgraphs of $D$, we directly consider the Eulerian orientations of $G$. Let $OE(G)$ and $EE(G)$ denote the sets of these with odd numbers and even numbers (respectively) of decreasing edges. As noted in the proof of Theorem 8.3, we have $\big|\, |EE(G)| - |OE(G)| \,\big| = \big|\, |EE(D)| - |OE(D)| \,\big|$. To show $|EE(G)| - |OE(G)| \neq 0$, we pair some elements of $EE(G)$ with elements of $OE(G)$, and discard them. For the remaining elements, we construct a 2-to-1 map onto the collection of sets $U$ in the statement of Lemma 8.16. Further, for each set $U$ the two elements mapped to $U$ are either both in $EE(G)$ or both in $OE(G)$. Since the number of such sets $U$ is odd, by Lemma 8.16, we conclude that $|EE(G)| - |OE(G)| \neq 0$.

Fix $D' \in EE(G) \cup OE(G)$. If $D'$ has any cyclically oriented triangle, then we can reverse its edges to get an orientation with opposite sign (and the same degrees). Formally, we label the

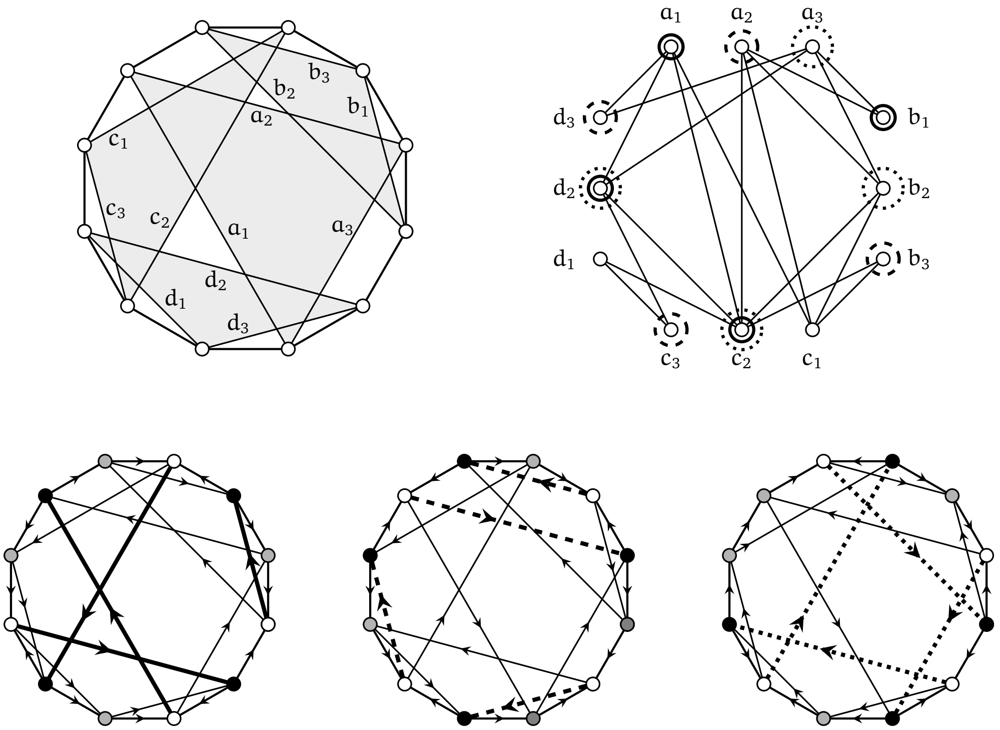



Figure 8.7: Top left: A cycle-plus-triangles graph G embedded in the plane. Top right: The graph J, where $V(J)$ is the set of triangle edges in G, and $e_1e_2 \in E(J)$ if edges $e_1$ and $e_2$ of G intersect. Three distinct transversals are shown, and each transversal induces an Eulerian subgraph. Bottom: An Eulerian orientation of G corresponding to each transversal of J shown above.

vertices (arbitrarily) and among all cyclically oriented triangles we choose the one with a vertex with the smallest label; this gives a parity-reversing involution (with no fixed points). So we assume that $D'$ has no triangle oriented cyclically; that is, every triangle is oriented transitively. See the bottom of Figure 8.7.

We color white (resp. black) each vertex with outdegree 2 (resp. 0) in its triangle; further, we mark as "bold" the edge in each triangle joining its black and white vertices. Given the black and white vertices of each triangle, we can extend to an orientation that has everywhere outdegree 2 if and only if the black and white vertices alternate around the long cycle. This is true if and only if each bold edge intersects an even number of bold edges.

Conversely, given a set of bold edges with each edge intersecting an even number of others, we can color vertices black and white and extend to the desired Eulerian orientation in precisely two ways; to get one from the other, we reverse all edges (and swap the colors black and white). Note that these two orientations have the same sign, since $(-1)^{6n} = 1$. Thus, to show that $|EE(G)| \neq |OE(G)|$, it suffices to show that the number of options for the bold edges is odd. This is precisely the statement of Lemma 8.16. $\square$

## 8.3 Planar Graphs are 5-AT

Now we will show that every planar graph G satisfies $\mathrm{AT}(G) \leqslant 5$. The proof is analogous to that of Thomassen's result that $\chi_\ell(G) \leqslant 5$; see Theorem 11.1. We use a strengthened induction hypothesis, which allows smaller lists (now smaller outdegree) for the vertices on the outer face. As before, we consider two cases: G has a chord, and G has no chord. The first is easy, but the second requires some clever counting.

nice for $(G, e)$

**Definition 8.18.** Let G be a plane graph and $e = v_1v_2$ be an edge on the boundary of the outer face of G. An orientation D of $G - e$ is *nice for* $(G, e)$ if the two conditions below hold:

1. $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v) \leqslant 2$ for every boundary vertex $v$, and $D_D^+(v) \leqslant 4$ for every interior vertex $v$; and
2. $|EE(D)| \neq |OE(D)|$.

It may help to think of $v_1$ and $v_2$ as the precolored vertices in the proof of Theorem 11.1.

**Theorem 8.19.** *If* G *is a plane graph and* $e$ *is a boundary edge of* G*, then* $(G, e)$ *has a nice orientation. Thus, AT*$(G) \leqslant 5$ *for every planar graph* G.

*Proof.* The second statement follows directly from the first, since $\mathrm{AT}(G) \leqslant 5$ if and only if G has an orientation[1] D with $\Delta^+(D) \leqslant 4$ and $|EE(D)| \neq |OE(D)|$. Now we prove the first.

Suppose the theorem is false and let $(G, e)$ be a counterexample with the fewest vertices. Let $v_1$ and $v_2$ be the endpoints of $e$, and let $v_1, \ldots, v_n$ be the boundary vertices of G in (clockwise) order. We can assume that $n \geqslant 3$ and G is a near triangulation.

**Case 1: G has a chord, $e'$.** Let $x$ and $y$ be the endpoints of $e'$. Let $G_1$ and $G_2$ be subgraphs of G such that $e \in E(G_1)$, $G_1 \cup G_2 = G$, and $V(G_1) \cap V(G_2) = \{x, y\}$. By the minimality of G, there exist orientations $D_1$ and $D_2$ such that $D_1$ is nice for $(G_1, e)$ and $D_2$ is nice for $(G_2, e')$. Let $D := D_1 \cup D_2$. Now $d_{D_2}^+(x) = d_{D_2}^+(y) = 0$, so no edge incident to $x$ or $y$ in $D_2$ is in a directed cycle. Thus $|EE(D)| = |EE(D_1)|\,|EE(D_2)| + |OE(D_1)|\,|OE(D_2)|$ and $|OE(D)| = |EE(D_1)|\,|OE(D_2)| + |OE(D_1)|\,|EE(D_2)|$. As a result, $|EE(D)| - |OE(D)| = |EE(D_1)|\,|EE(D_2)| + |OE(D_1)|\,|OE(D_2)| - |EE(D_1)|\,|OE(D_2)| - |OE(D_1)|\,|EE(D_2)| = (|EE(D_1)| - |OE(D_1)|)(|EE(D_2)| - |OE(D_2)|) \neq 0$. So D is a nice orientation for $(G, e)$.

$G'$

helper for $(G, e)$

**Case 2: G has no chord.** If $n = 3$, then let $E(D) := \{v_3v_1, v_3v_2\}$. Now D is a nice orientation of $(G, v_1v_2)$. So instead assume $n \geqslant 4$. Let $G' := G - v_n$, and let $v_1, w_1, \ldots, w_k, v_{n-1}$ denote the neighbors of $v_n$ in G. An orientation D of $G' - e$ is a *helper for* $(G, e)$ if it satisfies the two conditions below:

1. $d_D^+(v_1) = d_D^+(v_2) = 0$, $d_D^+(v_{n-1}) \leqslant 1$ and $d_D^+(w_i) \leqslant 3$ for each $i \in [k]$; and
2. $d_D^+(v_i) \leqslant 2$ for each $i \in [n-2]$ and $d_D^+(v) \leqslant 4$ for every interior vertex $v$ of G.

[1] Given an orientation $D'$ that is nice for $(G, e)$, we form D by orienting $e$ arbitrarily. Note that $|EE(D)| = |EE(D')|$ and $|OE(D)| = |OE(D')|$, since $d_{D'}^+(v_1) = d_{D'}^+(v_2) = 0$.

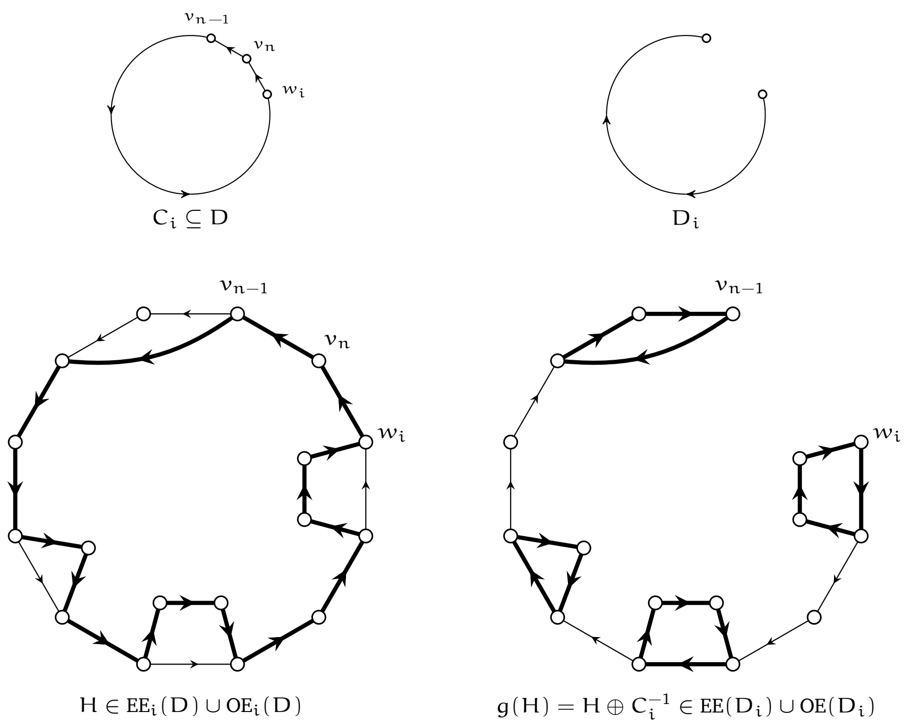



Figure 8.8: Top: The construction of $D_i$ from $C_i$. Bottom: The bijection from $EE_i(D) \cup OE_i(D)$ to $EE(D_i) \cup OE(D_i)$ near the end of Case 2 in the proof of Theorem 8.19.

If there exists a helper orientation $D'$ for $(G, e)$ with $|EE(D')| \neq |OE(D')|$, then we can extend $D'$ to a nice orientation $D$ for $(G, e)$ by adding the arcs $v_n v_1, v_{n-1} v_n, w_1 v_n, \ldots, w_k v_n$. Since $d^+_D(v_1) = 0$ and $N^+_D(v_n) = \{v_1\}$, none of these new edges lies on any directed cycle. So $D$ is a nice orientation for $(G, e)$, since $|EE(D)| = |EE(D')| \neq |OE(D')| = |OE(D)|$.

Now instead assume that every helper orientation $D'$ for $(G, e)$ has $|EE(D')| = |OE(D')|$. By the minimality of $G$, we know that $(G', e)$ has a nice orientation $D''$. To get an orientation $D$ of $G - e$, we add to $D''$ the arcs $v_n v_{n-1}, v_n v_1, w_1 v_n, \ldots, w_k v_n$. Let $EE_i(D) := \{H \in EE(D) : w_i v_n \in H\}$ and $OE_i(D) := \{H \in OE(D) : w_i v_n \in H\}$ for each $i \in [k]$. Note that $EE(D) = EE(D'') \cup \bigcup_{i=1}^k EE_i(D)$ and $OE(D) = OE(D'') \cup \bigcup_{i=1}^k OE_i(D)$. Furthermore, these are disjoint unions, since $v_n v_{n-1}$ is the only out-edge incident to $v_n$ that can possibly lie on a directed cycle. Recall that $|EE(D'')| \neq |OE(D'')|$, since $D''$ is nice. So, to show that $|EE(D)| \neq |OE(D)|$, it suffices to prove that $|EE_i(D)| = |OE_i(D)|$ for all $i \in [k]$. $EE_i$, $OE_i$

Our idea is to construct a helper orientation $D_i$, for each $i$, and bijectively map $EE_i(D) \cup OE_i(D)$ into $EE(D_i) \cup OE(D_i)$, either preserving all parities or else reversing all parities. By assumption, $|EE(D_i)| = |OE(D_i)|$. This implies that $|EE_i(D)| = |OE_i(D)|$, which completes the proof. All that remains is to construct each $D_i$ and specify the bijections.

Fix $i \in [k]$. If $EE_i(D) \cup OE_i(D) = \emptyset$, then trivially $|EE_i(D)| = |OE_i(D)|$, so we are done.

Assume instead that this union is non-empty. Since every edge in an Eulerian subgraph lies in a directed cycle, let $C_i$ be some directed cycle in $D$ containing $w_i v_n$ and $v_n v_{n-1}$. See Figure 8.8. Let $P_i := C_i \cap D''$, and note that $P_i$ is a directed path from $v_{n-1}$ to $w_i$. Form $D_i$ from $D''$ by reversing all edges of $P_i$. All degrees in $D_i$ are the same as in $D''$, except that $d^+_{D_i}(w_i) = d^+_{D''}(w_i) + 1$ and $d^+_{D_i}(v_{n-1}) = d^+_{D''}(v_{n-1}) - 1$. Since $D''$ is nice for $(G, e)$, we have $d^+_{D_i}(w_i) \leqslant 3$ and $d^+_{D_i}(v_{n-1}) \leqslant 1$. Thus, we conclude that $D_i$ is a helper for $(G, e)$. So, by assumption, $|EE(D_i)| = |OE(D_i)|$.

Finally, we construct the bijection from $EE_i(D) \cup OE_i(D)$ to $EE(D_i) \cup OE(D_i)$. Let $C_i^{-1}$ denote the reverse of $C_i$. For each $H \in EE_i(D) \cup OE_i(D)$, let $g(H) = H \oplus C_i^{-1}$, that is, the symmetric difference of $H$ and $C^{-1}$, formed by taking their union and removing all directed 2-cycles. It is straightforward to check that $g(H) \in EE(D_i) \cup OE(D_i)$. Further, this is a bijection: for any $H \in EE(D_i) \cup OE(D_i)$, let $g^{-1}(H) = H \oplus C_i$. Lastly, if $C_i$ has even length, then $H$ and $g(H)$ have the same parity. If $C_i$ has odd length, then $H$ and $g(H)$ have opposite parities. □

## 8.4 The Coefficient Formula

$f_{d_1,\dots,d_n}$ For a field $\mathbb{F}$ and a polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$, recall that $f_{d_1,\dots,d_n}$ denotes the coefficient in $f$ of $x_1^{d_1} \cdots x_n^{d_n}$. Our next theorem gives an alternate way to evaluate $f_{d_1,\dots,d_n}$.

**Theorem 8.20 (Coefficient Formula).** *Suppose* $f \in \mathbb{F}[x_1, \dots, x_n]$ *and* $d_1, \dots, d_n$ *are nonnegative integers with* $\sum d_i = \deg(f)$. *For any* $L_1, \dots, L_n \subseteq \mathbb{F}$ *with* $|L_i| = d_i + 1$ *we have*

$$f_{d_1,\dots,d_n} = \sum_{\substack{(\alpha_1,\dots,\alpha_n) \\ \in L_1 \times \cdots \times L_n}} \frac{f(\alpha_1, \dots, \alpha_n)}{N(\alpha_1, \dots, \alpha_n)}, \tag{8.2}$$

*where*

$$N(\alpha_1, \dots, \alpha_n) := \prod_{i \in [n]} \prod_{\beta \in L_i \setminus \{\alpha_i\}} (\alpha_i - \beta).$$

We call (8.2) the Coefficient Formula[2]. In general, computing the coefficient $f_{d_1,\dots,d_n}$ is difficult, even with the Coefficient Formula. However, one advantage it provides is that we get to pick whatever values of $L_1, \dots, L_n$ we like, to simplify the computation. For certain highly symmetric graphs (and specific choices of $L_1, \dots, L_n$) we will show that most of the terms in the sum evaluate to 0. In some cases, such as edge-coloring regular class 1 graphs, we show that each nonzero term in the sum has the same absolute value.

Our plan is to construct[3] a polynomial $F$ that agrees with $f$ at each point $(\alpha_1, \dots, \alpha_n) \in L_1 \times \cdots \times L_n$ and has the same degree as $f$. For such $F$, the (contrapositive of the) Combinatorial

[2] Various authors [213, 237, 281, 353] proved more general forms (the Notes give more details) that are stronger for non-homogeneous polynomials, i.e., those in which different terms may have distinct degrees. However, since the graph polynomial is always homogeneous, these more general statements typically offer no additional power for graph coloring. So we prefer the version above, which admits a simpler proof.

[3] In fact, our construction is simply the Lagrange interpolation polynomial for $f$ on the set $L_1 \times \cdots \times L_n$.

Nullstellensatz implies that $(f-F)_{d_1,\dots,d_n} = 0$; that is, $f_{d_1,\dots,d_n} = F_{d_1,\dots,d_n}$. To finish, we show that $F_{d_1,\dots,d_n}$ equals the right side of (8.2).

*Proof.* Let $\mathbf{1}_{(\alpha_1,\dots,\alpha_n)} : L_1 \times \cdots \times L_n \to \{0,1\}$ be the indicator function for $(\alpha_1,\dots,\alpha_n)$. That is, $\mathbf{1}_{(\alpha_1,\dots,\alpha_n)}(\alpha_1,\dots,\alpha_n) = 1$ and $\mathbf{1}_{(\alpha_1,\dots,\alpha_n)}(x_1,\dots,x_n) = 0$ for all $(x_1,\dots,x_n) \in L_1 \times \cdots \times L_n \setminus \{(\alpha_1,\dots,\alpha_n)\}$. Let

$$F(x_1,\dots,x_n) := \sum_{\substack{(\alpha_1,\dots,\alpha_n)\\ \in L_1\times\cdots\times L_n}} f(\alpha_1,\dots,\alpha_n)\mathbf{1}_{(\alpha_1,\dots,\alpha_n)}.$$

It is easy to see that $F$ agrees with $f$ on all points in $L_1 \times \cdots \times L_n$: for each such point, all terms in the sum are 0, except for one, which has the desired value.

To apply the Combinatorial Nullstellensatz, we must write $F$ as a polynomial. We do this as follows. For each $n$-tuple $\alpha_1,\dots,\alpha_n \in L_1 \times \cdots \times L_n$, we let

$$N(\alpha_1,\dots,\alpha_n) := \prod_{i\in[n]} \prod_{\beta\in L_i\setminus\{\alpha_i\}} (\alpha_i - \beta).$$

It is easy to check that

$$\mathbf{1}_{(\alpha_1,\dots,\alpha_n)}(x_1,\dots,x_n) = \frac{\prod_{i\in[n]}\prod_{\beta\in L_i\setminus\{\alpha_i\}}(x_i-\beta)}{N(\alpha_1,\dots,\alpha_n)}.$$

To see this, note that for each point $(x_1,\dots,x_n) \in L_1\times\cdots\times L_n \setminus \{(\alpha_1,\dots,\alpha_n)\}$, the numerator is 0 (since one of its factors is $(\beta-\beta)$). When the numerator is nonzero, its value is precisely $N(\alpha_1,\dots,\alpha_n)$. Now we show that $F_{d_1,\dots,d_n}$ has the desired value.

$$\begin{aligned}
F_{d_1,\dots,d_n} &= \left(\sum_{\substack{(\alpha_1,\dots,\alpha_n)\\ \in L_1\times\cdots\times L_n}} f(\alpha_1,\dots,\alpha_n)\mathbf{1}_{(\alpha_1,\dots,\alpha_n)}\right)_{d_1,\dots,d_n}\\
&= \sum_{\substack{(\alpha_1,\dots,\alpha_n)\\ \in L_1\times\cdots\times L_n}} \frac{f(\alpha_1,\dots,\alpha_n)}{N(\alpha_1,\dots,\alpha_n)}\left(\prod_{i\in[n]}\prod_{\beta\in L_i\setminus\{\alpha_i\}}(x_i-\beta)\right)_{d_1,\dots,d_n}\\
&= \sum_{\substack{(\alpha_1,\dots,\alpha_n)\\ \in L_1\times\cdots\times L_n}} \frac{f(\alpha_1,\dots,\alpha_n)}{N(\alpha_1,\dots,\alpha_n)}.
\end{aligned}$$

□

For certain graphs, clever choices of $L_1 \times \cdots \times L_n$ make evaluating the Coefficient Formula considerably simpler. The proof of our next corollary shows an example with $L_i := \{0,\dots,d_i\}$ for all $i$. But it is easier to digest if we first extract the following observation.

**Lemma 8.21.** *If* $L_i := \{0, \ldots, d_i\}$ *and* $\alpha \in L_i$, *then* $\prod_{\beta \in L_i \setminus \{\alpha\}} (\alpha - \beta) = (-1)^{d_i+\alpha} \binom{d_i}{\alpha}^{-1} d_i!$.

*Proof.* $$\prod_{\beta \in L_i \setminus \{\alpha\}} (\alpha - \beta) = \prod_{0 \leqslant \beta < \alpha} (\alpha - \beta) \prod_{\alpha < \beta \leqslant d_i} (\alpha - \beta) = \alpha!(d_i - \alpha)!(-1)^{d_i - \alpha} = (-1)^{d_i+\alpha} \binom{d_i}{\alpha}^{-1} d_i!.$$ □

**Corollary 8.22.** *Suppose* $f \in \mathbb{F}[x_1, \ldots, x_n]$ *and* $d_1, \ldots, d_n$ *are nonnegative integers such that* $\sum d_i = \deg(f)$. *The coefficient in* $f$ *of* $\prod_{i=1}^n x_i^{d_i}$ *is given by*

$$f_{d_1,\ldots,d_n} = \left( \prod_{i=1}^n d_i! \right)^{-1} \sum_{\alpha_1=0}^{d_1} \cdots \sum_{\alpha_n=0}^{d_n} (-1)^{d_1+\alpha_1} \cdots (-1)^{d_n+\alpha_n} \binom{d_1}{\alpha_1} \cdots \binom{d_n}{\alpha_n} f(\alpha_1, \ldots, \alpha_n).$$

*In particular, if* $d_i := d$ *for all* $i$, *then*

$$f_{d,\ldots,d} = (d!)^{-n} \sum_{\varphi} \left( \prod_{i=1}^n (-1)^{d+\varphi(x_i)} \binom{d}{\varphi(x_i)} \right) f(\varphi),$$

*where the sum ranges over all* $\varphi : \{x_1, \ldots, x_n\} \to \{0, \ldots, d\}$ *and* $f(\varphi)$ *denotes* $f$ *evaluated with* $x_i := \varphi(x_i)$ *for all* $i \in [n]$.

*Proof.* The proof consists entirely of using Lemma 8.21 to substitute into the Coefficient Formula, and simplifying. □

$K_{2\star n}$ Let $K_{2\star n} := K_{2n} - nK_2$. That is, $K_{2\star n}$ denotes the complete $n$-partite graph, with each part of size 2. Clearly, $\chi(K_{2\star n}) = n$. Using Hall's Theorem and induction, it is easy to show that $\chi_\ell(K_{2\star n}) = n$ (either two vertices in the same part have a common color, or every vertex can get its own color). Our next theorem strengthens this result further, to the Alon–Tarsi number.

**Theorem 8.23.** *AT*$(K_{2\star n}) = n$.

*Proof.* Let $G := K_{2\star n}$. Since $\deg(f_G) = 2n(n-1)$, proving that $\mathrm{AT}(G) = n$ is equivalent to showing that $f_{n-1,\ldots,n-1} \neq 0$. By Corollary 8.22, it suffices to show that

$$\sum_{\varphi} \prod_{v \in V(G)} (-1)^{n-1+\varphi(v)} \binom{n-1}{\varphi(v)} f_G(\varphi) \neq 0.$$

Recall that $f_G(\varphi) = 0$ if $\varphi$ is an improper coloring of $G$, so we restrict the sum to proper colorings. Note that $G$ has a unique proper $n$-coloring, up to permuting colors: the two vertices in each part get the same color. So $\prod_{v \in V(G)} (-1)^{n-1+\varphi(v)} \binom{n-1}{\varphi(v)} = \prod_{i=0}^{n-1} \binom{n-1}{i}^2$. In particular, the left side is constant over all colorings. For each proper coloring $\varphi$, we have

$$f_G(\varphi) = \prod_{0 \leqslant i < j \leqslant n-1} (i-j)^4.$$

Certain factors in this product may be negated, depending on the arbitrary orientation $D$ we use to define $f_G$. So for $D$ we orient the cliques in the top and bottom identically, and we orient each cross-edge opposite of the corresponding clique edges. Now each factor that appears four times is negated exactly twice. So $f_G(\varphi)$ is positive. $\square$

To help illustrate the preceeding techniques, we use them for a few concrete computations.

**Example 8.24.** We explicitly compute $f_{n-1,\dots,n-1}$ for each $n \in \{2,3\}$, first by the Coefficient Formula, and second by the Alon–Tarsi Theorem and parity-reversing involutions.

Corollary 8.22, with ideas in the proof of Theorem 8.23, gives the expression $f_{n-1,\dots,n-1} = ((n-1)!)^{-2n} \sum_\varphi \prod_{i=0}^{n-1} \binom{n-1}{i} \prod_{0\leqslant i<j\leqslant n-1}(i-j)^4$, where the sum ranges over all proper $n$-colorings $\varphi$ of $K_{2\star n}$. Thus $f_{1,1,1,1} = 1^{-4}\sum_\varphi \binom{1}{0}^2\binom{1}{1}^2(0-1)^4 = \sum_\varphi 1 = 2$. Similarly, letting $n-1 := 2$ gives $f_{2,2,2,2,2,2} = 2^{-6}\sum_\varphi \binom{2}{0}^2\binom{2}{1}^2\binom{2}{2}^2(0-1)^4(0-2)^4(1-2)^4 = \sum_\varphi 1 = 6$.

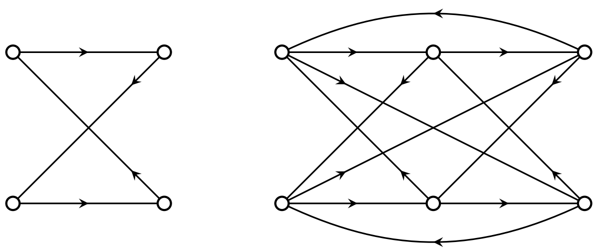


Figure 8.9: Eulerian orientations of $K_{2\star 2}$ and $K_{2\star 3}$ with everywhere outdegree equal to 1 (left) and 2 (right).

For an alternate perspective, we now compute the same values using the Alon–Tarsi Theorem. Note that $K_{2\star 2} \cong C_4$. It is easy to check that $C_4$ has exactly two orientations with all outdegrees equal to 1, and they differ on all four edges. Thus, $f_{1,1,1,1} = 2$.

For $n = 3$, we view $K_{2\star 3}$ as consisting of a "top part" and a "bottom part" (with each part inducing a clique), and we denote their vertex sets by $V_1$ and $V_2$; see the right of Figure 8.9. To build an Eulerian orientation $D$ of $K_{2\star 3}$, we must direct exactly half of the six edges between parts from $V_1$ to $V_2$. Call this set of 3 edges $S$.

If each edge of $S$ has a distinct endpoint in $V_1$, then $D$ restricted to $V_1$ is a directed 3-cycle. Reversing the edges of this 3-cycle gives another Eulerian orientation with opposite parity, so their net contribution to $f_{2,2,2,2,2,2}$ is 0. The same is true if $S$ has distinct endpoints in $V_2$, or in both $V_1$ and $V_2$. Thus, we can restrict our sum to choices of $S$ which induce $P_4$.

It is easy to check that there exist 6 such sets $S$. Further, if any of these 6 sets is oriented from $V_1$ to $V_2$, then we can extend to an Eulerian orientation in exactly one way. And that orientation is formed from the one shown in Figure 8.9 by reversing an even number of edges. Thus, we again conclude that $f_{2,2,2,2,2,2} = 6$. $\diamondsuit$

## 8.5 Exponentially Many List-Colorings

In this section we will prove that for every planar graph $G$ and every 5-assignment $L$ to $G$, the number of $L$-colorings of $G$ is at least $5^{|G|/4}$. This will follow easily from Theorem 8.26, together with the fact that every planar graph is 5-choosable, which we proved in Section 11.1 and is also implied by Theorem 8.19. (The same approach shows that if $G$ is planar with girth at least 5 and $L$ is a 3-assignment, then the number of $L$-colorings that $G$ admits is at least $3^{|G|/6}$; see Exercise 5.) To prove Theorem 8.26, we need the following easy lemma.

**Lemma 8.25.** *Let $a_1, \dots, a_n$ be positive integers, let $t := \max a_i$, and let $S := \sum_{i=1}^{n} a_i$. If $t \geqslant 2$, then*
$$\prod_{i=1}^{n} a_i \geqslant t^{\frac{S-n}{t-1}}.$$

*Proof.* Our proof is by induction on $n$. Let $P := \prod_{i=1}^{n} a_i$. If $n = 1$, then $P = a_1 = t = t^{\frac{S-1}{t-1}}$.

Now assume $n \geqslant 2$, and let $g(x) := t^{\frac{x-1}{t-1}}$. It is easy to check that $g(x)$ is convex when $x \in [1, t]$; its second derivative is positive. Further, $g(1) = 1$ and $g(t) = t$. Thus, the line $y = x$ lies above $g(x)$ on the interval $[1, t]$. That is, for all $x \in [1, t]$ we have
$$x \geqslant t^{\frac{x-1}{t-1}}. \tag{8.3}$$

Now let $r := \min a_i$. By induction (deleting one instance of $r$ from the set), we have $P/r \geqslant t^{\frac{(S-r)-(n-1)}{t-1}}$. Hence, combining this with (8.3) gives
$$P = \frac{P}{r} r \geqslant t^{\frac{(S-r)-(n-1)}{t-1}} t^{\frac{r-1}{t-1}} = t^{\frac{S-n}{t-1}}. \qquad \square$$

Now we use Lemma 8.25 to prove Theorem 8.26. We say that a polynomial $f(x_1, \dots, x_n)$ vanishes *vanishes* on a grid $L_1 \times \cdots \times L_n$ if $f$ takes the value 0 at every point of the grid.

**Theorem 8.26.** *Let $\mathbb{F}$ be an arbitrary field and let $L_1, \dots, L_n$ be non-empty subsets of $\mathbb{F}$. Let* $L, t, S$ *$L := L_1 \times \cdots \times L_n$. Let $t := \max |L_i|$ and let $S := \sum_{i=1}^{n} |L_i|$. Let $f(x_1, \dots, x_n)$ be a polynomial* $d$ *over $\mathbb{F}$ that does not vanish on all of $L$, and let $d := \deg f$. Now the number of points of $L$ where $f$ is nonzero is at least*
$$t^{\frac{S-n-d}{t-1}},$$
*provided that $S \geqslant n + d$ and $t \geqslant 2$.*

*Proof.* Our proof is by induction on $n$ and $d$.

If $d = 0$, then $f$ is a nonzero constant, so $f$ is nonzero for all points in $L$. The number of these is $\prod_{i=1}^{n} |L_i|$. So we are done by Lemma 8.25, taking $a_i := |L_i|$.

Next suppose $n = 1$ and $d \geqslant 1$. The number of roots of a degree $d$ (single-variable) polynomial is at most $d$. So the number of points in $L$ where $f$ is nonzero is at least $|L_1| - d =$

$t-d$. Since $n=1$, the hypothesis $S \geqslant n+d$ implies $t-d \geqslant 1$. Since $S=t$ when $n=1$, inequality 8.3, with $x := t-d$ gives

$$t-d \geqslant t^{\frac{t-d-1}{t-1}} = t^{\frac{S-1-d}{t-1}}.$$

Finally, assume that $n \geqslant 2$ and $d \geqslant 1$. Assume, by possibly permuting indices, that there exists $i \neq 1$ such that $|L_i| = t$. Fix an arbitrary $\alpha \in L_1$. We divide polynomial $f$ by $(x_1 - \alpha)$, so that polynomials $q_\alpha$ and $r_\alpha$ satisfy the following:

$$f(x_1, \dots, x_n) = (x_1 - \alpha)q_\alpha(x_1, \dots, x_n) + r_\alpha(x_2, \dots, x_n).$$

Now $\deg q_\alpha = (\deg f) - 1 = d-1$, and $\deg r_\alpha \leqslant d$, and $r_\alpha$ does not contain variable $x_1$. We will consider two cases: (a) $r_\alpha(x_1, \dots, x_n)$ vanishes on the grid $L_2 \times \cdots \times L_n$ (for some choice of $\alpha \in L_1$) and (b) $r_\alpha(x_2, \dots, x_n)$ does not vanish on $L_2 \times \cdots \times L_n$, for every choice of $\alpha \in L_1$. In Case (a) we will use induction once, and in Case (b) we will do so $|L_1|$ times.

(a) Suppose (for some $\alpha \in L_1$) that $r_\alpha(x_2, \dots, x_n)$ vanishes on the grid $L_2 \times \cdots \times L_n$. This implies that $|L_1| \geqslant 2$, since $f$ does not vanish on all of $L_1 \times \cdots \times L_n$. Furthermore, $q_\alpha$ does not vanish on all of $L_1 \setminus \{\alpha\} \times L_2 \times \cdots \times L_n$. Thus, by the induction hypothesis, the number of points of $L_1 \setminus \{\alpha\} \times L_2 \times \cdots \times L_n$ where $q_\alpha$ is nonzero is at least

$$t^{\frac{(S-1)-n-(d-1)}{t-1}} = t^{\frac{S-n-d}{t-1}}.$$

Now we are done, since also $f$ is nonzero at each such point.

(b) Suppose (for every $\alpha \in L_1$) that $r_\alpha(x_2, \dots, x_n)$ does not vanish on $L_2 \times \cdots \times L_n$. Let $j := |L_1|$. By induction, the number of points of $L_2 \times \cdots \times L_n$ where $r_\alpha$ is nonzero is at least $t^{\frac{(S-j)-(n-1)-d}{t-1}}$. Each point of $L_2 \times \cdots \times L_n$ where $r_\alpha$ is nonzero can be extended to a point of $L_1 \times \cdots \times L_n$ where $f$ is nonzero, by letting $x_1 := \alpha$. We can repeat this argument for each $\alpha \in L_1$. Thus, by (8.3), the number of points of $L_1 \times \cdots \times L_n$ where $f$ is nonzero is at least

$$j t^{\frac{(S-j)-(n-1)-d}{t-1}} \geqslant t^{\frac{j-1}{t-1}} t^{\frac{(S-j)-(n-1)-d}{t-1}} = t^{\frac{S-n-d}{t-1}}. \qquad \square$$

Theorem 8.26 gives the following as an immediate corollary.

**Theorem 8.27.** *If $G$ is planar and $L$ is a 5-assignment for $G$, then the number of $L$-colorings of $G$ is at least $5^{|G|/4}$.*

*Proof.* Fix a planar graph $G$ and a 5-assignment $L$ for $G$. To be clear, we say that $L := L_1 \times \cdots \times L_n$, where $n := |G|$. Let $f_G(x_1, \dots, x_n)$ be the graph polynomial for $G$. By Theorem 8.19 (or Thomassen's proof that all planar graphs are 5-choosable), $f_G$ does not vanish on all of $L$. Thus, we apply Theorem 8.26, with $a_i := |L_i|$ for all $i$ (so $t := 5$), with $S := 5n$, and with $d := \|G\| \leqslant 3n-6$. So the number of $L$-colorings of $G$ is at least

$$5^{\frac{5n-n-(3n-6)}{5-1}} = 5^{\frac{n+6}{4}} > 5^{\frac{n}{4}}. \qquad \square$$

## 8.6 Edge-Coloring Regular Graphs

The main result of this section is the following theorem.

**Theorem 8.28.** *If* $G$ *is a* $k$*-regular planar multigraph and* $\chi'(G) = k$*, then the line graph* $L(G)$ *has AT*$(L(G)) = k$.

$\mathcal{C}$ **Definition 8.29.** For a graph $G$, let $\mathcal{C}$ denote the set of proper edge-colorings of $G$ with colors
$AT'(G)$ $0, \ldots, k-1$. Let $AT'(G)$ denote $AT(L(G))$, the Alon–Tarsi number of the line graph of $G$. For
$f_D(x)$ an orientation $D$ of $L(G)$, let $f_D(\boldsymbol{x}) := \prod_{ee' \in D}(x_e - x_{e'})$. Let $\mathrm{sign}_D(\varphi) := f_D(\varphi)/|f_D(\varphi)|$;
$\mathrm{sign}_D(\varphi)$ so $\mathrm{sign}_D(\varphi) \in \{1, -1\}$ for all orientations $D$ of $L(G)$ and all proper edge-colorings $\varphi \in \mathcal{C}$.

We first prove Theorem 8.28, assuming the truth of Lemmas 8.30 and 8.31 (stated below). We will prove the lemmas soon. The first of these lemmas is particularly interesting, since it applies to all $k$-regular graphs with $\chi' = k$.

*Proof of Theorem 8.28.* By Lemma 8.30, to prove that $AT'(G) = k$, it suffices to show that $\sum_{\varphi \in \mathcal{C}} \mathrm{sign}_D(\varphi) \neq 0$. By Lemma 8.31, each term in this sum is equal. Since $\chi'(G) = k$, we know that $\mathcal{C} \neq \emptyset$; that is, the sum is non-empty. Thus, $\sum_{\varphi \in \mathcal{C}} \mathrm{sign}(\varphi) \neq 0$, as desired. □

**Lemma 8.30.** *Let* $G$ *be a* $k$*-regular graph with* $\chi'(G) = k$ *and let* $D$ *be an orientation of* $L(G)$*. Now* $AT'(G) = k$ *if and only if* $\sum_{\varphi \in \mathcal{C}} \mathit{sign}_D(\varphi) \neq 0$.

*Proof.* Since $G$ is regular, $L(G)$ is $2(k-1)$-regular (if 2 edges in $G$ are parallel, then they are joined by two parallel edges in $L(G)$). Let $n := |V(G)|$. So each monomial in the expansion of $f_D(\boldsymbol{x})$ has degree $|E(G)|(k-1) = nk(k-1)/2$. Thus, $AT'(G) = k$ if and only if the monomial $\prod_{i=1}^{|E(G)|} x_i^{k-1}$ has nonzero coefficient. Using Corollary 8.22 (with $L_i := \{0, \ldots, k-1\}$ for every $i$), this coefficient is

$$((k-1)!)^{-nk/2} \sum_{\varphi : E(G) \to \{0,\ldots,k-1\}} \left( \prod_{e \in E(G)} \binom{k-1}{\varphi(e)} (-1)^{k-1+\varphi(e)} \right) f_D(\varphi). \tag{8.4}$$

We can immediately restrict this sum to proper colorings $\varphi$, since if $\varphi$ is improper, then $f_D(\varphi) = 0$. We only care about whether this coefficient is 0 or not (rather than its actual value), so we ignore the factor $((k-1)!)^{-nk/2}$. Further, if $\varphi \in \mathcal{C}$, then exactly $n/2$ edges $e$ have $\varphi(e) = i$, for each $i$. Thus, $\prod_{e \in E(G)} \binom{k-1}{\varphi(e)} (-1)^{k-1+\varphi(e)}$ is constant over all $\varphi \in \mathcal{C}$. This holds precisely because every vertex of $G$ sees every color. More formally, let $E(v)$ denote
$Q_v$ the set of edges in $G$ that are incident to $v$, and let $Q_v$ denote the $k$-clique in $L(G)$ induced by $E(v)$. Let

$$f_{D,v}(\boldsymbol{x}) := \prod_{\substack{e,e' \in Q_v \\ ee' \in E(D)}} (x_e - x_{e'}).$$

Now $f_D(x) := \prod_{v\in V(G)} f_{D,v}(x)$. Thus, $|f_{D,v}(\varphi)| = \prod_{0\leqslant i<j\leqslant k-1}(j-i)$ for every $v$, since G is k-regular and $\varphi$ is proper. Hence, $|f_D(\varphi)| = \left(\prod_{0\leqslant i<j\leqslant k-1}(j-i)\right)^n$, for every $\varphi \in \mathcal{C}$. Pulling out this common factor from (8.4) proves the lemma. □

Now we prove that $\sum_{\varphi\in\mathcal{C}} \mathrm{sign}_D(\varphi) \neq 0$. In fact, we prove the stronger result that all $\varphi \in \mathcal{C}$ have the same sign.

**Lemma 8.31.** *If* $G$ *is a planar,* $k$*-regular multigraph with* $\chi'(G) = k$ *and* $D$ *is an orientation of* $L(G)$*, then sign*$_D(\varphi) = (-1)^{\binom{k-1}{2}n/2}$ *for all* $\varphi \in \mathcal{C}$.

*Proof.* For a 1-factor $M$ in $G$, let $D_M$ be the orientation of $L(G)$ defined as follows. At each vertex $v \in V(G)$ index the incident edges as $0, \ldots, k-1$, in clockwise order around $v$, beginning with the edge in $M$ as index 0. Form the orientation $D$ of $L(G)$ by directing each edge $ee' \in E(L(G))$ from the lower index to the higher index. So each edge $vw \in E(G)$ has two indices, one for $v$ and one for $w$. (Also, $Q_v$ is oriented transitively for each $v$.) $D_M$

Our proof consists of the following two claims.

**Claim 1.** *If* $\varphi \in \mathcal{C}$ *and* $M$ *is the* 1*-factor of edges colored* $k-1$ *by* $\varphi$*, then sign*$_{D_M}(\varphi) = (-1)^{\binom{k-1}{2}n/2}$.

**Claim 2.** *If* $M$ *and* $M'$ *are two* 1*-factors of* $G$*, then sign*$_{D_M}(\varphi) =$ *sign*$_{D_{M'}}(\varphi)$.

From these claims, the lemma follows immediately. Fix $\varphi \in \mathcal{C}$, let $M$ be the 1-factor consisting of the edges colored $k-1$ in $\varphi$, and let $M'$ be an arbitrary 1-factor. Now $\mathrm{sign}_{D_{M'}}(\varphi) = \mathrm{sign}_{D_M}(\varphi) = (-1)^{\binom{k-1}{2}n/2}$. Thus, it remains to prove the claims.

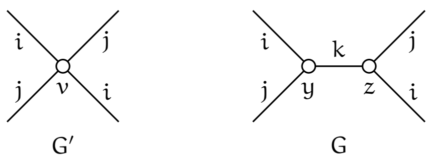



Figure 8.10: Left: Colors $i$ and $j$ cross at $v$ in $G'$.
Right: The corresponding edges in $G$.

Proof. [Proof of Claim 1] Form $G'$ from $G$ by contracting each edge in $M$. Fix colors $i, j \in \{0, \ldots, k-2\}$. The edges in $G'$ colored $i$ form a 2-factor (disjoint union of cycles), as do the edges colored $j$. Colors $i$ and $j$ *cross at* $v$ if the edges colored $i$ and $j$ incident to $v$ appear in the cyclic order $i, j, i, j$ around $v$ (that is, the monochromatic cycles colored $i$ and $j$, and containing $v$, cross at $v$); otherwise $i$ and $j$ do not cross at $v$. Let $yz$ be the edge in $G$ that is contracted to form $v$. If colors $i$ and $j$ cross at $v$, then $f_{D_M,y}(\varphi)$ and $f_{D_M,z}(\varphi)$ both have factors of $(i-j)$ or both have factors of $(j-i)$. If $i$ and $j$ do not cross at $v$, then one has a factor of $(i-j)$ and the other a factor of $(j-i)$. Let $r_v$ denote the number of pairs of colors that cross at $v$. cross at $v$ $r_v$

Now $\text{sign}_{D_M,y}(\varphi)\text{sign}_{D_M,z}(\varphi) = (-1)^{\binom{k-1}{2}-r_v}$. Since G is planar, each cycle of edges colored i crosses each cycle of edges colored j an even number of times. Thus, $\sum_{v\in V(G')} r_v \equiv 0 \pmod 2$. So

$$\begin{aligned}\text{sign}_{D_M}(\varphi) &= \prod_{w\in V(G)} \text{sign}_{D_M,w}(\varphi) = \prod_{yz\in M} \text{sign}_{D_M,y}(\varphi)\,\text{sign}_{D_M,z}(\varphi)\\ &= (-1)^{\binom{k-1}{2}n/2-\sum_{v\in V(G')} r_v} = (-1)^{\binom{k-1}{2}n/2}.\end{aligned}$$ ♢

$s_v$ $e_{M,v}, e_{M',v}$ $[e_{M,v}, e_{M',v})$ $[e_{M',v}, e_{M,v})$

Proof. [Proof of Claim 2] For each $v \in V(G)$, let $s_v$ denote the number of edges in $Q_v$ that are oriented differently in $D_M$ and $D_{M'}$. Let $e_{M,v}$ and $e_{M',v}$ be the edges incident to $v$ in M and M′. If $e_{M,v} = e_{M',v}$, then $s_v = 0$. Otherwise, let $[e_{M,v}, e_{M',v})$ denote the set of edges starting with $e_{M,v}$ and proceeding clockwise until $e_{M',v}$ (including $e_{M,v}$ and excluding $e_{M',v}$); define $[e_{M',v}, e_{M,v})$ similarly. So $s_v = |[e_{M,v}, e_{M',v})| \times |[e_{M',v}, e_{M,v})|$. If k is odd, then $s_v = |[e_{M,v}, e_{M',v})|$ and $|[e_{M',v}, e_{M,v})|$ have opposite parities. Thus, $s_v$ is even for all $v$, which finishes the proof.

So instead assume that k is even. Now $s_v \equiv |[e_{M,v}, e_{M',v})| \equiv |[e_{M',v}, e_{M,v})| \pmod 2$. The subgraph $M \cup M'$ consists of even cycles and isolated edges. Each endpoint $v$ of an isolated edge has $s_v = 0$. Let H be an arbitrary even cycle in $M \cup M'$. Now it suffices to show that $\sum_{v\in V(H)} s_v \equiv 0 \pmod 2$. Let $H_{in}$ denote the subgraph of G induced by vertices (strictly) inside H. Note that $\sum_{w\in V(H_{in})} d_G(w) = k|H_{in}| \equiv 0 \pmod 2$, since k is even. By the handshaking lemma, $\sum_{w\in V(H_{in})} d_{H_{in}}(w) \equiv 0 \pmod 2$. Thus, $\sum_{v\in V(H)} s_v = |E(V(H), V(H_{in}))| + \|H\| = \sum_{w\in V(H_{in})}(d_G(w) - d_{H_{in}}(w)) + \|H\| \equiv 0 \pmod 2$. ♢

Together, Claims 1 and 2 complete the proof of Lemma 8.31. □

## 8.7 Every Graph is Total Weight (2,3)-Choosable

total weighting

**Definition 8.32.** A *total weighting* of a graph G is a mapping $\varphi : V(G) \cup E(G) \to \mathbb{R}$ and a total weighting $\varphi$ is *proper* if every edge $vw$ of G satisfies

$$\varphi(v) + \sum_{u\in N_G(v)} \varphi(uv) \neq \varphi(w) + \sum_{u\in N_G(w)} \varphi(uw). \tag{8.5}$$

total weight $(a, b)$-choosable

For a list assignment L, an L-*total weighting* $\varphi$ is a proper total weighting such that $\varphi(z) \in L(z)$ for every $z \in V(G) \cup E(G)$. A graph G is *total weight* $(a, b)$-*choosable* if it has an L-total weighting for every list assignment L such that $|L(v)| = a$ and $|L(e)| = b$ for all $v \in V(G)$ and all $e \in E(G)$.

The main result of this section is the following.

**Theorem 8.33.** *Every graph is total weight* $(2, 3)$-*choosable.*

To prove this, we use the Combinatorial Nullstellensatz and the following definitions.

**Definition 8.34.** For a graph G, we fix an arbitrary orientation D of G. Naturally, we consider the polynomial

$$\prod_{vw \in D} \left( \left( x_v + \sum_{u \in N_G(v)} x_{uv} \right) - \left( x_w + \sum_{u \in N_G(w)} x_{uw} \right) \right), \tag{8.6}$$

which we call $f_G^{tot}(\{x_z : z \in V(G) \cup E(G)\})$, or simply $f_G^{tot}$ for short. For a total weighting $\varphi$, let $f_G^{tot}(\varphi)$ be the evaluation of $f_G^{tot}$ with $x_z = \varphi(z)$ for all $z \in V(G) \cup E(G)$. The key point motivating our definition of $f_G^{tot}$ is that $\varphi$ is a proper total weighting of G if and only if $f_G^{tot}(\varphi) \neq 0$. This is because $f_G^{tot}(\varphi) \neq 0$ if and only if (8.5) holds for every edge $vw$ in G. $f_G^{tot}$

An *index function* d of G assigns to each $z \in V(G) \cup E(G)$ a nonnegative integer $d(z)$. An index function d is *valid* if $\sum_{z \in V(G) \cup E(G)} d(z) = |E(G)|$. For a valid index function d, let $f_d$ denote the coefficient in $f_G^{tot}$ of $\prod_{z \in V \cup E} x_z^{d(z)}$. An index function d is *nonsingular* if there exists a valid index function $d'$ with $d'(z) \leqslant d(z)$ for all $z \in V(G) \cup E(G)$ and $f_{d'} \neq 0$. index function valid nonsingular

When an index function d is nonsingular, the Combinatorial Nullstellensatz implies that G has a proper L-total weighting whenever $|L(z)| > d(z)$ for all $z \in V(G) \cup E(G)$. This is why we study nonsingular index functions.

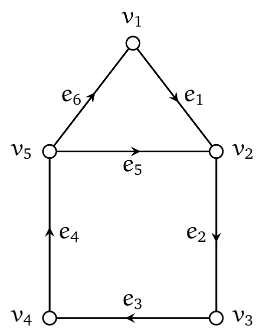
To prove that G is $(2,3)$-total weight choosable, we let $d(v) := 1$ and $d(e) := 2$ for all $v \in V(G)$ and $e \in E(G)$. As usual, the hard work is showing that d is nonsingular. In this case, we translate the problem into one about matrix permanents, which allows us to use tools from linear algebra. We encode the coefficients in each factor of (8.6) as entries in a matrix. Let $A_G[e,z]$ denote a matrix with its rows indexed by edges of G and its columns indexed by edges and vertices of G; for each $e = vw \in E(D)$ and $z \in V(G) \cup E(G)$, let

$$A_G[e,z] = \begin{cases} 1 & \text{if } z = v, \text{ or else } z \neq vw \text{ and } z \text{ is an edge incident to } v \\ -1 & \text{if } z = w, \text{ or else } z \neq vw \text{ and } z \text{ is an edge incident to } w \\ 0 & \text{otherwise} \end{cases}$$

Figure 8.11 shows an example. The point of this definition of $A_G[e,z]$ is that now

$$f_G^{tot} = \prod_{e \in E(D)} \sum_{z \in V(G) \cup E(G)} A_G[e,z] x_z. \tag{8.7}$$

Recall that the *permanent*, denoted *per*$(A)$, of an $n \times n$ matrix A, is defined as $\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i \in [n]} A[i, \sigma(i)]$, where $S_n$ is the set of all permutations of $[n]$. (The permanent is defined similarly to the determinant, but is missing the factor $\mathrm{sign}(\sigma)$, which turns out to make a big difference). Given a vertex or edge $z$, let $A_G[z]$ denote the column of $A_G$ indexed by $z$. For an index function d of G, let $A_G(d)$ be a matrix such that each of its columns is a column per$(A)$ $A_G[z]$ $A_G(d)$




| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | | $-1$ | | | $-1$ | $1$ | $1$ | $-1$ | | | |
| $e_2$ | $1$ | | $-1$ | | $1$ | | | $1$ | $-1$ | | |
| $e_3$ | | $1$ | | $-1$ | | | | | $1$ | $-1$ | |
| $e_4$ | | | $1$ | | $-1$ | $-1$ | | | | $1$ | $-1$ |
| $e_5$ | $-1$ | $-1$ | | $1$ | | $1$ | | $-1$ | | | $1$ |
| $e_6$ | $-1$ | | | $1$ | $1$ | | $-1$ | | | | $1$ |

Figure 8.11: Left: An orientation of a graph $G$. Right: The matrix $A_G[e, z]$ for $G$.

of $A_G$ and the column of $A_G$ indexed by $z$ appears exactly $d(z)$ times. So $A_G(d)$ is a square matrix, with each side of length $|E(G)|$, and it is unique up to permuting its columns. Thus, $\text{per}(A_G(d))$ is well-defined.

To translate this problem into the language of linear algebra, we need the following lemma and corollary.

**Lemma 8.35.** *Let $A$ be an $m \times s$ matrix. For the polynomial $\prod_{i=1}^{m} \sum_{j=1}^{s} a_{ij} x_j$, the coefficient of $x_1^{d_1} \cdots x_s^{d_s}$ equals $\frac{\text{per}(A(d))}{d_1! \cdots d_s!}$.*

*Proof.* In expanding the polynomial, each term corresponds to picking one column $j$ for each row $i$ of $A$. A term contributes to the coefficient of $x_1^{d_1} \cdots x_s^{d_s}$ precisely when each column $j$ is picked exactly $d_j$ times in the expansion. These terms arise naturally in the expansion of $\text{per}(A(d))$, except that a single term contributing to the coefficient of $x_1^{d_1} \cdots x_s^{d_s}$ corresponds to $d_1! \cdots d_s!$ terms in $\text{per}(A(d))$. This is because each of the $d_1!$ orders of the $d_1$ copies of $A[z_1]$ counts as different in $\text{per}(A(d))$, and similarly for copies of each $A[z_i]$. $\square$

Applying this lemma to $A_G(d)$ gives the following.

**Corollary 8.36.** *For a valid index function $d$ of $G$, we have $f_d \neq 0$ if and only if $\text{per}(A_G(d)) \neq 0$. In other words, $d$ is nonsingular if and only if $\text{per}(A_G(d)) \neq 0$.*

We need the following two propositions. The first is a simple property of permanents, and the second follows directly from the definition of $A_G$. We leave the proofs to Exercise 7.

**Proposition 8.37.** *Let $A$ be a square matrix and $\vec{v}$ one of its columns. Suppose that $\vec{v} = c'\vec{v}' + c''\vec{v}''$, where $c'$ and $c''$ are real numbers. If we form $A'$ and $A''$ from $A$ by replacing $\vec{v}$ with $\vec{v}'$ and $\vec{v}''$, respectively, then $\text{per}(A) = c'\text{per}(A') + c''\text{per}(A'')$.*

Since $A_G$ is an $m \times (m + n)$ matrix, its rank is at most $m$. Thus, its columns satisfy many linear dependencies. In particular, the following holds.

**Proposition 8.38.** *Each edge* $e = vw$ *of* $G$ *satisfies* $A_G[e] = A_G[v] + A_G[w]$.

**Lemma 8.39.** *Let* $d : V(G) \cup E(G) \to \{0, 1, \ldots\}$ *be an index function of* $G$. *Now* $d$ *is nonsingular if there exists a square matrix* $A$ *such that per*$(A) \neq 0$ *and the columns of* $A$ *can be expressed as linear combinations of columns of* $A_G$ *such that each column* $z$ *of* $A_G$ *has nonzero coefficient in at most* $d(z)$ *columns of* $A$.

*Proof.* Let $A$ be a square matrix, with per$(A) \neq 0$. Fix a representation of the columns of $A$ as linear combinations of the columns of $A_G$. This representation is not unique, precisely because of Proposition 8.38. Let $p = p(A)$ denote the number of columns of $A$ for which more than one column of $A_G$ appears in its representation (with nonzero coefficient). $p$

We use induction on $p$. The base case, $p = 0$, follows directly from Corollary 8.36. So assume $p \geqslant 1$. By symmetry, assume that column 1 of $A$ is a nontrivial linear combination of $q$ columns of $A_G$, with $q \geqslant 2$. Form $A_1, \ldots, A_q$ from $A$ by replacing column 1 with each of the columns that appears in its linear combination. Proposition 8.38 implies that per$(A_i) \neq 0$, for some $i \in [q]$. Now we are done by induction, since $p(A_i) < p(A)$. □

**Theorem 8.40.** *Let* $G$ *be a connected graph with* $|G| \geqslant 2$, *and let* $T$ *be a spanning tree of* $G$. *Let* $d(v) := 1$ *for all* $v \in V(G)$, $d(e) := 0$ *for all* $e \in T$ *and* $d(e) := 2$ *for all* $e \in E(G) \setminus T$. *There exists a matrix* $A$ *with columns that are columns of* $A_G$ *such that per*$(A) \neq 0$ *and each column* $z$ *of* $A_G$ *appears at most* $d(z)$ *times. Thus,* $G$ *is (3,2)-choosable.* $T$

We use induction on $|G|$. Let $T$ be a spanning tree of $G$, with a leaf $v$. We let $G := G - v$ and $T' := T - v$. By induction, we get the desired matrix $A'$ for $G'$ and $T'$. First we extend $A'$ to a matrix $A''$ with the same permanent as $A'$ but with columns corresponding to $G$, rather than $G'$. This is easy, but has the problem that some columns for $G$ appear too often in $A''$. Finally, we use linear dependencies among the columns for $G$, substituting some columns for others, to ensure that no column appears too often in $A$.

*Proof.* The final statement follows directly from the previous statement, by Corollary 8.36 and the Combinatorial Nullstellensatz. To prove the previous statement, we use induction on $|G|$. The base case $|G| = 2$ is easy, so assume that $|G| \geqslant 3$. Let $d(u) := 1$ for all $u \in V(G)$, $d(e) := 0$ for all $e \in T$, and $d(e) := 2$ for all $e \in E(G) \setminus T$. Let $v$ be a leaf of $T$, let $G' := G - v$, and let $T' := T - v$. By induction, there exist a matrix $A'$ such that per$(A') \neq 0$ and each column of $A'$ is a column of $A_{G'}$ with each column $z$ of $A_{G'}$ appearing at most $d(z)$ times as a column of $A'$. We now modify $A'$ to get the desired matrix $A$. $d$ $v$ $A'$

Let $s := d_G(v)$ and denote the neighbors of $v$ in $G$ by $w_1, \ldots, w_s$, where $vw_s \in E(G)$. By symmetry, we assume that the rows for edges $vw_1, \ldots, vw_s$ are at the bottom of $A_G$. Let $m := |E(G)|$ and $m' := |E(G')| = |E(G)| - s$. Let $W_0$ be an $m \times m$ matrix as follows; see Figure 8.12. Starting from $A'$, we extend each column from a column of $A_{G'}$ to the corresponding column of $A_G$ (by adding $s$ rows at the bottom); this gives an $m \times m'$ matrix. As the final $s$ columns of $W_0$, we add $s$ copies of the column $A_G[v]$. Let $B$ denote the $s \times s$ $s$

$$
\begin{array}{c|ccc|}
 & v_1 & v_2 & v_4 \\
\hline
e_1 & 1 & -1 & \\
e_2 & & 1 & \\
e_3 & & & -1 \\
\hline
\end{array}
\qquad \longrightarrow \qquad
\begin{array}{c|ccc|ccc|}
 & \overset{e_6-v_5}{v_1} & v_2 & \overset{e_4-v_5}{v_4} & v_5 & v_5 & v_5 \\
\hline
e_1 & 1 & -1 & & & & \\
e_2 & & 1 & & & & \\
e_3 & & & -1 & & & \\
\hline
e_4 & & & 1 & -1 & -1 & -1 \\
e_5 & & -1 & & 1 & 1 & 1 \\
e_6 & -1 & & & 1 & 1 & 1 \\
\hline
\end{array}
$$

$$A' \qquad\qquad W_0$$

$$\downarrow$$

$$
\begin{array}{c|ccc|ccc|}
 & e_6 & v_2 & e_4 & e_6 & e_4 & v_5 \\
\hline
e_1 & 1 & -1 & & 1 & & \\
e_2 & & 1 & & & & \\
e_3 & & & -1 & & -1 & \\
\hline
e_4 & -1 & & & -1 & & -1 \\
e_5 & 1 & -1 & 1 & 1 & 1 & 1 \\
e_6 & & & 1 & & 1 & 1 \\
\hline
\end{array}
\qquad \longleftarrow \qquad
\begin{array}{c|ccc|ccc|}
 & e_6 & v_2 & e_4 & \overset{e_6-v_1}{v_5} & \overset{e_4-v_4}{v_5} & v_5 \\
\hline
e_1 & 1 & -1 & & & & \\
e_2 & & 1 & & & & \\
e_3 & & & -1 & & & \\
\hline
e_4 & -1 & & & -1 & -1 & -1 \\
e_5 & 1 & -1 & 1 & 1 & 1 & 1 \\
e_6 & & & 1 & 1 & 1 & 1 \\
\hline
\end{array}
$$

$$W_3 \qquad\qquad W_2$$

Figure 8.12: Clockwise from top left: The matrices $A'$, $W_0$, $W_2$, and $W_3$ (when $G$ and $A_G$ are as in Figure 8.11, and $v_5$ is a leaf of $T$ with $N_T(v_5) = \{v_2\}$, so $E(T) = \{e_1, e_2, e_3, e_5\}$). Note that $\mathrm{per}(A') = -1$, $\mathrm{per}(W_0) = (-1)3!\mathrm{per}(A') = 6$, $\mathrm{per}(W_2) = \mathrm{per}(W_0) = 6$, and $\mathrm{per}(W_3) = 12$. The differences above some columns in $W_0$ and $W_2$ show the identities used in forming successive matrices.

submatrix in the lower right corner of $W_0$. Because of the many 0s in the final $s$ columns of $W_0$, every nonzero term in the expansion of $\mathrm{per}(W_0)$ is the product of a nonzero term in each of $\mathrm{per}(A')$ and $\mathrm{per}(B)$. Each of the $s!$ nonzero terms in the expansion of $\mathrm{per}(B)$ is precisely $(-1)^{d_D^-(v)} = \pm 1$; so $\mathrm{per}(B) = \pm s!$. Thus, $\mathrm{per}(W_0) = \mathrm{per}(A')\mathrm{per}(B) = \pm s!\mathrm{per}(A') \neq 0$.

We now repeatedly modify $W_0$, without changing its permanent, to reach a matrix in which each column is a column of $A_G$ and each column $z$ appears at most $d(z)$ times. For each $i \in [s-1]$, we let $e_i := vw_i$ and do the following.[4] If column $A_G[w_i]$ does not appear in $W_{i-1}$, let $W_i := W_{i-1}$. Otherwise, $A_G[w_i]$ appears once, and we replace it by $A_G[e_i]$. This does not change the permanent, as follows. By Proposition 8.36, replacing $A_G[w_i]$ by $A_G[e_i] - A_G[v]$ does not change the permanent. Further, replacing $A_G[w_i]$ by $A_G[v]$ yields a matrix with permanent equal to 0, since $s+1$ of its columns have nonzero entries only in the

[4]We cannot do this for the edge $vw_s$ since it lies in $T$ and thus, by definition, $d(vw_s) = 0$.

final $s$ rows. Thus, replacing $A_G[w_i]$ by $A_G[e_i]$ leaves the permanent unchanged.

In $W_{s-1}$ each column $A_G[w_i]$ appears 0 times, each $A[e_i]$ appears at most once, and $A_G[v]$ appears $s$ times. To get $W_s$ from $W_{s-1}$, for each $i \in [s-1]$, replace a copy of $A_G[v]$ by $A_G[e_i] - A_G[w_i]$. As in the proof of Lemma 8.39, we can modify $W_s$, while maintaining a nonzero permanent, to reach a matrix where each column $z$ is a column of $A_G$, and $z$ appears at most $d(z)$ times. □

## 8.8 Extending the Alon–Tarsi Theorem to Paintabililty

In this section we prove Schauz's extension of the Alon–Tarsi Theorem to paintability (see Definition 1.28). In some sense, the proof is similar to that of the Kernel Lemma. We use induction on the number of vertices and, given a set $V_L$ by Lister, we must paint an independent $V_L$
set $V_P$ so that we can invoke the induction hypothesis after we delete $V_P$ and decrease by one $V_P$
the number of tokens at each vertex in $V_L \setminus V_P$. But this is where the similarities end. Proving the Kernel Lemma is much easier, since our initial orientation is kernel-perfect, by hypothesis.

Here we will construct the set $V_P$ iteratively. To facilitate this process, we consider not only the set of orientations with prescribed outdegrees. We generalize to allow some vertices to have only lower bounds prescribed for their outdegrees. A *generalized degree sequence* $\mathbf{d}$ for $\mathbf{d}$
a graph $G$ assigns to each vertex $v$ an "outdegree" $\mathbf{d}_v$, but these values need not sum to $\|G\|$; $\mathbf{d}$, $\mathbf{d}_v$
specifically, we allow $\sum_{v \in V(G)} \mathbf{d}_v < \|G\|$. For a generalized degree sequence $\mathbf{d}$, we write $\mathbf{d} + \mathbb{N}^S$ to denote all degree sequences such that $d^+(v) \geqslant \mathbf{d}_v$ when $v \in S$ and $d^+(v) = \mathbf{d}_v$ $\mathbf{d} + \mathbb{N}^S$
when $v \notin S$. We write $\mathbf{1}_v$ to denote the vector indexed by $V(G)$ with 1 in coordinate $v$ and 0 $\mathbf{1}_v$
elsewhere. Fix an orientation $\vec{G}$ of $G$, and let $\mathbf{d}$ be the outdegree sequence of $\vec{G}$. Let $D_{\mathbf{d}}(\vec{G})$ $D_{\mathbf{d}}(\vec{G})$
be the set of orientations of $G$ with outdegree sequence $\mathbf{d}$. Let $DE_{\mathbf{d}}(\vec{G})$ (resp. $DO_{\mathbf{d}}(\vec{G})$) be the $DE_{\mathbf{d}}(\vec{G})$
subset of $D_{\mathbf{d}}(\vec{G})$ with each element formed from $\vec{G}$ by reversing an even (resp. odd) number $DO_{\mathbf{d}}(\vec{G})$
of edges. Let $D_{\mathbf{d}+\mathbb{N}^S}(\vec{G}) := \uplus_{\mathbf{d}' \in \mathbf{d}+\mathbb{N}^S} D_{\mathbf{d}'}(\vec{G})$; recall that $\uplus$ denotes disjoint union. Define $DE_{\mathbf{d}+\mathbb{N}^S}(\vec{G})$
$DE_{\mathbf{d}+\mathbb{N}^S}(\vec{G})$ and $DO_{\mathbf{d}+\mathbb{N}^S}(\vec{G})$ analogously. $DO_{\mathbf{d}+\mathbb{N}^S}(\vec{G})$

The next theorem is the main result of this section.

**Theorem 8.41.** *Let $\vec{G}$ be a directed graph. If $f(v) > d^+(v)$ for all $v \in V(G)$ and $|DE_{\mathbf{d}}(\vec{G})| \neq |DO_{\mathbf{d}}(\vec{G})|$, then $G$ is $f$-paintable.*

If $D \in DE_{\mathbf{d}}(G)$, then the subgraph induced by edges where $D$ and $\vec{G}$ differ is an element of $EE(\vec{G})$. Similarly for $D \in DO_{\mathbf{d}}(G)$ and $OE(\vec{G})$. Thus, $|DE_{\mathbf{d}}(\vec{G})| \neq |DO_{\mathbf{d}}(\vec{G})|$ if and only if $|EE(\vec{G})| \neq |OE(\vec{G})|$. So Theorem 8.41 is a paintability analogue of the Alon–Tarsi Theorem.

We begin by describing a winning algorithm for Painter. However, the two key steps will only be justified later, in Lemmas 8.42 and 8.43.

*Proof.* Given a set $V_L$ by Lister, we show how Painter can find an independent set $V_P \subseteq V_L$ such that Painter can win by induction on $\vec{G} - V_P$. By hypothesis $|DE_{\mathbf{d}}(\vec{G})| \neq |DO_{\mathbf{d}}(\vec{G})|$. In particular, $D_{\mathbf{d}}(\vec{G}) \neq \emptyset$. Thus, $\sum_{v \in V(G)} \mathbf{d}_v = |E(\vec{G})|$. So

$$\left|DE_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})\right| = \left|DE_{\mathbf{d}}(\vec{G})\right| \neq \left|DO_{\mathbf{d}}(\vec{G})\right| = \left|DO_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})\right|.$$

By Lemma 8.42, we can iteratively find an independent set $V_P \subseteq V_L$ and a generalized degree sequence $\mathbf{d}'$ with $0 \leqslant \mathbf{d}' \leqslant \mathbf{d}$ and $\mathbf{d}'_v < \mathbf{d}_v$ for all $v \in V_L \setminus V_P$ and $\mathbf{d}'_v = 0$ for all $v \in V_P$ such that

$$\left|DE_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})\right| \neq \left|DO_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})\right|.$$

This set $V_P$ will be the vertices that Painter paints. Thus, we are more interested in $\vec{G} - V_P$. By Lemma 8.43, there exists a generalized degree sequence $\mathbf{d}''$ with $0 \leqslant \mathbf{d}''_v \leqslant \mathbf{d}'_v$ for all $v \in V \setminus V_P$ and

$$\left|DE_{\mathbf{d}''}(\vec{G} \setminus V_P)\right| \neq \left|DO_{\mathbf{d}''}(\vec{G} \setminus V_P)\right|.$$

Thus $\vec{G} - V_P$ and $\mathbf{d}''$ satisfy the hypotheses, so we are done by induction on $|\vec{G}|$. □

Now we prove the two key lemmas in the previous proof.

**Lemma 8.42.** *If $|DE_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})| \neq |DO_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})|$, then there exists an independent set $V_P \subseteq V_L$ and generalized degree sequence $\mathbf{d}'$ with $0 \leqslant \mathbf{d}' \leqslant \mathbf{d}$ such that $|DE_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})| \neq |DO_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})|$ and $\mathbf{d}'_v = 0$ for all $v \in V_P$ and $\mathbf{d}'_v < \mathbf{d}_v$ for all $v \in V_L \setminus V_P$.*

*Proof.* To begin, we prove three easy claims.

**Claim 1.** *For all $v \in V_L$ we have $(\mathbf{d} - \mathbf{1}_v) + \mathbb{N}^{V_L} = \mathbf{d} + \mathbb{N}^{V_L} \uplus (\mathbf{d} - \mathbf{1}_v) + \mathbb{N}^{V_L \setminus v}$.*

Proof. On the right side, we partition the set of degree sequences on the left side based on whether $v$ has degree equal to $\mathbf{d}_v$ or larger than $\mathbf{d}_v$. ♢

**Claim 2.** *For all $v \in V(G)$, the following two equalities hold.*

$$DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L}}(\vec{G}) = DE_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G}) \uplus DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L \setminus v}}(\vec{G})$$
$$DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L}}(\vec{G}) = DO_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G}) \uplus DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L \setminus v}}(\vec{G})$$

Proof. This follows immediately from Claim 1. ♢

**Claim 3.** *If $|DE_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})| \neq |DO_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})|$, then $|DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L}}(\vec{G})| \neq |DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L}}(\vec{G})|$ or else $|DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L \setminus v}}(\vec{G})| \neq |DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L \setminus v}}(\vec{G})|$.*

Proof. Assume the conclusion is false. Now Claim 2 gives

$$\begin{aligned}|DE_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})| &= |DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L}}(\vec{G})| - |DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L\setminus v}}(\vec{G})| \\ &= |DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L}}(\vec{G})| - |DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_L\setminus v}}(\vec{G})| \\ &= |DO_{\mathbf{d}+\mathbb{N}^{V_L}}(\vec{G})|.\end{aligned}$$

The first and third equalities hold by Claim 2, and the second holds because the conclusion is false. This proves the contrapositive. ♢

Let $\mathbf{d}^{temp} := \mathbf{d}$ and $V_{temp} := V_L$. We repeatedly apply Claim 3, as long as there exists $v \in V_{temp}$ such that $\mathbf{d}_v^{temp} > 0$. We decrease $\mathbf{d}_v^{temp}$ by 1, and possibly remove $v$ from $V_{temp}$. Specifically, if $|DE_{\mathbf{d}^{temp}-\mathbf{1}_v+\mathbb{N}^{V_{temp}\setminus\{v\}}}(\vec{G})| \neq |DO_{\mathbf{d}^{temp}-\mathbf{1}_v+\mathbb{N}^{V_{temp}\setminus\{v\}}}(\vec{G})|$, then $\mathbf{d}^{temp} := \mathbf{d}^{temp} - \mathbf{1}_v$ and $V_{temp} := V_{temp} \setminus \{v\}$. Otherwise, Claim 3 implies that $|DE_{\mathbf{d}^{temp}-\mathbf{1}_v+\mathbb{N}^{V_{temp}}}(\vec{G})| \neq |DO_{\mathbf{d}^{temp}-\mathbf{1}_v+\mathbb{N}^{V_{temp}}}(\vec{G})|$, so we have $\mathbf{d}^{temp} := \mathbf{d}^{temp} - \mathbf{1}_v$ and $V_{temp}$ is unchanged. $\mathbf{d}^{temp}$ $V_{temp}$

When this process halts, let $V_P := V_{temp}$ and $\mathbf{d}' := \mathbf{d}^{temp}$. By design, for all $v \in V_P = V_{temp}$, we have $\mathbf{d}'_v = 0$. If $v \in V_L \setminus V_P$, then at some point $v$ was removed from $V_{temp}$, which implies that $\mathbf{d}'_v < \mathbf{d}_v$. Now we show that $V_P$ is an independent set. Suppose to the contrary that there exist $v, w \in V_P$ and $\overrightarrow{vw} \in E(\vec{G})$. Given any orientation in $DE_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})$, we can reverse edge $vw$ to get an orientation in $DO_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})$. This bijection shows that $|DE_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})| = |DO_{\mathbf{d}'+\mathbb{N}^{V_P}}(\vec{G})|$, a contradiction. Thus, $V_P$ is an independent set. □

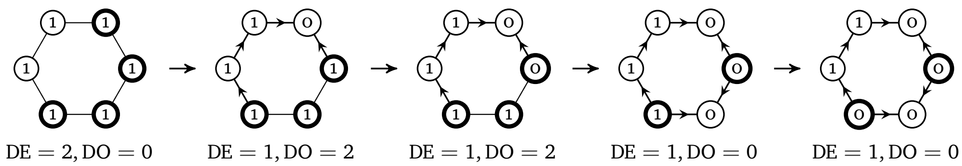



Figure 8.13: The proof of Lemma 8.42, when $G$ is $C_6$ and $V_L$ is four consecutive vertices. ($\vec{G}$ has all edges oriented clockwise.) Each vertex $v$ is labeled with $\mathbf{d}_v^{temp}$. Vertices in $V_{temp}$ are drawn in bold, and edges with orientations shown have those orientations determined by $\mathbf{d}^{temp}$ and $V_{temp}$.

Now we prove the second key lemma. Note the similarities to the proof of Lemma 8.7.

**Lemma 8.43.** *Let $\vec{G}$ be a directed graph, $\mathbf{d} \in \mathbb{N}^V$, and $V_P \subseteq V$. If $|DE_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})| \neq |DO_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})|$, then there exists $\mathbf{d}''$ such that $0 \leqslant \mathbf{d}'' \leqslant \mathbf{d}|_{V-V_P}$ and $|DE_{\mathbf{d}''}(\vec{G} - V_P)| \neq |DO_{\mathbf{d}''}(\vec{G} - V_P)|$.*

*Proof.* We again start with three easy claims.

**Claim 1.** *If $\overrightarrow{vw} \in E(\vec{G})$, then the following two equalities hold.*

$$|DE_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})| = |DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_P}}(\vec{G} - \overrightarrow{vw})| + |DO_{\mathbf{d}-\mathbf{1}_w+\mathbb{N}^{V_P}}(\vec{G} - \overrightarrow{vw})|$$
$$|DO_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})| = |DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_P}}(\vec{G} - \overrightarrow{vw})| + |DE_{\mathbf{d}-\mathbf{1}_w+\mathbb{N}^{V_P}}(\vec{G} - \overrightarrow{vw})|$$

Proof. On the right, we partition the set counted on the left by whether edge $vw$ is oriented as $\overrightarrow{vw}$ or $\overrightarrow{wv}$. $\diamondsuit$

**Claim 2.** *If $\overrightarrow{vw} \in E(\vec{G})$ and $|DE_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})| \neq |DO_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})|$, then $|DE_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_P}}(\vec{G}-vw)| \neq |DO_{\mathbf{d}-\mathbf{1}_v+\mathbb{N}^{V_P}}(\vec{G}-vw)|$ or $|DO_{\mathbf{d}-\mathbf{1}_w+\mathbb{N}^{V_P}}(\vec{G}-vw)| \neq |DE_{\mathbf{d}-\mathbf{1}_w+\mathbb{N}^{V_P}}(\vec{G}-vw)|$ (or both).*

Proof. If the conclusion is false, then both claimed inequalities are equalities. Adding these two equalities gives $\left|DE_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})\right| = \left|DO_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})\right|$, by Claim 1, a contradiction. $\diamondsuit$

**Claim 3.** *If $\left|DE_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})\right| \neq \left|DO_{\mathbf{d}+\mathbb{N}^{V_P}}(\vec{G})\right|$ and $E' \subseteq E$, then there exists a degree sequence $\mathbf{d}''$ such that $0 \leqslant \mathbf{d}''_v \leqslant \mathbf{d}_v$, for all $v$ such that $\left|DE_{\mathbf{d}''+\mathbb{N}^{V_P}}(\vec{G}-E')\right| \neq \left|DO_{\mathbf{d}''+\mathbb{N}^{V_P}}(\vec{G}-E')\right|$.*

Proof. This follows from Claim 2, by induction on $|E'|$. $\diamondsuit$

Now we use Claim 3 to finish the proof. Let $E' := E(V_P, V \setminus V_P)$. By Claim 3, we have $\mathbf{d}''$ such that $\left|DE_{\mathbf{d}''+\mathbb{N}^{V_P}}(\vec{G}-E')\right| \neq \left|DO_{\mathbf{d}''+\mathbb{N}^{V_P}}(\vec{G}-E')\right|$. However, each $v \in V_P$ has no incident edges in $\vec{G}-E'$. Thus, $d^+(v) = 0$ for each $v \in V_P$ and each orientation in $D_{\mathbf{d}''+\mathbb{N}^{V_P}}(\vec{G}-E')$. This implies that $\left|DE_{\mathbf{d}''}(\vec{G}-V_P)\right| = \left|DE_{\mathbf{d}''+\mathbb{N}^{V_P}}(\vec{G}-E')\right| \neq \left|DO_{\mathbf{d}''+\mathbb{N}^{V_P}}(\vec{G}-E')\right| = \left|DO_{\mathbf{d}''}(\vec{G}-V_P)\right|$, as desired. $\square$

## Notes

The Combinatorial Nullstellensatz (CN) was proved by Alon in 1999 [12]. However, it had roots in earlier work of Alon and Tarsi [20] and Alon, Nathanson, and Ruzsa [19]. The CN has been applied to a wide range of problems, particularly in combinatorics and number theory. For example, see [31, 194]. (In fact, [31] covers most of the material in this chapter, and much more.) This application is often called the *Polynomial Method*.

Our proof of the CN follows Michałek [299]. The Alon–Tarsi Theorem was proved in [20]; the authors mention that it extends work of Gessel [175], who proved the special case when $G$ is a complete graph. Example 8.5 also comes from [20]. Lemma 8.6 seems to be folklore. Lemma 8.7 has been proved by various authors, and our presentation follows [102, Lemma 1.6]. Theorem 8.8 is due to Hladký, Král, and Schauz [219]. Definition 8.10 and Lemma 8.11 are due to Cranston and Rabern [101, Lemma 17]. Theorem 8.9 is stated without proof in [91, end of §3.2] and we are unaware of Lemma 8.12 having appeared previously, although its proof draws heavily on ideas from the proof of [101, Lemma 20].

Theorem 8.14 is due to Kaul and Mudrock [239], and our presentation also draws on [31, Theorem 10]. This result was strengthened by Li, Shao, Petrov and Gordeev [283], who proved that $AT(C_s \square C_t)$ is 3 when $st$ is even and 4 when $st$ is odd. (It was conjectured [75] that $\chi_\ell(C_s \square C_t) = 3$ for all $s \geqslant 3$ and $t \geqslant 3$. This problem is solved by [283] when $st$ is even, but remains open when $st$ is odd.) Theorem 8.17 was originally proved by Fleischner and Stiebitz [164] using a long, tedious induction. Our proof follows Petrov [331], who was the first to state and prove Lemma 8.16. He proved something more general which we leave as Exercise 4.

Theorem 8.19, that every planar graph is 5-AT, is due to Zhu [436]. After presenting the proof in Section 8.3, his paper also reformulates the proof completely in terms of the graph polynomial. Gu and Zhu [192] give an alternate proof, as do Kozik and Podkanowicz [276].

Theorem 8.19 was generalized by Abe, Kim, and Ozeki [2], who showed that every $K_5$-minor-free graph is 5-AT. Versions of the Coefficient Formula were proved by Schauz [353, Theorem 3.2], Hefetz [213, Theorem 2.1], Lasoń [281, Theorem 3], and Karasev and Petrov [237, Theorem 4]. Our presentation follows Lasoń, but also incorporates simplifications of Rabern, since we only prove it for polynomials with all monomials of equal degree. Theorem 8.23 was proved by Huang, Wong, and Zhu [221].

Theorem 8.26 is due to Alon and Füredi [15]. In fact, they proved a slightly stronger statement with a harder proof. Theorem 8.27 and the precise formulations of Lemma 8.25 and Theorem 8.26, as well as all of their proofs, are due to Bosek, Grytczuk, Gutowski, Serra, and Zając [71] (drawing on ideas from the proof of the CN due to Michalek [299], which is what we presented in the proof of Theorem 8.1).

Recent applications of Theorem 8.26 include [240], [110], and [111]. Thomassen [380] proved that every planar graph of girth at least 5 is 3-choosable. He later showed [382] that every 3-assignment $L$ for every such graph $G$ admits at least $2^{|G|/10000}$ proper $L$-colorings. Bosek et al. note that this lower bound can be strengthened to $3^{|G|/6}$. This follows directly from 11.1, by an easy application of Theorem 8.26, akin to the proof of Theorem 8.27 (Exercise 5 asks the reader to provide details).

Theorem 8.28, that $AT'(G) = \Delta(G)$ for all regular planar multigraphs $G$ with $\chi'(G) = \Delta(G)$, is due to Ellingham and Goddyn [147]. Lemma 8.30 was also observed by Alon [11]. The proof in [147] of Lemma 8.31 is much longer than what we gave, although it also provides extra information. Our proofs of Lemmas 8.30 and 8.31 essentially follow a proof of Zhu [31]. Finally, Theorem 8.41, which implies that $\chi_p(G) \leqslant AT(G)$ for every graph $G$, is due to Schauz [355].

Recall Theorem 8.19, that every planar graph is 5-AT. Since there exist planar graphs that are not 4-choosable (see Theorem 2.3 and Exercise 9), this bound is sharp. So if we insist on lists of size less than 5, then we must in some way weaken our coloring requirements. A graph $G$ is $a$-*defective* $b$-*colorable* if it has a coloring from $[a]$ such that each monochromatic subgraph has maximum degree at most $b$. Similarly, $G$ is $a$-*defective* $b$-*choosable* if given any list assignment $L$ with $|L(v)| = a$ for all $v \in V(G)$ there exists an $L$-coloring $\varphi$ (not necessarily proper) such that each monochromatic subgraph of $G$ colored by $\varphi$ has maximum degree at most $b$. Equivalently, we require that for all $L$ with $|L(v)| = a$ for all $v \in V(G)$ there exists $H \subseteq G$ such that $\Delta(H) \leqslant b$ and $G - E(H)$ has a proper $L$-coloring. Eaton and Hull [142] and Škrekovski [407] showed that every planar graph is 2-defective 3-choosable. Cushing and Kierstead [109] showed that every planar graph is 1-defective 4-choosable. Finally, Škrekovski [406] also showed that every triangle-free planar graph is 1-defective 3-choosable.

It is natural to ask whether these results extend to Alon–Tarsi number. Grytczuk and Zhu [191] proved that every planar graph $G$ has a matching $M$ such that $AT(G-M) \leqslant 4$; see [31, Theorem 16] and [192] for alternate proofs. This result was strengthened by Abe, Kim, and Ozeki [2], who proved the same conclusion for every $K_5$-minor-free graph. That paper also shows that for every $K_5$-minor-free graph $G$ there exists a forest $F$ such that $AT(G-E(F)) \leqslant 3$. (For every planar graph $G$, [276] showed that there exists a forest $F$ such that $\text{col}(G-E(F)) \leqslant 3$.) In contrast, Kim, Kim, and Zhu [256] constructed a planar graph $G$ such that for all $H \subseteq G$ with $\Delta(H) \leqslant 3$ we have $AT(G-E(H)) \geqslant \chi_\ell(G-E(H)) > 3$. Finally, Xu and Zhu [419] showed that every triangle-free planar graph $G$ decomposes into a matching $M$ and a 2-degenerate graph. Thus, $AT(G-M) \leqslant 3$.

A *weighting* of a graph assigns real numbers to its edges, and the weight of each vertex is the sum of weights on its incident edges. A weighting is *proper* if it gives the endpoints of each edge distinct weights. A graph is *nice* if it has no isolated edges. (Note that every weighting of a non-nice graph is improper.) The 1-2-3-Conjecture posits that every nice graph has a proper weighting using only the weights 1, 2, and 3. It is this context that motivated Theorem 8.33. There we generalize to choosability, but we also incorporate vertex weights, each chosen from two options (rather than the single weight 0, in the 1-2-3-Conjecture). The best result on this conjecture [236] is that every nice graph has a proper weighting with weights 1 through 5.

For more than a decade it was an open problem whether there exists a positive integer $k$ such that every nice graph is total weight $(1,k)$-choosable. Eventually, Cao answered this question affirmatively, showing that $k = 17$ suffices [78]. This result was subsequently improved by Zhu, who showed that $k = 5$ suffices, i.e., every nice graph is total-weight $(1,5)$-choosable [437]. Related work includes [292], which proves sufficient conditions for a graph to be total-weight $(1,4)$-choosable and total-weight $(1,3)$-choosable.

Historically, most applications of the Alon–Tarsi Theorem have either been to very structured graphs, where we can often find parity-reversing bijections, or to fairly small graphs, where our enumeration of OE and EE is ad hoc, perhaps by hand. However, in 2023 Dvořák [129] published an efficient implementation of the Alon–Tarsi Theorem. His abstract states:

> We describe an efficient way to implement this approach, making it feasible to test choosability of graphs with around 70 edges. We also show that in case that Alon–Tarsi method fails to show that the graph is choosable, further coefficients of the graph polynomial provide constraints on the list assignments from which the graph cannot be colored. This often enables us to confirm colorability from a given list assignment, or to decide choosability by testing just a few list assignments.

## Exercises

8.1. Show that $\chi'(K_{3,3}) = 3$, but $AT(L(K_{3,3})) > 3$. [In Theorem 5.11, we in fact use the Kernel Lemma to show that $\chi'_\ell(K_{3,3}) = 3$.]

8.2. Show that $AT(C_{3k}^2) = 3$ for every integer $k \geqslant 2$. Further, prove that $\chi_\ell(C_9^2) = 3$ cannot be proved via the Kernel Lemma. [Together with the previous exercise, this shows that neither the Alon–Tarsi Theorem nor the Kernel Lemma implies the other.]

8.3. (a) Let $G := K_s \vee K_{2\star t}$ (where $\vee$ denotes join), let $A$ be an $(s+t)$-clique in $G$, and let $B = V(G) \setminus A$. Now $G$ is $f$-AT whenever $f(v) \geqslant s+t$ for all $v \in A$ and $f(v) \geqslant t$ for all $v \in B$. (b) Use part (a) and Lemma 8.7 to prove the following corollary. Let $G$ be the complement of a bipartite graph with parts $A$ and $B$. If $f(v) \geqslant \omega(G)$ for all $v \in A$ and $f(v) \geqslant |B|$ for all $v \in B$, then $G$ is $f$-AT.

8.4. In his proof of the Cycle-Plus-Triangles Theorem [331], Petrov actually proved the following more general version of Lemma 8.16. Let $W$ be a finite set partitioned into disjoint subsets $W_1, \ldots, W_n$ such that $|W_i|$ is odd for each $i$. Let $G$ be a graph with $W$ as its vertex set such that each $W_i$ is an independent set in $G$ and the induced subgraph $G[W_i \cup W_j]$ is Eulerian (all degrees even) for each pair $i, j \in [n]$. Consider subsets $U \subset W$ such that $|U \cap W_i| = 1$ for all $i$ and the subgraph induced by $U$ is Eulerian. The number of such sets $U$ is odd. Adapt the proof of Lemma 8.16 to prove this more general version.

8.5. Thomassen [380] proved that every planar graph of girth 5 is 3-choosable. He later extended this result to show that, if $G$ is a girth 5 planar graph and $L$ is a 3-assignment, then $G$ admits at least $2^{|G|/10000}$ proper $L$-colorings. Strengthen this lower bound to $3^{|G|/6}$ by combining the result of [380] with Theorem 8.26. [71]

8.6. A *Halin graph* is a plane graph consisting of a tree $T$ and a cycle $C$, where no vertex of $T$ has degree 2, at least one vertex of $T$ has degree at least 3, and $C$ connects the leaves of $T$ in the cyclic order determined by the drawing of $T$. Determine the Alon–Tarsi number of every Halin graph. [284]

8.7. Prove Propositions 8.37 and 8.38.

# Chapter 9

# The Activation Strategy

Purpose provides activation energy for living.
—Mihaly Csikszentmihalyi

In this chapter we study the activation strategy, which was designed to help Alice play the Marking Game. Somewhat surprisingly, this simple strategy enables Alice to perform nearly optimally on a wide range of graph classes, and is also well-suited for variations of the standard Marking Game. Further, the strategy serves nicely for one non-game problem.

## 9.1 An Introduction to Coloring Games

For each graph $G$, we consider the *Chromatic Game* played between two players, Alice and Bob, who construct a proper coloring of $G$. The players alternate turns, starting with Alice, and on each turn a player colors a (previously uncolored) vertex with some color from $[k]$, such that the resulting coloring is proper. Alice's goal is to construct a proper coloring of $G$, while Bob's goal is to stop her, by reaching the situation that some vertex $v$ is uncolored, but each color in $[k]$ is used on one of its colored neighbors.

The *game chromatic number* of $G$, denoted $\chi_g(G)$, is the minimum value $k$ such that Alice has a winning strategy. Clearly, $\chi(G) \leqslant \chi_g(G) \leqslant \Delta(G)+1$. So, for example, $\chi_g(K_n) = n$. But there do exist bipartite graphs with arbitrarily large game chromatic number.

**Example 9.1.** Form $B_n$ from the complete bipartite graph $K_{n,n}$ by removing a perfect matching $nK_2$. Now $\chi_g(B_n) = n$. For the lower bound, Bob uses the following easy strategy. Each time that Alice colors some vertex $v$, Bob uses the same color on the unique nonneighbor $v'$ of $v$ in the other part of $B_n$. This ensures that each color is used at most twice, so $\chi_g(B_n) \geqslant |V(B_n)|/2 = n$. Conversely, $\chi_g(B_n) \leqslant \Delta(B_n)+1 = n$.

It is easy to check that $\chi_g(K_{n,n}) = 3$. So $K_{n,n}$ and $B_n$ show that $\chi_g$ is not weakly decreasing when taking subgraphs. In fact, the increase in $\chi_g$ can be arbitrarily large. $\diamondsuit$

$\chi_g(\mathcal{G})$ For a graph class $\mathcal{G}$, let $\chi_g(\mathcal{G}) := \max_{G \in \mathcal{G}} \chi_g(G)$. In this section, we will seek bounds on $\chi_g(\mathcal{G})$ for various well-known classes of graphs $\mathcal{G}$, such as trees, outerplanar graphs, chordal graphs, and planar graphs. Typically, upper bounds will require more effort than lower, so that is where we will focus most of our attention.

(It is somewhat arbitrary that Alice moves first. We could also define a variant $\chi_g^B$ where Bob moves first. Certainly, for individual graphs G we may have either $\chi_g^B(G) > \chi_g(G)$ or else $\chi_g(G) > \chi_g^B(G)$. However, if $\mathcal{G}$ is a graph class that is closed under (a) taking disjoint unions and (b) adding an isolated vertex, then $\chi_g^B(\mathcal{G}) = \chi_g(\mathcal{G})$. We prove this in Lemma A.7.)

Our main topic of study in this chapter is the activation strategy (and its variants), which prescribes a way for Alice to play against Bob. We begin with an orientation $\vec{G}$ of our graph G induced by a linear order L of $V(G)$ (with $\overrightarrow{vw} \in E(\vec{G})$ if $vw \in E(G)$ and $v >_L w$). Before formally describing the activation strategy, we offer the following informal overview.

Alice keeps a set of "activated" vertices (initially empty), which includes all colored vertices, as well as typically some others. Each time that Bob plays and colors some vertex $b$, Alice "activates" $b$. Alice then follows an out-edge from $b$ to some vertex $w$. If $w$ is already activated, then Alice colors $w$ (with the smallest color unused on $N(w)$) and stops. Otherwise, she follows an out-edge from $w$ and recurses. If at some point, Alice reaches a vertex $w$ that has no uncolored outneighbors, then Alice colors $w$, even if $w$ was unactivated when she reached it. If Bob happens to color some vertex with no uncolored outneighbors, then Alice chooses the smallest uncolored vertex $v$ under L, activates $v$ (if $v$ is not already activated), and colors $v$.

As a warm-up, we show that the activation strategy does well when G is a forest.

**Lemma 9.2.** *If* G *is a forest, then there exists an orientation of* G *such that Alice wins the chromatic game, with* $k$ *colors, using the activation strategy whenever* $k \geqslant 4$.

*Proof.* We assume that G is a tree. If not, then on each turn, Alice responds in the component where Bob played most recently. If a component is completely colored, then Alice plays in an arbitrary component; we will see that these "extra" moves never make life harder for Alice.

It is well-known that every tree T has an orientation $\vec{T}$ with $\Delta^+(\vec{T}) \leqslant 1$. For example, orient an edge out from some leaf $v$, and recursively orient $T - v$. Figure 9.1 shows an example.[1] Now consider the maximum number of colored neighbors of some uncolored vertex $w$.

If $x$ is an inneighbor of $w$ and $x$ is colored, then $x$ is activated, so $w$ is also activated. Furthermore, $w$ has at most one colored inneighbor $x$, except possibly for a second such neighbor that was just colored by Bob; the second time that we would visit $w$ to activate it, we would instead color $w$. Finally, $w$ has at most one outneighbor, so at most one *colored* outneighbor. Thus, the total number of colored neighbors of $w$ is at most 3. So Alice can successfully color $w$ whenever $k \geqslant 4$.

(Recall from above that if Bob colors a vertex $b$ with no uncolored outneighbors, then Alice colors the smallest uncolored vertex. Clearly, Alice's job is no harder than usual. The only difference is that she need not worry about $b$ as a colored inneighbor.) □

[1] Technically, Alice should start, rather than Bob. To address this, we can add a single isolated vertex $v$ that comes first in L. On Alice's initial turn, she activates and marks $v$. Thereafter, play proceeds as shown in Figure 9.1.

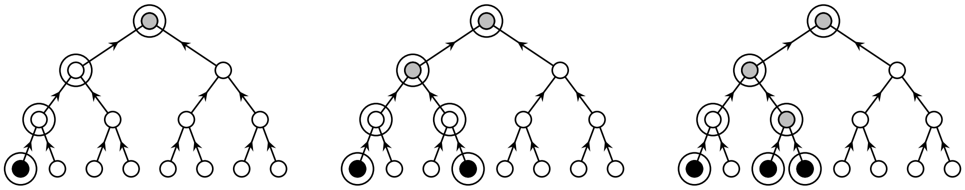


Figure 9.1: A sequence of moves by Bob (in Black) and responses by Alice (in Ash) according to the activation strategy. Each circled vertex has been activated by Alice.

Because the activation strategy is rather simple, it may surprise the reader to learn that this strategy gives an optimal upper bound on $\chi_g(\mathcal{F})$, where $\mathcal{F}$ is the class of all forests. It is straightforward to construct trees that need 4 colors, so we leave this to Exercise 2.

It is interesting to observe that our analysis in the proof of Lemma 9.2 did not rely on which colors were used on which vertices. In fact, we proved the stronger statement that each vertex is colored after at most 3 of its neighbors are colored. This motivates the following definition.

**Definition 9.3.** The *marking game*[2] on a graph $G$ is played between two players, Alice and Bob, alternating turns. Each player, on its turn, marks a (previously unmarked) vertex of $G$. The score of the game is $1 + \max_{v \in V(G)} N(v) \setminus U_v$, where $U_v$ is the set of vertices that are unmarked at the time that $v$ is marked. The *game coloring number* of $G$, denoted $col_g(G)$, is the minimum $k$ such that Alice can ensure the score in the marking game is at most $k$. For a graph class $\mathcal{G}$, let $\mathrm{col}_g(\mathcal{G}) := \max_{G \in \mathcal{G}} \mathrm{col}_g(G)$.

marking game

game coloring number

$\mathrm{col}_g(G)$, $\mathrm{col}_g(\mathcal{G})$

In fact, the proof of Lemma 9.2 shows the following.

**Corollary 9.4.** *If $\mathcal{F}$ is the class of all forests, then $\chi_g(\mathcal{F}) \leqslant col_g(\mathcal{F}) \leqslant 4$.*

**Remark 9.5.** Unlike $\chi_g$, the parameter $\mathrm{col}_g$ is weakly decreasing when taking subgraphs. That is, if $H \subseteq G$, then $\mathrm{col}_g(H) \leqslant \mathrm{col}_g(G)$. When $H$ is a spanning subgraph of $G$, this is true trivially. For the general case, by induction on $|G|$, we can assume that $|H| = |G| - 1$. Here Alice uses a typical "strategy-stealing" argument, playing a game on $H$ by simulating a game on $G$, for which she knows a strategy. We defer the details of the proof to Lemma A.8.

We now present Alice's activation strategy, for the marking game. Always $A$ is the set of activated vertices, and $U$ is the set of unmarked ones. Initially, $A := \emptyset$ and $U := V(G)$.

Algorithm 9.1 below precisely describes the activation strategy. But what does it imply about $\mathrm{col}_g$? For our first upper bound, we need the notion of 2-coloring number. Given a linear

[2] Given its roots in the chromatic game, one natural name might be the "coloring game". But the standard terminology is "marking game" since it more accurately reflects that each vertex is actually not assigned a color, but simply "marked" once a player selects it. However, we do use the term "game coloring number", which highlights the analogy with the coloring number (1 more than the degeneracy).

**Algorithm 9.1:** Activation Strategy for Alice to play the Marking Game

**Input :** An orientation $\vec{G}$ of a graph $G$ induced by a linear order $L$ of $V(G)$.
**Output:** A move for Alice in the Marking Game

**let** $x := b$ (Bob's most recent move)
**while** $x \notin A$
  **let** $A := A \cup \{x\}$
  **let** $s(x) := \min_L((N^+[x] \cap U) \cup \{b\})$
  **let** $x := s(x)$
**if** $x == b$
  **let** $x := \min_L U$
  **let** $A := A \cup \{x\}$
**mark** $x$

order $L$ of $V(G)$, a *backward neighbor* of a vertex $v$ is any vertex $w$ such that $vw \in E(G)$ and $w <_L v$; that is, $w$ is a neighbor of $v$ that precedes $v$ in $L$. A *loose backward neighbor* of $v$ is a vertex $w$ such that either (a) $w$ is a backward neighbor of $v$ or (b) there exists a vertex $x$ such that $vx, wx \in E(G)$ and $w <_L v <_L x$. Figure 9.2 illustrates these definitions with an example. The 2-*coloring number* of $G$, denoted $\mathrm{col}_2(G)$, is defined as the smallest integer $k$ such that there exists a linear order $L$ of $V(G)$ for which each vertex $v$ has fewer than $k$ loose backward neighbors w.r.t. $L$.

Although $\mathrm{col}_2(G)$ can be challenging to compute exactly, it is easy to upper bound. Indeed, every linear order $L$ clearly witnesses some upper bound on $\mathrm{col}_2(G)$. (Contrast this with $\mathrm{col}_g(G)$, for which it is initially unclear how to get any upper bound stronger than $\Delta(G) + 1$.)

**Lemma 9.6.** *Every graph* $G$ *satisfies* $\mathrm{col}_g(G) \leqslant 3\mathrm{col}_2(G) - 1$.

*Proof.* Alice fixes a linear order $L$ on $V(G)$ that witnesses $\mathrm{col}_2(G)$, and she plays the activation strategy w.r.t. $L$. For each uncolored vertex $v$, we must bound the number of neighbors of $v$ that

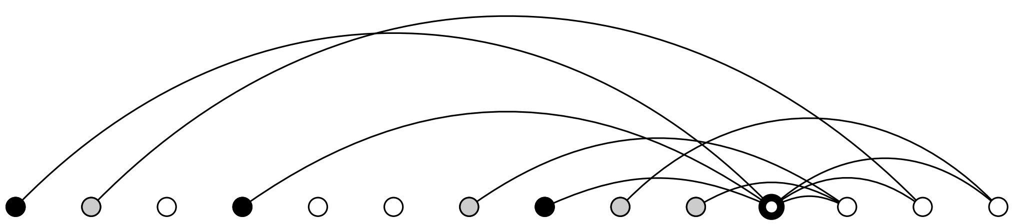


Figure 9.2: Vertices of $G$ are ordered from left (earliest) to right. The bold vertex $v$ has 3 backward neighbors, in black, and 4 additional loose backward neighbors, in gray. Thus $v$ has loose backward degree 7, w.r.t. the vertex order shown. Other edges of $G$ are omitted for clarity.

are marked before $v$. Our plan is to charge each such neighbor to a loose backward neighbor $w$, so that each such $w$ is charged at most 3 times.

As above, we orient each edge of $G$ into its endpoint earlier in $L$. Whenever Bob marks a vertex $x$ in $N^-(v)$, Alice immediately moves to its smallest (in $L$) uncolored neighbor $w$, activating $w$ if it is not already activated, and otherwise coloring $w$. Each such $w$ is visited at most twice by Alice, and each $w$ is also a loose backward neighbor of $v$. Thus, the number of vertices in $N^-(v)$ marked before $v$ is at most $2(k-1)$, where $k = \mathrm{col}_2(G)$; actually, this omits the final move of Bob before Alice marks $v$. Also, $|N^+(v)| \leqslant k-1$, since each outneighbor of $v$ is a loose backward neighbor. So when Alice marks $v$, its number of colored neighbors is at most $2(k-1)+1+(k-1) = 3k-2$. Hence, as claimed we have $\mathrm{col}_g(G) \leqslant 3\mathrm{col}_2(G)-1$. □

Lemma 9.6 is powerful for showing that $\mathrm{col}_g(\mathcal{G})$ is bounded (by some absolute constant) for various graph classes $\mathcal{G}$. Namely, this is true whenever $\mathcal{G}$ is a graph class of bounded expansion; such classes include all $\mathcal{G}$ defined by forbidden minors. We discuss this further in the Notes. But for many classes $\mathcal{G}$ this bound is not sharp. For example, when $\mathcal{G}$ consists of all forests, Lemma 9.6 implies $\mathrm{col}_g(\mathcal{G}) \leqslant 3(2)-1 = 5$, rather than the sharp upper bound of 4 we proved above. (This is unsurprising, since our proof used the inequality $|N^+(v)| \leqslant \mathrm{col}_2(G)-1$, which rarely holds with equality.)

To better analyze what upper bounds are implied by the activation strategy, we introduce a new parameter called rank, denoted $r(G)$. Given a graph $G$ and linear order $L$ of $V(G)$, the orientation $\vec{G}_L$ of $G$ is formed from $G$ by directing each edge towards its endpoint that appears earlier in $L$. For each vertex $v$, let $V^+(v) := \{x : x <_L v\}$, $V^+[v] := V^+(v) \cup \{v\}$, $V^-(v) := \{x : x >_L v\}$, $V^-[v] := V^+(v) \cup \{v\}$, $N^+(v) := V^+(v) \cap N(v)$, $N^+[v] := V^+(v) \cup \{v\}$, $N^-(v) := V^-(v) \cap N(v)$, and $N^-[v] := V^-(v) \cup \{v\}$.

$V^+(v)/V^+[v]$
$V^-(v)/V^-[v]$
$N^+(v)/N^+[v]$
$N^-(v)/N^-[v]$

Let $m(v, L, G) := \max |M| + |N|$, where $M$ and $N$ are, respectively, matchings from $N^-(v)$ to $V^+[v]$ and from $N^-[v]$ to $V^+(v)$. We also require $V(M) \cap V(N) \cap N^-(v) = \emptyset$. That is, the restrictions of $M$ and $N$ to $N^-(v)$ are disjoint. Let

$m(v, L, G)$

$$
\begin{aligned}
r(v, L, G) &:= d^+_{\vec{G}_L}(v) + m(v, L, G) \\
r(L, G) &:= \max_{v \in V} r(v, L, G) \\
r(G) &:= \min_{L} r(L, G),
\end{aligned}
$$

with the minimum over all linear orders $L$ of $V(G)$. The most important result on the activation strategy is the following theorem.

**Theorem 9.7.** *For any graph $G$ and linear ordering $L$ of $V(G)$, if Alice plays the marking game on $G$ using the activation strategy, w.r.t. $L$, then the game's score will be at most $1 + r(L, G)$. In particular, $\mathrm{col}_g(G) \leqslant 1 + r(G)$.*

We defer the proof of Theorem 9.7 to the next section. In the remainder of this section, we show that Theorem 9.7 implies sharp upper bounds on $\mathrm{col}_g(\mathcal{G})$ when $\mathcal{G}$ is the class of forests, chordal graphs, or outerplanar graphs. The same statement holds also for interval graphs, which we consider in Exercise 6. In the next section, after proving Theorem 9.7, we see what upper bound it implies for planar graphs. Although the resulting bound there is not sharp, it is nearly the best known. (The best known bound for planar graphs is only 1 better, and is proved using a refinement of the activation strategy.)

**Corollary 9.8.** *Every forest* $T$ *satisfies* $\mathrm{col}_g(T) \leqslant 4$.

We have already proved this in Corollary 9.4. But here we give an alternate proof, in the unified framework of Theorem 9.7.

*Proof.* We assume $T$ is connected; otherwise, we handle each component separately. Fix an arbitrary vertex $w$ of $T$ and let $L$ be an order of $V(T)$ by non-decreasing distance from $w$. For each vertex $w$, we have $d^+(w) \leqslant 1$. So it suffices to show that also $m(w, L, T) \leqslant 2$ for each $w$. In a matching $M$ from $N^-[w]$ to $V^+(w)$, every edge has its tail at $w$, so $|M| \leqslant 1$. In a matching $N$ from $N^-(w)$ to $V^+[w]$, every edge has its head at $w$, so $|N| \leqslant 1$. Thus, $m(w, L, T) \leqslant |M| + |N| \leqslant 2$. So $r(L, T) \leqslant 1 + 2 = 3$, and $\mathrm{col}_g(T) \leqslant 4$. □

chordal

outerplanar

Recall that a graph $G$ is *chordal* if every cycle in $G$ of length at least 4 has a chord. Equivalently (see Exercise 1.3), $G$ is chordal if there exists a linear ordering $L$ of $V(G)$ such that $N^+[v]$ is a clique for every $v \in V(G)$. And $G$ is *outerplanar*[3] if $G$ can be drawn in the plane with all vertices on the outer face.

**Corollary 9.9.** *If* $G$ *is outerplanar, then* $\mathrm{col}_g(G) \leqslant 7$.

*Proof.* By Remark 9.5 we assume that all face of $G$ are triangles, except the outer one; if not, then we add edges until this is true. Thus, $G$ is chordal. So $G$ has a linear order $L$ of $V(G)$ such that $d^+(w) \leqslant 2$ for all $w$ and $N^+[w]$ is a clique. Fix matchings $M$ and $N$ attaining the maximum in the definition of $r(w, L, G)$. Thus, if $xy \in M \cup N$ and $x \in N^-[w]$, then $w \in N^+[x]$ and $y \in N^+[x]$ and $y \in V^+[w]$, so $y \in N^+[w]$ since $N^+(x)$ is a clique. Let $S := (V(M) \setminus V^+[w]) \cup (V(N) \setminus V^+(w))$. Thus, $S \subseteq N^-[w]$. Now contracting $N^+(w)$ to a new vertex $p$ gives a copy of $K_{2,|S|-2}$, with one part being $\{w, p\}$ and the other part being $S \setminus \{x, w\}$, where $xw \in M$ and $wy \in N$ if such $x$ and $y$ exist. Since $G$ is outerplanar, $G$ is $K_{2,3}$-minor free. Thus, $|M| + |N| \leqslant 4$. So $\mathrm{col}_g(G) \leqslant 1 + r(G) \leqslant 1 + 2 + 4 = 7$. □

**Theorem 9.10.** *If* $\mathcal{Q}$ *is the class of outerplanar graphs, then* $\mathrm{col}_g(\mathcal{Q}) = 7$.

*Proof.* Corollary 9.9 gives the upper bound. Now we construct an outerplanar graph $G$ with
$v_i$ $\mathrm{col}_g(G) \geqslant 7$. We triangulate a 20-cycle $v_1v_2 \cdots v_{20}$ by adding path $v_{20}v_2v_{19}v_3 \cdots v_{12}v_{10}$.
$x_i$ For each $i \in [19] \setminus \{10\}$, add a new vertex $x_i$ adjacent to $v_i$ and $v_{i+1}$; see Figure 9.3, with cycle edges in bold and cycle vertices starting from top left. Let $C := \{v_i : i \in [20]\}$ and

[3]Equivalently, $G$ is outerplanar if and only if $G \vee K_1$ is planar. Here $G \vee H$ denotes the join of graphs $G$ and $H$, formed from their disjoint union by adding all possible edges with one endpoint in $V(G)$ and the other in $V(H)$.

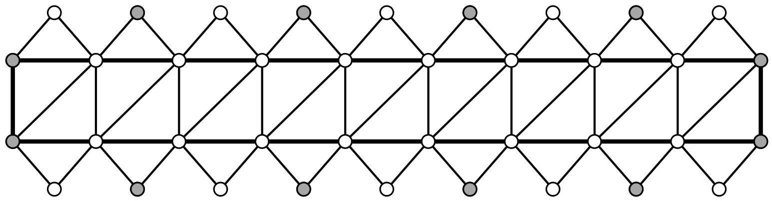


Figure 9.3: An outerplanar graph G with $\mathrm{col}_g(G) = 7$.

$C' := C \setminus \{v_1, v_{10}, v_{11}, v_{20}\}$. Here C is the set of "cycle vertices" and $C'$ excludes those vertices of C with fewer than 4 neighbors in C (so all vertices of $C'$ have exactly 4 neighbors in C). $C, C'$

Bob plays as follows. (1) Let $F := \{v_1, v_{10}, v_{11}, v_{20}\} \cup \{x_{2i} : i \in [9] \setminus \{5\}\}$; vertices of F are shown as gray. As long as there exists an unmarked vertex in F (for "first") Bob marks such a vertex. (2) Let $S := \{x_{2i-1} : i \in [10]\}$. After step (1), as long as there exists an unmarked vertex in S (for "second") Bob marks such a vertex (although we will specify soon how he chooses which one to mark). F

Note that when Bob finishes (1), there are still unmarked vertices in $C'$. This is because $|F| = 4+8 < 20-4 = |C'|$. If some vertex $v$ in $C'$ is the last vertex of G to be marked, then we are done, since $v$ has 4 neighbors in C and another 2 neighbors outside C, for a total of 6. So we may assume instead that Bob can continue marking vertices of S until the end of the game.

At some point on Bob's turn, there exists an unmarked vertex $v$ in $C'$ that is isolated in $G[C']$; this holds because $C'$ is finite, and Bob is never marking vertices in $C'$. Such a vertex $v$ has at most one unmarked neighbor $x_i$ in S, because vertices in S are pairwise at distance greater than 2. If such an $x_i$ exists, then Bob marks $x_i$; otherwise he marks an arbitrary vertex in S. Now whenever $v$ is marked, it will have 6 marked neighbors. □

**Corollary 9.11.** *Every chordal graph* G *satisfies* $\mathrm{col}_g(G) \leqslant 3\omega(G) - 1$.

*Proof.* Every chordal graph G has a vertex order L such that for each vertex $w$, the subset $N^+[w]$ is a clique; thus, $d^+(w) \leqslant \omega(G) - 1$. So we must show that $m(w, L, G) \leqslant 2\omega(G) - 1$. Consider a matching M from $N^-[w]$ to $V^+(w)$. Fix $xy \in M$ with $x \in N^-[w]$ and $y \in V^+(w)$. Since $w, y \in N^+[x]$ and $N^+[x]$ is a clique, also $y \in N^+(w)$. Since $xy$ was arbitrary in M, we get $|M| \leqslant |N^+(w)| \leqslant \omega(G) - 1$. Similarly, if N is a matching from $N^-(w)$ to $V^+[w]$, then each edge in N has its head in $N^+[w]$, so $|N| \leqslant |N^+[w]| \leqslant \omega(G)$. Thus, $r(G) \leqslant \max_{w \in V(G)} d^+(w) + m(w, L, G) \leqslant \omega(G) - 1 + \omega(G) - 1 + \omega(G)$. So $\mathrm{col}_g(G) \leqslant 1 + r(G) \leqslant 3\omega(G) - 1$, as desired. □

**Definition 9.12.** An *interval graph* is a graph G to which each vertex can be assigned an interval on the real line such that two vertices are adjacent in G if and only if their intervals intersect; Figure 9.4 shows an example. Let $\mathcal{I}_k$ denote the set of all interval graphs with clique number k. Let $\mathcal{C}_k$ denote the set of all chordal graphs with clique number k. It is straightforward to check that every interval graph is chordal; that is, $\mathcal{I}_k \subseteq \mathcal{C}_k$.

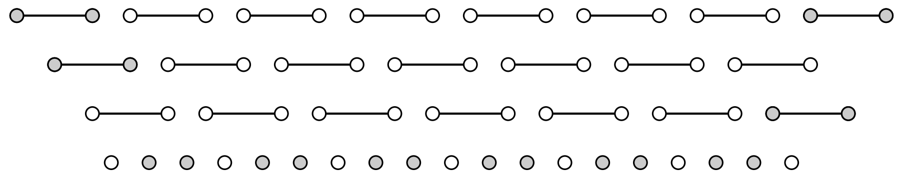


Figure 9.4: An interval graph $G_2$ with $\omega(G_2) = 2 + 1$ and $\mathrm{col}_g(G_2) \geqslant 3(2) + 1 = 7$. Vertices drawn in gray are those that Bob prioritizes marking before he marks any others.

**Theorem 9.13.** *For every positive integer* $k$*, there exists an interval graph* $G_k$ *with clique number* $k+1$ *such that* $col_g(G_k) = 3k+1$*. Thus,* $3k-2 \leqslant col_g(\mathcal{I}_k) \leqslant col_g(\mathcal{C}_k) \leqslant 3k-1$.

*Proof.* The second statement follows from the first, by Corollary 9.11.

The proof of the first statement is very similar to that of Theorem 9.10. Thus, we reuse
$t, v_i$ names from that proof. We give each vertex as its interval. Let $t := (3k+1)(k+1)+1$, let
$C, C'$ $v_i := [i, i+k]$, and let $C := \{v_i : i \in \{1, \ldots, t\}\}$. Let $C' := C \setminus \{v_1, \ldots, v_k, v_{t-k+1}, \ldots, v_t\}$. Let
$w_i, F$ $w_i := [i+1/2, i+1/2]$ such that $k < i < t$. (So each $w_i$ has length 0.) Let $F := \{v_i : i \in$
$S$ $\{1, \ldots, k, t-k+1, \ldots, t\}\} \cup \{w_i : (k+1) \text{ does not divide } i\}$. And let $S := \{w_i : (k+1) \text{ divides } i\}$. Figure 9.4 shows the case when $k = 2$; again $F$ is in gray.

Bob first repeatedly marks vertices of $F$ until they are are all marked, and then he repeated marks vertices of $S$ until they are are all marked. Note that $|F| + 2 = 2k + (t - (k+1)) - (3k + 1) + 2 = t - 2k = t - |C'|$. Thus, when Bob finishes marking $F$, at least two vertices in $C'$ remains unmarked. As before, if the last vertex $v$ to be marked is not in $S$, then $v$ will have $2k$ marked neighbors in $C$ and $k$ more marked neighbors $w_i$. So we assume that Bob can mark vertices in $S$ until the end of the game.

Again, at some point on Bob's turn, the subgraph induced by unmarked vertices in $C'$ will contain an isolated vertex $v$. Such a $v$ has at most one unmarked neighbor $w_i$, since the pairwise distance between vertices $w_i$ excluded from $F$ is at least 3. If such a $w_i$ exists, then Bob marks it. Now whenever $v$ is marked, it has $2k$ marked neighbors in $C'$ and $k$ marked neighbors $w_j$, for a total of $3k$ marked neighbors; this proves the theorem. $\square$

**Remark 9.14.** The lower bound on $\mathrm{col}_g(\mathcal{I}_k)$ in Theorem 9.13 is sharp; see Exercise 6. But the lower bound on $\mathrm{col}_g(\mathcal{C}_k)$ is not. The construction proving this is similar to that proving Theorem 9.13, but more involved, so we direct the interested reader to [417, Theorem 3].

## 9.2 Proving Theorem 9.7 and Coloring Planar Graphs

We begin this section by proving Theorem 9.7, which we used earlier to give sharp or nearly sharp upper bounds on $\mathrm{col}_g$ for forests, outerplanar graphs, interval graphs ($\mathcal{I}_k$), and chordal graphs ($\mathcal{C}_k$). For easy reference, we restate the theorem.

**Theorem 9.7.** For any graph $G$ and linear ordering $L$ of $V(G)$, if Alice plays the marking game on $G$ using the activation strategy, w.r.t. $L$, then the game's score will be at most $1 + r(L, G)$. In particular, $\mathrm{col}_g(G) \leqslant 1 + r(G)$.

Recall that $A$ is the set of activated vertices (updated throughout the game), and that $r(L, G) := \max_{v \in V(G)} r(v, L, G) = \max_{v \in V(G)}\{d^+_{\vec{G}_L}(v) + m(v, L, G)\}$. As above, $m(v, L, G) :=$ $\max |M| + |N|$, where $M$ (resp. $N$) is a matching from $N^-(v)$ (resp. $N^-[v]$) to $V^+[v]$ (resp. $V^+(v)$) and $V(M) \cap V(N) \cap N^-(v) = \emptyset$; that is, the restrictions of $M$ and $N$ to $N^-(v)$ are disjoint. $A$ $r(L, G)$ $m(v, L, G)$ $M, N$

When a vertex $v$ is being colored, clearly its number of colored outneighbors is at most $d^+_{\vec{G}_L}(v)$. We need 1 new color for $v$, so our main task is proving that $|N^-(v) \cap A| \leqslant m(v, L, G)$. Our plan is to partition $N^-(v) \cap A$ into two parts $X$ and $Y$ and construct matchings from each of $X$ and $Y$ (saturating $X$ and $Y$) onto $N^+[v]$. This essentially proves the theorem, although numerous details must be provided.

*Proof.* Recall the function $s(\cdot)$ from Algorithm 9.1; we call it the *successor* function, and it plays a crucial role in our proof. Let $X := \{w : w \text{ is activated before } s(w)\}$ and let $Y := \{w : w \text{ is activated after } s(w)\}$. Note that $X$ and $Y$ need not partition $V(G)$, since $X \cup Y$ omits each vertex $x$ such that $s(x) = x$. However, if $v \in U$ and $x \in N^-(v) \cap A$, then $s(x) \neq x$ (since $v \in N^+[x]$ and $x >_L v$); thus, $x \in X \cup Y$. Our first step is proving the following claim. $s(\cdot)$ $X$ $Y$

**Claim 1.** *The function* $s(\cdot)$ *is injective when restricted to* $X$ *and when restricted to* $Y$.

<u>Proof.</u> Consider $x \in X$. At some point Alice activates $x$, and then immediately activates $s(x)$. So there cannot exist $x' \in X$ with $x' \neq x$ but $s(x') = x$, since there is no time to activate $x'$ between activating $x$ and $s(x)$. Thus, $s(\cdot)$ is injective when restricted to $X$.

Similarly, consider $y \in Y$. After Alice activates $y$, she immediately visits $s(y)$, which is already activated, by the definition of $Y$. So Alice marks $s(y)$, since it is activated, removing the possibility that $s(y') = s(y)$ for some $y' \in Y$ with $y' \neq y$. Thus, $s(\cdot)$ is injective when restricted to $Y$. $\diamondsuit$

We consider two matchings $M$ and $N$ as follows.

$$M := \{(x, s(x)) : x \in X \cap N^-(v) \cap A\}$$
$$N := \{(y, s(y)) : y \in Y \cap N^-(v) \cap A\}$$

Let $X' := X \cap N^-(v) \cap A$ and let $Y' := Y \cap N^-(v) \cap A$. Now $|N^-(v) \cap A| = |X'| + |Y'| = |M| + |N|$, and $X' \cap Y' = \emptyset$. So we are done unless there exist $x \in X'$ and $y \in Y'$ with $s(x) = v = s(y)$. This creates a problem, since both $M$ and $N$ are matchings onto $N^+[v]$, but in the definition of $m(v, L, G)$ one of the matchings must be from $N^-[v]$ to $N^+(v)$. In this case, we slightly modify one of mathcings $M$ and $N$, as follows.

Since $s(x) = v$ and also $s(y) = v$, we cannot have $s(v) = v$. (If so, then the first time that Alice visited $v$, immediately after $x$, she would mark $v$, excluding the possibility that also

$s(y) = v$.) Thus, $v \in X \cup Y$. If $v \in X$, then let $M' := M - xv \cup \{(v, s(v))\}$. Now $M'$ matches $X' \setminus \{x\} \cup \{v\}$ onto $V^-(v)$, so matchings $M'$ and $N$ suffice. If instead $v \in Y$, then let $N' := N - yv \cup \{(v, s(v))\}$. Now $N'$ matches $Y' \setminus \{y\} \cup \{v\}$ onto $V^-(v)$, so matchings $M$ and $N'$ suffice. By the definition of $m(v, L, G)$, this proves the theorem. □

**Theorem 9.15.** *Every planar graph* $G$ *satisfies* $col_g(G) \leqslant 18$.

Our plan is to build a linear order $L$ of $V(G)$ such that $r(L, G) \leqslant 17$. Then we will be done by Theorem 9.7. At each step we choose as $v_i$, the next vertex in $L$, some vertex with $r(v_i, L, G)$ small. To find $v_i$ we use discharging. We naturally first try—as we often do when coloring inductively, by discharging—deleting each $v_i$ after it is chosen. But of course this attempt fails, since deleting $v_i$ decreases (incorrectly) the value of $r(v_j, L, G)$ for various unchosen $v_j$.

So instead after a vertex $v_i$ is put into $L$, we try keeping $v_i$ in $G$, but we must ensure that $v_i$ finishes with enough charge to avoid serving as the reducible configuration in $G$ that is "found" by the discharging (since it was already put into $L$, and cannot be put in again). We let $U$ be the set of "unchosen" vertices, those not yet added to $L$. When $v$ has already been added to $L$ and $d_U(v) \geqslant 4$, finding sufficient charge for $v$ is feasible. But when $d_U(v) \leqslant 3$, this is too difficult. So in this case we do delete $v$, but also add edges to ensure that $r(v_j, L, G)$ does not decrease for any unchosen vertex $v_j$.

*Proof.* We build our linear order $L$ inductively, in reverse. We have a subset $C$ of vertices that have already been put into $L$ (so-called "chosen" vertices); initially $C = \emptyset$ and always $U := V(G) \setminus C$ (vertices of $U$ are "unchosen"). Suppose that we have chosen $v_n, v_{n-1}, \ldots, v_{i+1}$, and now we must choose $v_i$. We construct from graph $G$ a new graph $H$ as follows.

(1) Delete all edges with both endpoints in $C$.

(2) Delete all vertices in $C$ with at most 3 neighbors in $U$.

(3) For each vertex $w$ deleted in (2), add an edge between each pair of vertices in $U \cap N_G(v)$, if the edge is not yet present.

Note that $H$ is a (simple) plane graph, inheriting its embedding from $G$.

Let $C' := \{v \in C : d_H(v) \geqslant 4\}$. For each $v \in U$, note that $N_U(v)$ and $N_{C'}(v)$ partition $N_H(v)$. Denote the sizes of these sets by $d_U(v)$ and $d_{C'}(v)$. For each $v \in C'$, we have $d_H(v) = d_U(v)$. And for each $v \in U$, we have $d_H(v) = d_U(v) + d_{C'}(v)$. We give each $v \in V(H)$ initial charge $d_H(v)$, and use the following single discharging rule:

(R1) Each vertex $v \in C'$ takes charge 0.5 from each neighbor in $U$.

Denote the resulting charge of each vertex $w$ by $ch^*(w)$. The sum of the initial charges in $H$ is $\sum_{v \in V(H)} d_H(v) = 2|E(H)| < 6|V(H)|$. Since the sum is unchanged by discharging, there exists a vertex $x$ such that $\mathrm{ch}^*(x) < 6$. Namely, $\mathrm{ch}^*(x) \leqslant 5.5$. For each vertex $v \in C'$, we have $\mathrm{ch}^*(v) = 1.5d_U(v) \geqslant 6$. Thus, $x \in U$. We will let $v_i := x$. So it suffices to verify that $r(x, L, G) = d^+_{\vec{G}_L}(x) + m(x, L, G) \leqslant 17$.

From our discharging, we know that

$$\begin{aligned} 5.5 \geqslant \mathrm{ch}^*(x) &= d_H(x) - 0.5d_{C'}(x) \\ &= d_U(x) + 0.5d_{C'}(x). \end{aligned}$$

Multiplying this inequality by 3 gives $16.5 \geqslant 3d_U(x)+1.5d_{C'}(x)$. Thus, $16 \geqslant 3d_U(x)+d_{C'}(x)$. We have not yet fully constructed L, but we do know that $d^+_{\vec{G}_L}(x) = d_U(x)$. So it suffices to show that

$$m(x, L, G) \leqslant 1 + 2d_U(x) + d_{C'}(x). \tag{9.1}$$

We bound $m(x, L, G)$ as follows. Consider matchings $M$ from $N^-(x)$ to $V^+[x]$ and $N$ from $N^-[x]$ to $V^+(x)$ such that $V(M) \cap V(N) \cap N^-(V) = \emptyset$ and $|M| + |N| = m(x, L, G)$. See Figure 9.5. Let $M' := M - \{yx \in M\}$. Now consider an arbitrary edge $yz \in M' \cup N$ with $y \in N^-[x]$. If $y \in V(H)$, then edge $yz$ is counted by $d_{C'}(x)$, since $V(M) \cap V(N) \cap N^-(v) = \emptyset$. But if $y \notin V(H)$, then $y$ was deleted in step (2) while constructing H. So in step (3), we added edge $xz$. Thus, edge $yz$ is counted by $2d_U(x)$, since possibly $V(M') \cap V(N) \cap N^+(x) \neq \emptyset$. As desired, this gives $m(x, L, G) = |M| + |N| \leqslant 1 + |M'| + |N| \leqslant 1 + (2d_U(x) + d_{C'}(x))$. Combining this with the argument above gives

$$r(x, L, G) = d_U(x) + m(x, L, G) \leqslant d_U(x) + 1 + (2d_U(x) + d_{C'}(x)) \leqslant 17. \qquad \square$$

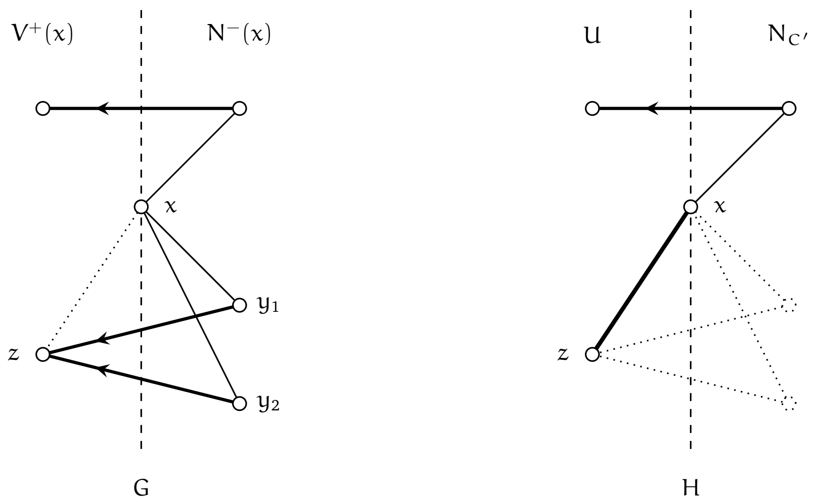



Figure 9.5: The proof of inequality (9.1). Edges of $M \cup N$ that persist (top) when creating H from G get counted by $d_{C'}(x)$. Edges of $M \cup N$ that are deleted (bottom) when creating H from G get counted by $2d_U(x)$. Possibly two deleted edges, one each from M and N, get charged to the same neighbor $z$ of $x$ in H.

## 9.3 The Harmonious Strategy

In this section we consider a variation of the activation strategy known as the harmonious strategy. (This name refers to the balance the strategy must strike between marking a vertex $v$ sooner, which makes more trouble for neighbors of $v$, and marking $v$ later, which makes more trouble for $v$ itself.) We present two examples of its usefulness. The first bounds $\mathrm{col}(G^2)$ in terms of $\mathrm{mad}(G)$ and $\Delta(G)$. The second considers an asymmetric version of the marking game that we saw earlier.

### 9.3.1 The Degeneracy of Squares for Graphs with Bounded Mad

**Theorem 9.16.** *If* $k \in \mathbb{Z}^+$ *and* $mad(G) \leqslant 2k$, *then* $col(G^2) \leqslant (2k-1)\Delta(G) + 2k + 1$.

Before proving the theorem, we first present a proof sketch. Since $\mathrm{mad}(G) \leqslant 2k$, Lemma 5.4 guarantees that $G$ has an orientation $\vec{G}$ with $\Delta^+(\vec{G}) \leqslant k$. We will mark the vertices in $V(G)$, and must show that at the time each vertex is marked it has at most $(2k-1)\Delta(G) + 2k$ of its neighbors in $G^2$ already marked.

$N^+(v)$ $N^-(v)$ $N^{++}(v)$ $N^{+-}(v)$ $N^{-+}(v)$ $N^{--}(v)$

Given an orientation $\vec{G}$, we denote by $N^+(v)$ and $N^-(v)$, the sets of all outneighbors and inneighbors of $v$. We extend this notation as follows. Whenever $a, b \in \{+, -\}$, let $N^{ab}(v)$ denote the vertices reachable from $v$ by a walk of length 2, where the first (resp. second) edge of $v$ is traversed as oriented if $a = +$ (resp. $b = +$) and otherwise is traversed opposite its orientation. (So, for example $v \in N^{+-}(v) \cap N^{-+}(v)$ whenever $d^+(v) \geqslant 1$ and $d^-(v) \geqslant 1$. But $v \notin N^{++}(v) \cup N^{--}(v)$ except when $v$ lies on a directed 2-cycle.)

Consider a vertex $v$ and its set $U_v$ of unmarked neighbors in $G^2$ at the time that we mark $v$. Note that $N_{G^2}(v) \subseteq N^+(v) \cup N^-(v) \cup N^{++}(v) \cup N^{+-}(v) \cup N^{-+}(v) \cup N^{--}(v)$. It is straightforward to bound the size of the union of all sets but the last by some value less than $(2k-1)\Delta(G)$. In fact, we cannot even necessarily bound $|N^{--}(v)|$ by less than $\Delta(G)^2$, which is far too big. Fortunately, it is enough to bound $|N^{--}(v) \setminus U_v|$. So our strategy is designed to focus on minimizing this quantity; more precisely, we aim to minimize its maximum over all $v$.

$\vec{G}^2$

Form $\vec{G}^2$ from $\vec{G}$ by adding, for each directed path $vwx$ the directed edge $\vec{vx}$. We walk through $\vec{G}^2$ following each arc at most once. (Each time that we traverse an arc, we delete it.) When we reach a vertex $v$ with outdegree at most 1, we mark $v$. If $v$ has outdegree 1, then we continue on to its outneighbor $w$, deleting arc $\vec{vw}$. Otherwise, we pick a new vertex and continue our walk from there.

The key idea is that if a vertex $w \in N^-(v) \cup N^{--}(v)$ is marked before $v$, then our walk traversed the edge $\vec{wv}$ and hence also traversed some out-edge from $v$ in $\vec{G}^2$. Thus, the number of such vertices is at most $|N^+(v) \cup N^{++}(v)| \leqslant k + k^2$. Now we provide the details.

*Proof.* Fix an arbitrary linear order $L$ on $V(G)$. Let $U$ be the set of unmarked vertices; initially, $U := V(G)$ and $U$ updates automatically, as we mark vertices. The bound on $\mathrm{col}(G)$ increases with $k$, so we assume $2k - 2 < \mathrm{mad}(G)$; otherwise, let $k := k - 1$. In particular, $\Delta(G) \geqslant k$. Fix an orientation $\vec{G}$ of $G$ with $\Delta^+(\vec{G}) \leqslant k$. Now we run Algorithm 9.2.

**Algorithm 9.2:** Marking vertices to show that $\mathrm{col}(G^2) \leqslant \Delta(G)(2k-1)+2k+1$

**Input :** An orientation $\vec{G}$ of a graph $G$ such that $\Delta^+(\vec{G}) \leqslant k$ and a linear order $L$ of $V(G)$.

**Output:** A vertex order witnessing that $\mathrm{col}(G^2) \leqslant \Delta(G)(2k-1)+2k+1$.

$v := \min_L U$ and $\vec{G}_{\mathrm{curr}} := \vec{G}^2$<br>
**while** $U \neq \emptyset$<br>
    **while** $N^+_{\vec{G}_{\mathrm{curr}}}(v) \cap U \neq \emptyset$<br>
        **let** $w := \min_L N^+_{\vec{G}_{\mathrm{curr}}}(v) \cap U$<br>
        **delete** $vw$<br>
        **if** $N^+_{\vec{G}_{\mathrm{curr}}}(v) = \emptyset$<br>
            **mark** $v$<br>
        $v := w$<br>
    $v := \min_L U$<br>
**output** order in which vertices were marked

As before, let $U_v$ be the set of neighbors of $v$ in $G^2$ that are not marked before $v$.

**Claim 1.** *For each* $v \in V(G)$, *we have* $|(N^-(v) \cup N^{--}(v)) \setminus U_v| \leqslant |N^+(v) \cup N^{++}(v)| \leqslant k^2 + k$.

Proof. Fix $v \in V(G)$ and consider a vertex $w \in N^-(v) \cup N^{--}(v)$ that is marked before $v$. Before (or immediately after) $w$ is marked, our walk visits $v$. Each time that $v$ is visited but not marked, the value of $N^+_{\vec{G}_{\mathrm{curr}}}(v) \cup N^{++}_{\vec{G}_{\mathrm{curr}}}(v)$ decreases by 1. This value starts at $N^+_{\vec{G}^2}(v) \cup N^{++}_{\vec{G}^2}(v)$ and always stays nonnegative. This proves the first inequality. The second inequality holds because $\Delta^+(\vec{G}) \leqslant k$. ♢

The rest of the proof is just (careful) algebra.

$$\begin{aligned}
|N_{\vec{G}^2}(v) \setminus U_v| &= |(N^+(v) \cup N^-(v) \cup N^{++}(v) \cup N^{+-}(v) \cup N^{-+}(v) \cup N^{--}(v)) \setminus U_v| \\
&\leqslant |(N^-(v) \cup N^{--}(v)) \setminus U_v| + |(N^+(v) \cup N^{++}(v) \cup N^{+-}(v)| + |N^{-+}(v)| \\
&\leqslant k^2 + k + d^+_{\vec{G}}(v)\Delta(G) + (\Delta(G) - d^+_{\vec{G}}(v))(k-1) \qquad (9.2) \\
&= k^2 + k + d^+_{\vec{G}}(v)(\Delta(G) - k + 1) + \Delta(G)(k-1) \\
&\leqslant k^2 + k + k(\Delta(G) - k + 1) + \Delta(G)(k-1) \\
&= \Delta(G)(2k-1) + 2k.
\end{aligned}$$

Each step in the chain of inequalities is straightforward, except for (9.2), which follows directly from Claim 1. This completes the proof. □

**Remark 9.17.** We mention that, for all integers $k$ and $D$ with $D \geqslant k \geqslant 2$ there exist graphs $H_{D,k}$ that are $k$-degenerate and have maximum degree $D$ but such that $H^2_{D,k}$ is not $((2k -$

$1)D - k^2 - 1)$-degenerate. Thus, the coefficient on $D$ in Theorem 9.16 is best possible. We briefly sketch the construction when $k = 2$; we leave the details, as well as the generalization to larger $D$, to Exercise 4.

Start with a $D$-regular graph and subdivide each edge once; call these newly created vertices $S$. For each vertex $v$ in $S$, add edges to (exactly) $D - k$ other vertices in $S$ with which $v$ has no common neighbor. Finally, subdivide each of these newly added edges. It is easy to check that this construction succeeds when the order of the original $D$-regular graph is sufficiently large and satisfies certain divisibility criteria.

### 9.3.2 The Asymmetric Marking Game

The Marking Game that we considered in Sections 9.1 and 9.2 has a natural asymmetric generalization. On each turn, Alice marks $a$ vertices, and then Bob marks $b$ vertices, for some positive integers $a$ and $b$. Otherwise, the rules are the same. The $(a, 1)$-game coloring number,
$(a,1)\text{-col}_g(G)$ denoted $(a, 1)$-$col_g(G)$, is the least integer $s$ such that Alice has a strategy to win with $s$ "colors" (that is, when each vertex is marked, it has at most $s - 1$ marked neighbors). In this section, we consider this asymmetric game when $b = 1$, although we remark briefly about the more general case at the end. We prove the following.

**Theorem 9.18.** *Let $a$ and $k$ be positive integers such that $k \leqslant a$. If $G$ is a graph with $mad(G) \leqslant 2k$, then $(a, 1)$-$col_g(G) \leqslant 2k + 2$. More specifically, Alice is guaranteed to win with $2k + 2$ colors by playing the strategy in Algorithm 9.3.*

The strategy in Algorithm 9.3 is another variation of the harmonious strategy that we saw in the previous subsection. Informally, there are a few main differences. (1) Alice is walking along edges in $G$, rather than in $G^2$. (2) Every time that Alice marks a vertex, she restarts her walk from vertex $v_B$, the one most recently marked by Bob. (3) Every time that Alice marks $a$ vertices, she must stop and let Bob take another turn. Otherwise, this version is quite similar to the version we saw above. Figure 9.6 shows an example of Alice using this strategy to play the $(2, 1)$-marking game.

*Proof of Theorem 9.18.* Since $mad(G) \leqslant 2k$, Lemma 5.4 guarantees there exists an orientation $\vec{G}$ with $\Delta^+(\vec{G}) \leqslant k$. Fix an arbitrary linear order $L$ of $V(G)$. We will show that when Alice plays according to the harmonious strategy given by Algorithm 9.3, at every point in time each unmarked vertex has at most $2k+1$ marked neighbors in $G$ (not just in $\vec{G}_{curr}$). This immediately implies the theorem. It is convenient to prove the following claim.

**Claim 1.** *Each time after Alice marks a vertex $v$, the following invariants all hold. (i) For each unmarked vertex $x$, the numbers of deleted arcs of the forms $\overrightarrow{wx}$ and $\overrightarrow{xy}$ are equal. (ii) $N^+_{\vec{G}_{curr}}(v) \cap U = \emptyset$. (iii) If Alice has completed her turn, then every marked vertex $x$ satisfies $N^+_{\vec{G}_{curr}}(x) \cap U = \emptyset$.*

Proof. Before each visit to $x$ an inedge is deleted and after each visit an outedge is. This proves (i). If $N^+_{\vec{G}_{curr}}(v) \cap U \neq \emptyset$, then the while at line 7 continues and $w$ is not marked at line 11. This proves (ii). If $x$ was marked by Alice, then (iii) holds by (ii). Otherwise, $x = b$. Recall that $d^{+}_{\vec{G}}(b) \leqslant k$. Either $U = \emptyset$, so (iii) holds, or $i = a$. In the latter case, (iii) holds by lines 3–5. $\diamondsuit$

Now we consider an unmarked vertex $x$. Consider the situation after the last vertex $v$ marked by Alice. Each vertex marked by Alice has no unmarked outneighbors in $\vec{G}_{\mathrm{curr}}$, by (ii). And also each vertex marked by Bob, except possibly the one marked most recently, has no unmarked outneighbors in $\vec{G}_{\mathrm{curr}}$, by (iii). Combining these facts with (i), and the fact that $d^{+}_{\vec{G}} \leqslant k$, shows that $x$ has at most $k+1$ marked inneighbors, including the one marked most recently by Bob. Since $x$ has at most $k$ outneighbors, in total $x$ has at most $2k+1$ marked neighbors, as claimed. This proves the theorem. $\square$

Finally, we briefly consider the general $(a, b)$-game coloring number, as promised.

**Corollary 9.19.** *Let $a$, $b$, and $k$ be positive integers such that $k \leqslant a/b$. If $G$ is a graph with mad$(G) \leqslant 2k$, then $(a, b)$-col$_g(G) \leqslant 2k + b + 1$.*

*Proof.* The proof is essentially the same as for Theorem 9.18. Suppose that Bob plays $b$ moves in a row. Now Alice spends $k$ of her moves responding to Bob's first move, followed by $k$ responding to Bob's second move, etc. The main difference is that when Alice marks an unmarked vertex, it might have $b$ neighbors that have all been marked by Bob since the last time that Alice completed a turn, rather than only 1 neighbor marked by Bob, as above. Thus, the term "2" in Theorem 9.18 is replaced by "$b + 1$". $\square$

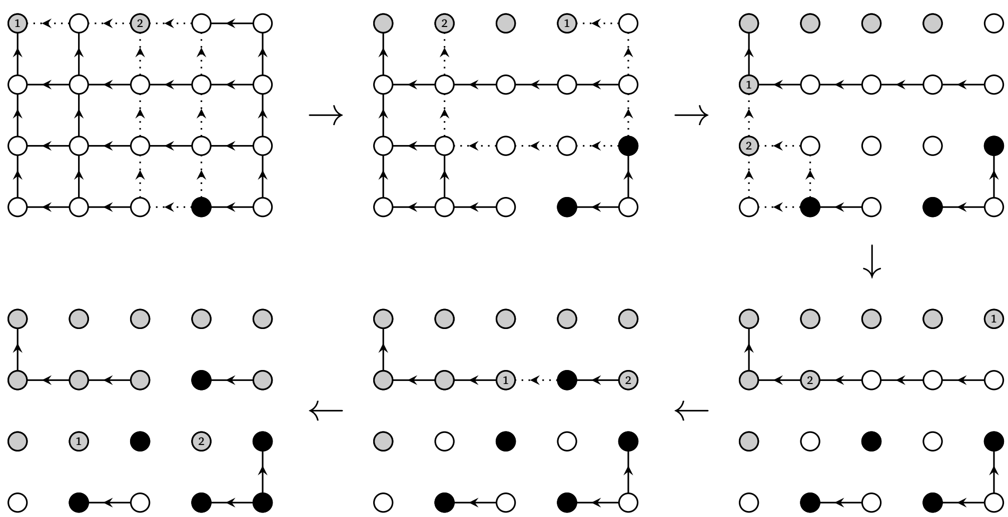


Figure 9.6: A $(2, 1)$-marking game played on the $4 \times 5$ grid, where Alice uses the harmonious strategy. Vertices are ordered top-to-bottom and, subject to that, left-to-right. Bob's moves are Black, and Alice's moves are Ash, with 1 and 2 denoting her 1st and 2nd move on each turn. Dotted paths denote the edges deleted by Alice. (For clarity, Bob starts.)

**Algorithm 9.3:** Moves for Alice in the $(a, 1)$-Marking Game

**Input :** Bob's previous move $v_B$, orientation $\vec{G}$ s.t. $\Delta^+(\vec{G}) \leqslant k$, and linear order $L$ of $V(G)$.

**Output:** Moves for Alice in the $(a, 1)$-Marking Game

**let** $i := 0$
**while** ($U \neq \emptyset$ and $i < a$)
  **if** $N^+_{\vec{G}_{\text{curr}}}(b) \cap U \neq \emptyset$
    **let** $w := \min_L N^+_{\vec{G}_{\text{curr}}}(b) \cap U$
    **delete** $vw$
  **else let** $w := \min_L U$
  **while** $N^+_{\vec{G}_{\text{curr}}}(w) \cap U \neq \emptyset$
    **let** $x := \min_L N^+_{\vec{G}_{\text{curr}}}(w) \cap U$
    **delete** $wx$
    **let** $w := x$
  **mark** $w$ (since $w$ now has no outneighbors)
  **let** $i := i + 1$
**output** set of vertices marked by Alice

## 9.4 The Defective Coloring Game

In this final section, we return to the coloring game, with which we started the chapter. But there is a twist. Rather than constructing a proper coloring, Alice and Bob now construct a defective (not necessarily proper) coloring.

defect

$d$-relaxed proper $r$-coloring

$d$-relaxed chromatic number

$d$-$\chi(G)$

Given a coloring $\varphi$ of a graph $G$, the *defect*, $\text{def}_\varphi(v)$, of a vertex $v$ is the degree of $v$ in the subgraph induced by vertices colored with $\varphi(v)$. And $\text{def}_\varphi(G) := \max_{v \in V(G)} \text{def}_\varphi(v)$. A $d$-*relaxed proper* $r$-*coloring* is an $r$-coloring $\varphi$ of $G$ such that $\text{def}_\varphi(G) \leqslant d$. So a 0-relaxed proper $r$-coloring is simply a proper $r$-coloring. The $d$-*relaxed chromatic number* of $G$, denoted $d$-$\chi(G)$, is the minimum $r$ such that $G$ has a $d$-relaxed proper $r$-coloring.

$(r, d)$-relaxed coloring game

d-relaxed game chromatic number

In the $(r, d)$-*relaxed coloring game*, Alice and Bob jointly construct a $d$-relaxed proper $r$-coloring. As before, the players alternate turns coloring the vertices of $G$ from $[r]$, and after each turn the subgraph induced by each color class must have maximum degree at most $d$. Alice wins if eventually all vertices are colored. And Bob wins if eventually some vertex $v$ is uncolored, but coloring $v$ with each color $i$ creates a vertex of degree at least $d + 1$ in the subgraph induced by vertices colored $i$. Analogous to the non-game problem, the $d$-*relaxed game chromatic number*, $d$-$\chi_g(G)$, of a graph $G$ is the minimum $r$ such that Alice has a strategy on $G$ to win the $(r, d)$-relaxed coloring game.

Given a graph $G$, it is natural to ask for which pairs $(r, d)$ Alice can always win the $(r, d)$-relaxed coloring game; that is, for which pairs $(r, d)$ is $d$-$\chi_g(G) \leqslant r$. More concretely, if we fix r (resp. d), we ask for the minimum d (resp. r) such that $d\text{-}\chi_g(G) \leqslant r$. For each single graph G, both questions have finite answers, since $0\text{-}\chi_g(G) \leqslant \Delta(G) + 1$ and $\Delta(G)\text{-}\chi_g(G) \leqslant 1$. But, as with the marking game, we typically ask about the worst case (maximum) over some infinite family of graphs. Given a graph class $\mathcal{G}$, we want to determine the minimum r such that there exists some d for which $d\text{-}\chi_g(G) \leqslant r$ for all $G \in \mathcal{G}$. In this case, we say that $d\text{-}\chi_g(\mathcal{G}) \leqslant r$. $d\text{-}\chi_g(\mathcal{G})$

**Remark 9.20.** It is important to note that when (Alice or Bob is) attempting to color a vertex $v$ with a color $\alpha$ in the d-relaxed coloring game, the move is legal if and only if both

(i) $v$ has at most d neighbors colored $\alpha$; and

(ii) each neighbor of $v$ colored $\alpha$ has at most $d - 1$ neighbors colored $\alpha$.

It is this second requirement that makes analyzing this game much different than analyzing the coloring (or marking) game.

The following example may be surprising.

**Example 9.21.** Recall that $\chi_g(K_{n,n}) = 3$, as follows. Let X and Y denote the parts of $K_{n,n}$. By symmetry, we assume that Alice starts by using color 1 on X. For the lower bound, note that Bob can respond with 2 on X, rendering colors 1 and 2 unavailable on Y. Now we prove the upper bound. Regardless of Bob's first move, Alice next uses color 2 or 3 on Y. Now all of X can be colored with 1. And all of Y can be colored with 2 or 3. Thus, $\chi_g(K_{n,n}) \leqslant 3$.

In contrast, $1\text{-}\chi_g(K_{n,n}) = n$, as follows. The upper bound follows from coloring greedily. Now we prove the lower bound. Each time that Alice uses a color $\alpha$ on one part, Bob immediately replies by using $\alpha$ on the other part, thus rendering it unavailable for all remaining vertices. Hence, $1\text{-}\chi_g(K_{n,n}) \geqslant |V(K_{n,n})|/2 = n$. ◊

So in general $d\text{-}\chi_g(G)$ is not weakly decreasing as d grows. But most of our arguments will prove constant upper bounds on $d\text{-}\chi_g(\mathcal{G})$ that hold for all sufficiently large values of d.

### 9.4.1 Partial k-trees

Recall that a k-*tree* is a graph with a vertex order $v_1, \dots, v_n$ such that $v_1, \dots, v_k$ is a clique and for all $i > k$ the set $N(v_i) \cap \{v_1, \dots, v_{i-1}\}$ is also a clique $K_k$. A *partial* k-*tree* is a subgraph of a k-tree. For example, each tree is a 1-tree and each forest is a partial 1-tree. Below we sketch Alice's strategy to play the defective coloring game on a k-tree. Alice's strategy to play on a partial k-tree G will be to play on a supergraph H such that H is a k-tree. Once we describe her strategy in more detail, it is straightforward to check that this approach succeeds. k-tree partial k-tree

Each k-tree is chordal, so has a simplicial elimination order L. Note that $\Delta^+(\overrightarrow{G_L}) = k$. (Recall that $\overrightarrow{G_L}$ orients each edge toward its endpoint appearing earlier in L.) When picking a vertex $v$ to color, it is natural for Alice to use the activation strategy. But how does she choose a color for $v$? To play on a k-tree using $k+1$ colors, Alice always avoids using a color $\alpha$ on a vertex $v$ if $\alpha$ is already used on some outneighbor of $v$.

The activation strategy guarantees that at most $2k+1$ inneighbors of $v$ are colored before $v$. This ensures that (1) in Remark 9.20 holds when $d \geqslant 2k+1$. But ensuring (2) in that remark is more challenging. In fact, to do so we must extend the activation strategy.

Intuitively, we are concerned that some vertex $v$ colored $\alpha$ will have many of its inneighbors colored $\alpha$ by Bob, after $v$ is colored. Perhaps we will reach a partial coloring where $\mathrm{def}_\varphi(v) = d$. Now Alice will want to color some outneighbor $w$ with $\alpha$, but this will be forbidden. Our solution is to also consider $w$ as a candidate for where Alice starts her walk after Bob colors each inneighbor of $v$.

Alice's walk may not start at $w$ the first time that Bob colors such an inneighbor of $v$. But $v$ has at most $k$ outneighbors, each of which will start a walk at most twice, being colored immediately on the second time. Thus, Bob can color at most $2k+1$ inneighbors of $v$ with $\alpha$, after $v$ is colored (but before all of its outneighbors are colored). As noted above, $v$ has at most $2k-1$ inneighbors colored $\alpha$ before $v$ is colored. Since $d^+_{\overrightarrow{G_L}}(v) \leqslant \Delta^+(\overrightarrow{G_L}) \leqslant k$, we thus get $\mathrm{def}_\varphi(v) \leqslant (2k+1)+(2k-1)+k = 5k$. This approach allows us to prove the following.[4]

**Theorem 9.22.** *If* $G$ *is a partial* $k$*-tree, then* $5k$-$\chi_g(G) \leqslant k+1$. *Furthermore,* $d$-$\chi_g(G) \leqslant k+1$ *for all* $d \geqslant 5k$.

With more careful analysis, we can weaken the final hypothesis to $d \geqslant 4k-1$. Clearly, forests are partial 1-trees. And it is easy to check that outerplanar graphs are partial 2-trees. This implies the following.

**Corollary 9.23.** *The following both hold:*

*(a) If* $G$ *is a forest, then* $d$-$\chi_g(G) \leqslant 2$ *whenever* $d \geqslant 3$.

*(b) If* $G$ *is outerplanar, then* $d$-$\chi_g(G) \leqslant 3$ *whenever* $d \geqslant 7$.

We would like to prove that all planar graphs are partial $k$-trees for some fixed $k$. Unfortunately, this is false. The minimum $k$ such that a graph $G$ is a partial $k$-tree is called the *treewidth* of $G$. And it is well-known that there exist planar graphs with arbitrarily large treewidth. (For example, the grid $P_n \square P_n$ has treewidth $n$; see Exercise 7.) Thus, to handle all planar graphs, we need a new idea, pseudo partial $k$-trees. The definition, given below, is rather technical.

So why do these graph classes matter? In Theorem 9.25, which is the most important result in Section 9.4, we show that if $G$ is an $(a,b)$-pseudo partial $k$-tree, then for sufficiently large $d$ we have $d$-$\chi_g(G) \leqslant k+1$. And in Lemma 9.30, we prove that every planar graph $G$ is a $(3,8)$-pseudo partial 2-tree. Thus, $d$-$\chi_g(G) \leqslant 3$ for every planar $G$ and every sufficiently large $d$. We also get stronger bounds for outerplanar graphs, as well as getting constant bounds for the graphs embeddable in each fixed surface.

[4]We omit the additional details needed to prove Theorem 9.22, because we are most interested in Corollary 9.23, which is strengthened (for $d \geqslant 30$) by Corollary 9.28.

**Definition 9.24.** Let $a$ and $b$ be integers such that $0 \leqslant a \leqslant b$. A connected graph $G$ is an $(a, b)$-*pseudo chordal graph* if there exist two oriented graphs, $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$, both on vertex set $V(G)$, with $\overrightarrow{E_1} := E(\overrightarrow{G_1})$ and $\overrightarrow{E_2} := E(\overrightarrow{G_2})$ such that the following 4 conditions hold:

$(a, b)$-pseudo chordal graph

(1) $E_1$ and $E_2$ partition $E(G)$, where $E_i$ is formed from $\overrightarrow{E_i}$ by omitting orientations.

(2) $\overrightarrow{G_1}$ is acyclic.

(3) $\Delta^+(\overrightarrow{G_2}) \leqslant a$ and $\Delta(\overrightarrow{G_2}) \leqslant b$.

(4) For each $v \in V(G)$, the vertex subset $N^+_{\overrightarrow{G_1}}(v)$ induces a transitive tournament in $\overrightarrow{G_1} \cup \overrightarrow{G_2}$

A graph $G$ is an $(a, b)$-*pseudo partial* $k$-*tree* if $G$ is a subgraph of an $(a, b)$-pseudo chordal graph in which the directed graph $\overrightarrow{G_1}$ satisfies $\Delta^+(\overrightarrow{G_1}) \leqslant k$.

$(a, b)$-pseudo partial $k$-tree

Now we can state the central result of Section 9.4.

**Theorem 9.25.** *Let $G$ be an $(a, b)$-pseudo partial $k$-tree and let* $f(a, b, k) = 2k^2 + 3k + 2ak + 2bk + 2ab + 3b + 2$. *Now* $d\text{-}\chi_g(G) \leqslant k + 1$ *for all* $d \geqslant f(a, b, k)$.

We defer the proof of Theorem 9.25 to Section 9.4.3. First we show how to use this result to deduce the desired consequences for outerplanar graphs and for planar graphs.

### 9.4.2 Outerplanar and Planar Graphs as Pseudo Partial $k$-trees

The main goal of this subsection is to prove that every planar graph is a $(3, 8)$-pseudo partial 2-tree. Combined with Theorem 9.25, this lemma will imply that $d\text{-}\chi_g(G) \leqslant 3$ for every planar graph $G$, when $d$ is sufficiently large. As a warm-up, we first prove that every outerplanar graph is a $(1, 3)$-pseudo partial 1-tree. Combining this lemma with Theorem 9.25 will imply that $d\text{-}\chi_g(G) \leqslant 2$ for every outerplanar graph $G$, again when $d$ is big enough. Recall that a graph is *outerplanar* if it can be drawn in the plane so that every vertex appears on the outer (unbounded) face.

**Lemma 9.26.** *If $G$ is outerplanar, then $G$ has either (i) a vertex $v$ such that $d(v) \leqslant 1$ or (ii) an edge $vw$ such that $d(v) = 2$ and $d(w) \leqslant 4$.*

*Proof.* Given an arbitrary graph $G_0$, we extend it to a near triangulation (all faces are triangles, except possibly the outer face) with the same vertex set, calling this new graph $G$. It suffices to prove that $G$ contains an edge $vw$ such that $d_G(v) = 2$ and $d_G(w) \leqslant 4$, as follows. If $d_{G_0}(v) = 2$, then $vw \in E(G_0)$, so $G_0$ contains an instance of (ii). Otherwise, $d_{G_0}(v) \leqslant 1$, so $G_0$ contains an instance of (i).

To find the desired edge $vw$, consider the weak dual of $G$ (the dual, excluding the vertex corresponding to the outer face), which is a tree, $T^*$. Let $vwx$ be a face of $G$ corresponding to the end of a longest path $P$ in $T^*$. By symmetry, we assume $d_G(v) = 2$. Note that $\min\{d_G(w), d_G(x)\} \leqslant 4$. Otherwise, we can extend $P$ to a longer path in $T^*$ by following a path of triangles incident to $w$ or $x$; this contradicts our choice of $P$. $\square$

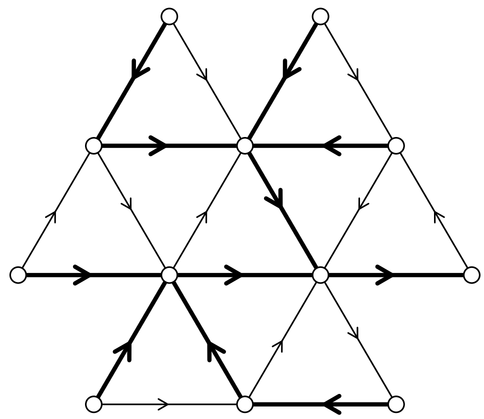


Figure 9.7: An outerplanar graph drawn as a $(1,3)$ partial 1-tree. Here $\overrightarrow{E_1}$ is in bold and all other edges are in $\overrightarrow{E_2}$. This (underlying undirected) graph shows that not every outerplanar graph is a $(1,2)$ partial 1-tree.

**Lemma 9.27.** *If $G$ is an outerplanar graph, then $G$ is a $(1,3)$-pseudo partial 1-tree. That is, there exist edge sets $E_1, E_2$ and orientations $\overrightarrow{E_1}, \overrightarrow{E_2}$ inducing digraphs $\overrightarrow{G_1}$, $\overrightarrow{G_2}$ such that*

*(a)* $E_1 \cap E_2 = \emptyset$ *and* $E(G) \subseteq E_1 \cup E_2$,

*(b)* $\Delta^+(\overrightarrow{G_1}) \leqslant 1$,

*(c)* $\Delta^+(\overrightarrow{G_2}) \leqslant 1$,

*(d)* $\Delta(\overrightarrow{G_2}) \leqslant 3$,

*(e)* $\overrightarrow{G_1}$ *is acyclic, and*

*(f)* $N^+_{\overrightarrow{G_1}}(v)$ *induces a transitive tournament in* $\overrightarrow{G_1} \cup \overrightarrow{G_2}$ *for all* $v \in V(G)$.

*Proof.* (Figure 9.7 shows an example.) We prove the slightly stronger statement using instead $(d')$: $d_{\overrightarrow{G_2}}(v) \leqslant \max\{3, d_G(v) - 1\}$ for all $v \in V(G)$. Our proof is by induction on $|G|$, and the base case $|G| = 1$ is trivial. For the induction step, Lemma 9.26 implies that $G$ has either (i) a vertex $v$ such that $d(v) \leqslant 1$ or (ii) an edge $vw$ such that $d(v) = 2$ and $d(w) \leqslant 4$. In each case $G - v$ has the desired edge sets $E_1'$, $E_2'$ and orientations $\overrightarrow{E_1'}$ and $\overrightarrow{E_2'}$. If $d(v) \leqslant 1$, then we orient outward its at most one incident edge, and add it to $\overrightarrow{E_1'}$. If $d(v) = 2$, then we orient outward both incident edges, adding $\overrightarrow{vw}$ to $\overrightarrow{E_2'}$ and the other incident edge to $\overrightarrow{E_1'}$. Now the resulting orientations $\overrightarrow{E_1}$ and $\overrightarrow{E_2}$ prove the lemma. □

**Corollary 9.28.** *If $G$ is an outerplanar graph, then $d$-$\chi_g(G) \leqslant 2$ if $d \geqslant 30$.*

*Proof.* This follows directly from Theorem 9.25 and Lemma 9.27, taking $(a, b, k) := (1, 3, 1)$, since $2(1)^2+3(1)+2(1)(1)+2(3)(1)+2(1)(3)+3(3)+2 = 2+3+2+6+6+9+2 = 30$. □

Now we prove that every planar graph is a $(3, 8)$-pseudo partial 2-tree. We need the following well-known lemma for planar graphs. An edge $vw$ with $d(v) \leqslant d(w)$ is a *light edge* if (i) $d(v) = 3$ and $d(w) \leqslant 10$ or (ii) $d(v) = 4$ and $d(v) \leqslant 7$ or (iii) $d(v) = 5$ and $d(w) \leqslant 6$. light edge

**Lemma 9.29.** *If* $G$ *is a planar graph with* $\delta(G) \geqslant 3$*, then* $G$ *contains a light edge.*

*Proof.* We actually prove the lemma for the slightly larger class of plane maps (which allow parallel edges, but no 2-faces). Suppose the lemma is false and let $G$ be a counterexample that minimizes $|G|$ and, subject to that, maximizes $\|G\|$. We first show that it suffices to consider the case that $G$ is a triangulation. If not, then $G$ contains some facial walk $W$ of length at least 4. If every vertex on $W$ has degree at least 6, then we add an arbitrary diagonal of $W$, which cannot be a light edge; this contradicts the maximality of $\|G\|$. So assume that $W$ contains a $5^-$-vertex $w$. Since $G$ has no light edge, $d(v) \geqslant 6$ and $d(x) \geqslant 6$, where $vwx$ is a portion of $W$. Now we add edge $vx$, which again cannot be light, contradicting the maximality of $\|G\|$. Thus, we assume $G$ is a triangulation.

We use vertex charging; that is, we give each vertex $v$ charge $d(v) - 6$ (and give no charge to faces). Recall that $\sum_{v \in V(G)}(d(v) - 6) = 2(3|V(G)| - 6) - 6|V(G)| = -6 < 0$.

Now each vertex distributes its charge equally among incident edges with an endpoint of degree 5 or less. Since $G$ is a triangulation with no light edges, no vertex $v$ has $5^-$-neighbors that appear successively in the cyclic order around $v$. Thus, every $7^+$-vertex $v$ sends at least $(d(v) - 6)/\lfloor d(v)/2 \rfloor$ to each incident edge $vw$ with $d(w) \leqslant 5$.

If $d(w) = 3$ and $vw \in E(G)$, then $d(v) \geqslant 11$, so $vw$ finishes with at least $(3 - 6)/3 + (11 - 6)/\lfloor 11/2 \rfloor = 0$. If $d(w) = 4$ and $vw \in E(G)$, then $d(v) \geqslant 8$, so $vw$ finishes with at least $(4 - 6)/4 + (8 - 6)/\lfloor 8/2 \rfloor = 0$. If $d(w) = 5$ and $vw \in E(G)$, then $d(v) \geqslant 7$, so $vw$ finishes with at least $(5 - 6)/5 + (7 - 6)/\lfloor 7/2 \rfloor > 0$. Each other edge starts and ends with 0. Thus, the total charge after discharging is nonnegative, a contradiction. □

**Lemma 9.30.** *If* $G$ *is planar, then* $G$ *is a* $(3, 8)$ *pseudo partial* 2*-tree. That is, there exist edge sets* $E_1$, $E_2$ *and orientations of them* $\overrightarrow{E_1}$, $\overrightarrow{E_2}$ *inducing digraphs* $\overrightarrow{G_1}$ *and* $\overrightarrow{G_2}$ *such that*

*(a)* $E_1 \cap E_2 = \emptyset$ *and* $E(G) \subseteq \overrightarrow{E_1} \cup \overrightarrow{E_2}$*,*

*(b)* $\Delta^+(\overrightarrow{G_1}) \leqslant 2$*,*

*(c)* $\Delta^+(\overrightarrow{G_2}) \leqslant 3$*,*

*(d)* $\Delta(\overrightarrow{G_2}) \leqslant 8$*,*

*(e)* $\overrightarrow{G_1}$ *is acyclic, and*

*(f)* $N^+_{\overrightarrow{G_1}}(v)$ *induces a transitive tournament in* $\overrightarrow{G_1} \cup \overrightarrow{G_2}$ *for all* $v \in V(G)$.

*Proof.* Our proof is by induction on $|E(G)|$. We prove the lemma with the slightly stronger properties (c′) $d^{+}_{\overrightarrow{G_2}}(v) \leqslant \max\{3, d_G(v) - 2\}$ and (d′) $d_{\overrightarrow{G_2}}(v) \leqslant \max\{8, d_G(v) - 2\}$. The base case $|E(G)| \leqslant 1$ is easy: let $E_1 := E(G)$ and $E_2 := \emptyset$, and orient $E_1$ arbitrarily. We assume that $|E(G)| \geqslant 2$, and we consider the following two cases for our induction step.

**Case 1: $\delta(G) \leqslant 2$.** Let $v$ be a $2^-$-vertex. Form $G'$ from $G$ by deleting $v$ and adding edge $wx$ (if $wx \notin E(G)$), where $\{w, x\} = N(v)$).[5] By induction, this smaller graph $G'$ has the desired digraphs $\overrightarrow{G_1'}$ and $\overrightarrow{G_2'}$. To extend these digraphs from $G'$ to $G$, we simply direct all edges incident to $v$ outward and add them to $E(\overrightarrow{G_1'})$.

**Case 2: $\delta(G) \geqslant 3$.** Now Lemma 9.29 guarantees that $G$ contains a light edge $vw$ with $d(v) \leqslant 5$ and $d(w) \leqslant 10$. Let $G' := G - vw$. Again, by induction, $G'$ has the desired graphs $\overrightarrow{G_1'}$ and $\overrightarrow{G_2'}$. To extend to $G$, direct edge $vw$ as $\overrightarrow{vw}$ and add it to $E(\overrightarrow{G_2'})$.

It is straightforward to check that this process yields the desired graphs $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$. Clearly, (b) and (e) are maintained in Case 1 (and unaffected in Case 2). Similarly, (c′) and (d′) are maintained in Case 2; in fact, that is why we use the strengthened versions. Note that (a) holds trivially, and (f) holds because we possibly add an edge when deleting a 2-vertex, as remarked in the footnote. □

**Corollary 9.31.** *If $G$ is a planar graph, then $d$-$\chi_g(G) \leqslant 3$ if $d \geqslant 132$.*

*Proof.* This follows directly from Theorem 9.25 and Lemma 9.30. We take $(a, b, k) := (3, 8, 2)$, so $2k^2+3k+2ak+2kb+2ab+3b+2 = 2(2)^2+3(2)+2(3)(2)+2(2)(8)+2(3)(8)+3(8)+2 = 8 + 6 + 12 + 32 + 48 + 24 + 2 = 132$. □

### 9.4.3 Proving Theorem 9.25

To conclude this chapter, we prove Theorem 9.25. For reference, we restate it below.

**Theorem 9.32.** *Let $G$ be an $(a, b)$-pseudo partial $k$-tree and let $f(a, b, k) := 2k^2 + 3k + 2ak +$
$f(a, b, k)$ $2bk + 2ab + 3b + 2$. Now $d$-$\chi_g(G) \leqslant k + 1$ for all $d \geqslant f(a, b, k)$.*

Before giving a formal proof, we provide an overview. Throughout the game, we denote the current (partial) coloring by $\varphi$. Alice will follow an "extended" activation strategy. When Bob colors some vertex $b$, Alice must decide where to start her walk for the activation strategy. In addition to the uncolored vertices in $N_1^+(b)$, she also considers as candidates the vertices $w \in N_1^+(b)$ such that $\varphi(w) = \varphi(b)$, but $w$ has some uncolored outneighbor in $\overrightarrow{G_1}$ or some uncolored neighbor (in- or out-) in $\overrightarrow{G_2}$. This extended set of candidates will be necessary to ensure that Bob does not push too high the defect of some colored vertex with an uncolored outneighbor. Once Alice selects a vertex $x$ to color, we must also specify how she chooses a color for $x$. Alice simply picks any color not already used on $N_1^+(x)$. Since $\Delta^+(\overrightarrow{G_1}) \leqslant k$, this strategy will produce a coloring $\varphi$ of $G$ whenever the game is played with at least $k + 1$ colors.

[5]This step is needed to ensure that (f) holds.

So most of the work consists of bounding the defect of the resulting coloring $\varphi$. Of course, we also must specify Alice's strategy more precisely.

*Proof.* Because $\overrightarrow{G_1}$ is acyclic, there exists a linear order $L$ on $V(G)$ such that $\overrightarrow{vw} \in E(\overrightarrow{G_1})$ implies $v >_L w$; we say that $w$ is *smaller* than $v$. (When selecting among possible vertices to color, Alice will always pick the one that is smaller.) We need a few definitions. For a vertex $x$, each vertex $y \in N_1^+(x)$ is a *major parent* of $x$. Likewise, each vertex $y \in N_1^-(x)$ is a *major child* of $x$. Furthermore, each vertex $y \in N_2(x)$ is a *minor relative* of $x$. The *mother,* $m(x)$, of a vertex $x$ is the smallest uncolored vertex in $N_1^+[x]$. The smallest uncolored minor relative of $x$ is denoted $r(x)$. Note that $m(x)$ and $r(x)$ may each possibly be undefined. However, if $x$ is uncolored, then $m(x)$ is defined, since $x$ is a candidate for $m(x)$. The *father,* $f(x)$, of $x$ is the smallest vertex $y \in N_1^+(x)$ such that either (i) $y = m(x)$ or (ii) $x$ and $y$ are both colored, $\varphi(y) = \varphi(x)$, and at least one of $m(y)$ and $r(y)$ is defined. Clearly, each of $m(x)$, $r(x)$, and $f(x)$ can change as the game proceeds and more vertices are colored. (It is also worth noting that possibly $f(x) = m(x)$ and/or $m(x) = x$.)

smaller
major parent
major child
minor relative
mother, $m(x)$
$r(x)$
father, $f(x)$

Now we can specify Alice's strategy. It consists of a search stage, a recursive stage, and a coloring stage. In the search stage, Alice chooses, based on Bob's most recent move $b$, from which vertex $v$ to start her walk in $G$. If $f(b)$ is uncolored, then $v := f(b)$. If $f(b)$ is colored, but $m(f(b))$ is uncolored, then $v := m(f(b))$. Otherwise, $v := r(f(b))$.

In the recursive stage, if Alice moves to a vertex that is activated, then she enters the coloring stage. But if she moves to a vertex $v$ that is not activated, then Alice activates $v$, and sets $v := m(v)$. Finally, in the coloring stage, Alice simply colors her current vertex $v$ with any color $\alpha$ that is not used on $N_1^+(v)$.

If $G$ is an $(a, b)$-pseudo partial $k$-tree, then let $H$ be a supergraph of $G$ such that $H$ is an $(a, b)$-pseudo chordal graph satisfying $\Delta^+(\overrightarrow{G_1}) \leqslant k$. (We assume $V(G) = V(H)$.) Rather than playing on $G$, Alice will play on $H$. So we must show that the strategy described above allows Alice to win on $H$. Let $f(a, b, k) := 2k^2 + 3k + 2ak + 2bk + 2ab + 3b + 2$. (The exact value of $f$ is relatively unimportant. It is primarily chosen to be an upper bound on twice the size of various sets. The key is only that $f$ is constant, given $a$, $b$, and $k$.)

We must ensure that $\text{def}_\varphi(G) \leqslant \text{def}_\varphi(H) \leqslant f(a, b, k)$. To guarantee this final inequality holds, we will prove 2 claims. Recall that a color $\alpha$ is *eligible* for use on a vertex $v$ if $\alpha$ is unused on $N_1^+(v)$; otherwise, $\alpha$ is *ineligible* for use on $v$.

eligible

Claim 1: If Alice follows her strategy, then each uncolored vertex $x$ is adjacent to fewer than $f(a, b, k)$ vertices colored with eligible colors. Claim 2: If Alice follows her strategy and some vertex $x$ is colored with $\alpha$ and $\text{def}_\varphi(x) \geqslant f(a, b, k)$, then $\alpha$ is ineligible for use on every major parent of $x$ and on every minor relative of $x$.

Note that Claim 1 ensures that $\text{def}_\varphi(x) \leqslant f(a, b, k)$ at the time that each vertex $x$ is colored. And Claim 2 ensures that once a vertex is colored the inequality $\text{def}_\varphi(x) \leqslant f(a, b, k)$ will persist regardless of how neighbors are colored. Claim 2 handles major parents and minor relatives; major children are handled implicitly by the strategy. This ensures that any move prescribed for Alice by her strategy will be legal. Likewise, Bob can always play the next move that would be prescribed by Alice's strategy. Of course, Bob need not follow Alice's strategy. But it suffices to show that he has some legal move. Thus, it suffices for us to prove Claims 1 and 2.

The proof of each claim is similar. Each outdegree in $\overrightarrow{G_1}$ is bounded, as is each degree in $\overrightarrow{G_2}$. What may be unbounded are the indegrees in $\overrightarrow{G_1}$. Fortunately, we do not need to bound these indegrees. For example, when proving Claim 1, it suffices to bound the number of inneighbors that are already colored. As usual with the activation strategy, our plan is to charge the colored inneighbors to a set of outneighbors (and neighbors in $\overrightarrow{G_2}$). The key is that each such outneighbor will be charged at most twice.

**Claim 1.** *If Alice follows her strategy, then each uncolored vertex $x$ is adjacent to fewer than $f(a,b,k)$ vertices colored with eligible colors.*

Proof. Fix a time in the game when a vertex $x$ is uncolored. Note that $N(x) = N_1^+(x) \cup N_1^-(x) \cup N_2(x)$. By definition, no vertex in $N_1^+(x)$ is colored with an eligible color. And by hypothesis, $|N_2(x)| \leqslant b$. Let $S$ be the set of vertices in $N_1^-(x)$ that are colored with eligible colors for $x$. Let

$$Q := N^+[x] \cup \bigcup_{z \in N_2^+(x)} \left(N^+(z) \cup (N_2^-(z) \setminus \{x\})\right).$$

We will show that each time a vertex in $S$ is colored Alice takes a distinct action at some vertex in $Q$. She takes at most 2 distinct actions (activation and coloring) at each vertex. Thus, $|S| \leqslant 2|Q| \leqslant 2(k + a + 1 + a(k + b - 1)) = 2k + 2a + 2 + 2ak + 2ab - 2a < f(a,b,k) - b$.

Fix $y \in S$. First suppose that Bob colors $y$. See the left of Figure 9.8. Since $x$ is uncolored, $x$ is a candidate for $f(y)$, which means that $f(y)$ is defined. If $f(y)$ is uncolored, then $f(y) \in N^+(x)$, since $N_1^+(y)$ induces a transitive tournament in $\overrightarrow{G_1} \cup \overrightarrow{G_2}$. So Alice takes an action (activation or coloring) at $f(y)$, and we are done. If instead $f(y)$ is colored, then $\varphi(f(y)) = \varphi(y)$, by the definition of $f(y)$. Again $f(y) \in N_1^+(y)$. If $f(y) \in N_1^+(x)$, then $\varphi(y)$ is used on $N_1^+(x)$, which makes $\varphi(y)$ ineligible for $x$, a contradiction. Thus, $f(y) \in N_2^+(x)$. So the next vertex where Alice takes an action is in $N_1^+[f(y)]$, as desired.

Now assume instead that Alice colors $y$. Since $x$ is uncolored and $x \in N_1^+(y)$, Alice must have previously visited and activated $y$. When she did, she continued on to some other vertex. Since $y$ was uncolored, this next vertex is $m(y)$. Since $m(y) \in N_1^+(y)$ and $x \in N_1^+(y)$, we have $m(y) \in N^+[x]$, as desired. ◇

**Claim 2.** *If Alice follows her strategy and some vertex $x$ is colored with some color $\alpha$ and $def_\varphi(x) \geqslant f(a,b,k)$, then $\alpha$ is ineligible for use on every major parent of $x$ and on every minor relative of $x$.*

Proof. Fix a time in the game when a vertex $w$ is uncolored, and let $x$ be a major child of $w$ or a minor relative of $w$; assume $x$ is colored with $\alpha$. See the right of Figure 9.8. It suffices to show that if $\alpha$ is eligible for $w$, then $\mathrm{def}_\varphi(x) < f(a,b,k)$. Let $S$ be the subset of $N_1^-(x)$ colored with $\alpha$. Clearly, $\mathrm{def}_\varphi(x) \leqslant |N_1^+(x) \setminus \{w\}| + |N_2(x)| + |S|$. Since $|N_1^+(x)| \leqslant k$ and $|N_2(x)| \leqslant b$, it suffices to show that $|S| \leqslant f(a,b,k) - (k-1) - b$. Let

$$Q := N^+[x] \cup N_2^-(x) \cup \bigcup_{z \in N^+(x) \setminus \{w\}} \left(N^+(z) \cup N_2^-(z)\right).$$


Figure 9.8: Left: The proof of Claim 1. An uncolored vertex $x$, and the subset $S$ of its major children colored with colors eligible for $x$. Right: The proof of Claim 2. A vertex $x$ colored $\alpha$, the subset $S$ of its major children also colored $\alpha$, and an uncolored vertex $w \in N_2(x) \cup N_1^+(x)$.

Again, we show that for every vertex $y \in S$ Alice takes a distinct action at some vertex of $Q$. Thus, $|S| \leqslant 2|Q| \leqslant 2(k+1+b+(a+k-1)(k+b)) = 2k+2+2b+2ak+2ab+2k^2+2bk-2k-2b = 2ak+2ab+2k^2+2bk+2 < f(a,b,k)-k-b+1$.

Fix $y \in S$. First suppose that Bob colored $y$. At that time $f(y)$ was defined, since $x$ was a candidate for $f(y)$: either $x$ was uncolored or else $x$ was colored, $\varphi(x) = \varphi(y)$, and $w$ was uncolored. Thus, $f(y) \in N^+(y)$, which implies $f(y) \in N^+[x]$. If $f(y)$ was uncolored, then Alice took an action in $N^+[x]$. If instead $f(y)$ was colored, then the next vertex $m(f(y))$ where Alice took action was in $N_1^+(f(y)) \cup N_2(f(y))$. So we are done.

Now suppose instead that Alice colored $y$. At the time Alice colored $y$, vertex $x$ was uncolored (or else $\alpha$ would have been ineligible for $y$). So the fact that Alice colored $y$ means that she must have previously activated $y$. After she did, she was in the recursive stage, so continued to $m(y)$. Since $m(y) \in N_1^+(y)$ and $x \in N_1^+(y)$, we get that $m(y) \in N^+[x]$. Thus, Alice took action in $N^+[x]$ as claimed. ♢

Together, Claims 1 and 2 complete the proof. □

## Notes

The chromatic game was introduced by Brams, in the context of coloring faces of planar maps. This game was published by Martin Gardner [172] in his *Mathematical Games* column in 1981, but was largely ignored by graph theorists for the next decade. In 1991, it was reinvented by Bodlaender [42], who defined $\chi_g$ and proved that $\chi_g(F) \leqslant 5$ for every forest $F$. He also constructed trees $T$ such that $\chi_g(T) = 4$; see Exercise 2. Bodlaender further conjectured that $\chi_g(G)$ is bounded by some constant for all planar graphs $G$.

Faigle, Kern, Kierstead, and Trotter [157] improved Bodlaender's upper bound, showing that $\chi_g(F) \leqslant \mathrm{col}_g(F) \leqslant 4$ for every forest $F$. This is our Lemma 9.2 and Corollary 9.4. Their proof introduced the activation strategy, but only for forests. They also showed that every interval graph $G$ satisfies $\mathrm{col}_g(G) \leqslant 3\omega(G) - 2$ and[6] that there exist interval graphs $G$ with $\chi_g(G) \geqslant 2\omega(G) - 2$. Lemma 9.6 is a simplified (and thus weakened) version of Theorem 9.7, and it first appeared in [33].

[6] This first inequality was phrased only for $\chi_g$, but the proof gives the same bound for $\mathrm{col}_g$.

Guan and Zhu [193] proved Corollary 9.9, that $\mathrm{col}_g(G) \leqslant 7$ for every outerplanar graph $G$. And the outerplanar graph $G$ with $\mathrm{col}_g(G) = 7$, constructed in Theorem 9.10, was found by Kierstead and Yang [254]. Zhu [431] proved Corollary 9.11, that $\mathrm{col}_g(G) \leqslant 3\omega(G) - 1$ for every chordal graph $G$. The interval graphs $I_k$ for which $\mathrm{col}_g(I_k) = 3\omega(I_k) - 2$, constructed in Theorem 9.13, were again found by Kierstead and Yang [254]. The chordal graphs $G_k$ in Remark 9.14, for which $\mathrm{col}_g(G_k) = 3\omega(G) - 1$, were discovered by Wu and Zhu [417].

The activation strategy, as we present it in Algorithm 9.1, and Theorem 9.7 together form the heart of Sections 9.1 and 9.2. These are both due to Kierstead [246]. In the same paper, he proved Theorem 9.15, as well as provided the proofs we present for Corollaries 9.9 and 9.11, which were proved previously as noted above. (He also gave a corollary of Theorem 9.7 implying the bound $\mathrm{col}_g(G) \leqslant 3\omega(G) - 2$ for interval graphs, as well as a few other interesting corollaries.) Previously, Zhu [430] proved that $\mathrm{col}_g(G) \leqslant 19$ for all planar graphs $G$.

Zhu's result was a breakthrough, giving the first constant upper bound on $\mathrm{col}_g$ for all planar graphs, as well as showing how to extend the activation strategy used for trees [157] to all planar graphs. Kierstead's proof of Theorem 9.15 further simplified this strategy and generalized it to all graphs, improving by 1 the upper bound for planar graphs. This upper bound was improved 1 more by Zhu [434]. Using a refinement of the activation strategy, he showed that every planar graph $G$ satisfies $\mathrm{col}_g(G) \leqslant 17$.

The harmonious strategy, which we considered in Section 9.3, was introduced by Kierstead and Yang [254] to prove Theorem 9.18. Later they, together with Yi [255], adopted the strategy to prove Theorem 9.16. Other applications of this strategy include game colorings of directed graphs [422], game colorings of squares [421], and strong edge colorings of sparse graphs [423]. More recently, van den Heuvel and Kierstead [396] applied these ideas in their work on generalized coloring numbers.[7]

Defective coloring has been widely studied, starting at least in the mid 1980's. A related topic, clustered coloring, imposes a bound on the order of every component in each monochromatic subgraph, rather than simply on its maximum degree. For both defective and clustered coloring, the main focus has been on (a) finding graph classes for which a constant number of colors allow defect or clustering to be bounded by a constant and (b) determining the minimum number of colors that allows this. (Actually computing the minimum defect or clustering has been comparatively de-emphasized.) We mirror this focus in our treatment of the defective coloring game. We revisit clustered coloring in Section 10.5 and defective coloring in Section 12.3. For more on both topics, we recommend the excellent survey by Wood [416].

The defective game chromatic number was introduced in [84]. Key early work on this topic included [124]. Theorem 9.25, the main result of Section 9.4, was proved by Dunn and Kierstead [125]. But most of the lemmas in that section were proved earlier. Theorem 9.22 and Corollary 9.23 are due to Zhu [431], as are the definitions of $(a,b)$-pseudo chordal graph and $(a,b)$-pseudo partial $k$-tree. Lemma 9.26 is perhaps folklore, but it was proved in a stronger

[7] Informally, generalized coloring numbers are akin to the 2-coloring number we considered in Lemma 9.6, but they allow longer paths between a vertex and its set of "neighbors". For more on this topic, we recommend [396].

form in [195] and [156, Theorem 5]. Lemma 9.29 is due to Borodin [51], who proved a much stronger form; see Exercises 8 and 9. Lemmas 9.27 and 9.30 are from [431] and Corollaries 9.28 and 9.31 are from [125].

## Exercises

9.1. Determine the game chromatic number of the octahedron and the icosahedron.

9.2. Construct trees T for which $\mathrm{col}_g(T) \geqslant \chi_g(T) \geqslant 4$.

9.3. Computer $\chi_g^B(K_{n,n} - nK_2)$. (Here $\chi_g^B$ is a variant of $\chi_g$ where Bob plays first.) Contrast this value with $\chi_g(K_{n,n} - nK_2)$.

9.4. For all integers D and k with $D \geqslant k \geqslant 1$, construct a graph $H_{D,k}$ that is k-degenerate with maximum degree D such that $H^2_{D,k}$ is *not* $((2k-1)D - k^2 - 1)$-degenerate. [108]. [So Theorem 9.16 is optimal up to an additive constant.]

9.5. Corollary 9.4 shows that $\mathrm{col}_g(G) \leqslant 4$ whenever G is a forest. Extend this result to all pseudoforests (graphs in which each component has at most one cycle).

9.6. Use Theorem 9.7 to prove the following two results. (a) If G is an interval graph, then $\mathrm{col}_g(G) \leqslant 3\omega(G) - 2$. [157] (b) Let G be a graph with maximum degree $\Delta$ and let $\vec{G}$ be an acyclic orientation of G such that $\Delta^+(\vec{G}) \leqslant k$, for some positive integer k. If H is the line graph of $\vec{G}$, then $\mathrm{col}_g(H) \leqslant \Delta + 3k - 1$. [76]

9.7. The *treewidth* of a graph G, denoted tw(G), is given by $\mathrm{tw}(G) := \min_H \omega(H) - 1$, over all chordal supergraphs H of G. Prove that $\mathrm{tw}(P_n \Box P_n) = n$; here $P_n \Box P_n$ is the $n \times n$ grid.

9.8. Show that all of the numerical parameters in Lemma 9.29 are best possible. That is, for each of the 3 conditions that follow, construct a graph G satisfying it: if $v \in V(G)$ with $d(v) \leqslant 5$ and $vw \in E(G)$, then (i) $d(v) = 3$ and $d(w) \geqslant 10$ or (ii) $d(v) = 4$ and $d(w) \geqslant 7$ or (iii) $d(v) = 5$ and $d(w) \geqslant 6$. [50]

9.9. Consider a planar graph G such that $\delta(G) \geqslant 3$ and for each edge $vw$ (i) if $d(v) = 3$, then $d(w) \geqslant 10$ and (ii) if $d(v) = 4$, then $d(w) \geqslant 7$ and (iii) if $d(v) = 5$, then $d(w) \geqslant 6$. Let $e_{i,j}$ denote the number of edges $vw$ such that $d(v) = i$ and $d(w) = j$. Determine the minimum coefficients $a, b, c$ such that always $ae_{5,6} + be_{4,7} + ce_{3,10} \geqslant 1$. [This is a special case of a more general result, that requires only that $\delta(G) \geqslant 3$, but adds a term in the inequality for $e_{3,9}$, as well as terms for each $e_{i,j}$ with $i \geqslant j \geqslant 3$ and $i + j \leqslant 11$. [51]]

9.10. Corollaries 9.28 and 9.31 show, respectively, that (a) if G is outerplanar, then $d\text{-}\chi_g(G) \leqslant 2$ when $d \geqslant 30$ and (b) if G is planar, then $d\text{-}\chi_g(G) \leqslant 3$ when $d \geqslant 132$. Show that both results are sharp, even in the non-game setting. That is, show that for every positive integer there exist (a) an outerplanar graph $G_d$ such that $d\text{-}\chi(G_d) \geqslant 2$ and (b) a planar graph $H_d$ such that $d\text{-}\chi(H_d) \geqslant 3$.

# Chapter 10

# The Vertex Shuffle

If something cannot go on forever, it will stop.

—Herbert Stein

A vertex shuffle starts with a coloring that doesn't quite achieve what we want and moves vertices between color classes (shuffles them) to improve the coloring. This process has many variations, depending on the specific problem we are considering. Although we do not emphasize it here, one nice consequence of this algorithmic viewpoint is that our proofs typically yield polynomial-time algorithms to find the colorings.[1]

## 10.1 An Introduction to the Vertex Shuffle

In this section, we restrict ourselves to recoloring a single vertex at a time, and we focus on graphs for which $\chi$ is close to $\Delta$. We begin with a classic partitioning result.

**Theorem 10.1.** *Let* $G$ *be a graph,* $s$ *be a positive integer, and* $r_1, \dots, r_s$ *be nonnegative integers. If* $\sum_{i=1}^{s} r_i \geqslant \Delta + 1 - s$*, then there exists a partition of* $V(G)$ *into parts* $V_1, \dots, V_s$ *such that for all* $i \in [s]$ *we have* $\Delta(G[V_i]) \leqslant r_i$.

We want to put each vertex $v$ into a part $V_i$ where $d_{V_i}(v) \leqslant r_i$. So if some vertex $v$ in part $i$ has more than $r_i$ neighbors in its part, then we should move $v$ to another part where it causes less trouble. When all $r_i$ are equal, each such move decreases the total number of edges within parts, so the process must end. For general $r_i$ we need a more refined metric.

*Proof.* Given a partition $\mathcal{P} = \{V_1, \dots, V_s\}$ of $V(G)$, let

$$f(\mathcal{P}) := \sum_{i=1}^{s} (\|G[V_i]\| - r_i\,|V_i|).$$

[1]Sometimes this requires significant optimization, but often the translation is fairly straightforward. We discuss algorithms a bit more in the Notes.

We choose $\mathcal{P}$ to minimize $f(\mathcal{P})$, and we show that $\mathcal{P}$ satisfies the desired conclusion. Suppose to the contrary that there exists an index $i \in [s]$ and a vertex $v \in V_i$ such that $d_{V_i}(v) > r_i$. Since $\sum_{i=1}^{s}(r_i+1) \geqslant \Delta+1$, there exists an index $j$ such that $v$ has at most $r_j$ neighbors in $V_j$. Moving $v_i$ to $V_j$ decreases $f$, since it decreases the $i$th term in the sum and does not increase the $j$th term. This decrease of $f$ contradicts the minimality of $\mathcal{P}$. Thus instead, $d_{V_i}(v) \leqslant r_i$ for all $i \in [s]$ and $v \in V_i$. □

Above we choose $\mathcal{P}$ to minimize $f(\mathcal{P})$. Alternatively, we can start from an arbitrary partition and move vertices between parts, at each step decreasing $f$. This gives rise to the name Vertex Shuffle, and it implies an efficient algorithm for finding such a partition. Next, we apply Theorem 10.1 to prove an upper bound on the chromatic number of a graph $G$ with no large clique. When $\ell = \Delta+1$, this simplifies to Brooks' Theorem.

**Theorem 10.2.** *If* $G$ *is a graph with* $\omega(G) < \ell$ *for some integer* $\ell \geqslant 4$ *then*

$$\chi(G) \leqslant \Delta + 1 - \left\lfloor \frac{\Delta+1}{\ell} \right\rfloor.$$

Given a partition $V_1, \ldots, V_s$ guaranteed by Lemma 10.1, the idea is to apply Brooks' Theorem to each subgraph $G[V_i]$.

*Proof.* We assume that $\ell \leqslant \Delta+1$, since otherwise the result is trivial. Let $s := \lfloor \frac{\Delta+1}{\ell} \rfloor$. Let $r_i := \ell-1$ for all $i \in [s-1]$ and $r_s := \Delta-\ell(s-1)$. Note that $\sum_{i=1}^{s} r_i = (s-1)(\ell-1)+\Delta-\ell(s-1) = \Delta+1-s$. Thus, $r_1, \ldots, r_s$ satisfy the hypothesis of Theorem 10.1. Let $V_1, \ldots, V_s$ be a partition of $V(G)$ such that $\Delta(G[V_i]) \leqslant r_i = \ell-1$ for all $i \in [s-1]$ and $\Delta(G[V_s]) \leqslant r_s = \Delta-\ell(s-1)$. Since $\omega(G) < \ell$, also $\omega(G[V_i]) < \ell$. So Brooks' Theorem implies $\chi(G[V_i]) \leqslant \ell-1$ for all $i \in [s-1]$. Similarly, $r_s = \Delta-\ell(s-1) \geqslant \Delta-\ell\left(\frac{\Delta+1}{\ell}-1\right) = \Delta-(\Delta+1)+\ell = \ell-1$. So $\Delta(G[V_s]) \geqslant \omega(G[V_s])$, which implies that $\chi(G[V_s]) \leqslant \Delta(G[V_s])$. Now using disjoint colors sets on the different $G[V_i]$ gives

$$\chi(G) \leqslant \sum_{i=1}^{s} \chi(G[V_i]) \leqslant \sum_{i=1}^{s} r_i = \Delta+1-s = \Delta+1-\left\lfloor \frac{\Delta+1}{\ell} \right\rfloor.$$ □

k-choosable with defect d

A (possibly improper) coloring has *defect* $d$ if every monochromatic subgraph has maximum degree at most $d$. A graph $G$ is $k$*-choosable with defect* $d$ if every $k$-assignment $L$ admits an $L$-coloring with defect $d$. *From here until the end of Section 10.1 we allow all* $L$*-colorings to be improper.* Our final main result of Section 10.1 is Theorem 10.3. Note that when $r = 0$ we recover the simple fact that every graph $G$ is properly $\lfloor \text{mad}(G)+1 \rfloor$-choosable, by degeneracy.

**Theorem 10.3.** *If mad*$(G) < \frac{2r+2}{r+2}k$*, then* $G$ *is* $k$*-choosable with defect* $r$ *(whenever* $k$ *and* $r$ *are integers with* $k \geqslant 1$ *and* $r \geqslant 0$*).*

To prove Theorem 10.3, we want to show that certain subgraphs are reducible, by Theorem 10.1. Thus, we adapt its statement and proof to list-coloring. (Here we only consider the case where all $r_i$ are equal, but the statement holds more generally; see Exercise 4.)

**Lemma 10.4.** *Let* $L$ *be a list assignment for a graph* $G$*. If* $(r+1)|L(v)| > d(v)$ *for all* $v \in V(G)$*, then* $G$ *has an* $L$*-coloring with defect* $r$*.*

*Proof.* Pick an $L$-coloring $\varphi$ of $G$ minimizing the number of monochromatic edges. Suppose some vertex $v$ has at least $r+1$ neighbors $w$ with $\varphi(w) = \varphi(v)$. Since $(r+1)|L(v)| > d(v)$, some $\alpha \in L(v)$ is used by $\varphi$ at most $r$ times on $N(v)$. Now recoloring $v$ with $\alpha$ decreases the number of monochromatic edges, contradicting our choice of $\varphi$. Thus, $\varphi$ has defect $r$. $\square$

**Lemma 10.5.** *Let* $L$ *be a list assignment for a graph* $G$*, and let* $A, B$ *be a partition of* $V(G)$*. If* $\varphi$ *is an* $L$*-coloring of* $G[A]$ *with defect* $r$ *and* $(r+1)|L(v)| > (r+1)d_A(v) + d_B(v)$ *for all* $v \in B$*, then* $G$ *has an* $L$*-coloring with defect* $r$*.*

*Proof.* For each $v \in B$, form $L'(v)$ from $L(v)$ by removing each color used by $\varphi$ on $N_A(v)$. Now $(r+1)|L'(v)| \geqslant (r+1)(|L(v)| - d_A(v)) > d_B(v)$, so $G[B]$ has an $L$-coloring $\varphi'$ with defect $r$, by Lemma 10.4. By design, $\varphi(a) \neq \varphi'(b)$ for each $ab \in E(G)$ with $a \in A$ and $b \in B$. Thus, $\varphi \cup \varphi'$ is an $L$-coloring of $G$ with defect $r$. $\square$

To prove Theorem 10.3, we look for $B \subseteq V(G)$ such that we can $L$-color $G - B$ by induction and extend the $L$-coloring to $B$ by Lemma 10.5. If no such $B$ exists, then we show that $2\|G\|/|G| \geqslant \frac{2r+2}{r+2}k$, which contradicts the hypothesis on $\mathrm{mad}(G)$.

*Proof of Theorem 10.3.* Fix integers $k$ and $r$ with $k \geqslant 1$ and $r \geqslant 0$, and let $L$ be a $k$-assignment for $G$. Assume $\mathrm{mad}(G) < \frac{2r+2}{r+2}k$. We use induction on $|G|$. Let $v_1, \ldots, v_t$ be a maximal sequence of distinct vertices in $V(G)$ with $(r+1)|L(v_i)| \leqslant (r+1)d_{A_i}(v_i) + d_{B_i}(v_i)$ where $A_i := \{v_1, \ldots, v_{i-1}\}$ and $B_i := V(G) \setminus A_i$. Let $A := \{v_1, \ldots, v_t\}$ and $B := V(G) \setminus A$.

First assume $t < |G|$. By induction, $G[A]$ has an $L$-coloring $\varphi$ with defect $r$, because $|G[A]| < |G|$. Since $v_1, \ldots, v_t$ is maximal, each $w \in B$ has $(r+1)|L(w)| > (r+1)d_A(v) + d_B(v)$. Thus, by Lemma 10.5, we can extend $\varphi$ to an $L$-coloring of $G$ with defect $r$.

Now assume instead that $t = |G|$. This gives

$$\begin{aligned}
(r+1)k|G| &= \sum_{v_i \in V(G)} (r+1)k \\
&\leqslant \sum_{v_i \in V(G)} \left((r+1)d_{A_i}(v_i) + d_{B_i}(v_i)\right) \\
&= \sum_{v_i \in V(G)} \left(r d_{A_i}(v_i) + d_G(v_i)\right) \\
&= (r+2)\|G\|.
\end{aligned}$$

The initial and final terms in this chain give $\frac{2r+2}{r+2}k \leqslant \frac{2\|G\|}{|G|}$, contradicting the hypothesis. $\square$

## 10.2 Hitting Sets

hitting set

In a graph $G$, a *hitting set* is an independent set that intersects every maximum clique. The main result of this section is the following theorem.

**Theorem 10.6.** *If* $G$ *is a graph with* $\omega(G) > \frac{2}{3}(\Delta(G)+1)$*, then* $G$ *has a hitting set.*

The proof of this result combines various pieces. One of these pieces, Corollary 10.14 has a proof that is fairly involved. Thus, in this section we prove Theorem 10.6 assuming the corollary; and in the next section we prove the corollary.

Before proving Theorem 10.6, we show how it is useful. Recall the Borodin–Kostochka Conjecture: If $G$ is a graph with $\Delta(G) \geqslant 9$ and $\omega(G) \leqslant \Delta(G)-1$, then $\chi(G) \leqslant \Delta(G)-1$. That is, a $\Delta(G)$-clique is the only obstruction to coloring with $\Delta(G)-1$ colors. Reed's Conjecture is similar, but more far-reaching: Every graph $G$ satisfies $\chi(G) \leqslant \lceil \frac{1}{2}(\omega(G)+\Delta(G)+1)\rceil$.

**Lemma 10.7.** *If* $G$ *is a counterexample to Reed's Conjecture with the fewest vertices, then* $\omega(G) \leqslant \frac{2}{3}(\Delta(G)+1)$.

*Proof.* Assume, to the contrary, that $G$ is a minimum counterexample to Reed's Conjecture and $\omega(G) > \frac{2}{3}(\Delta(G)+1)$. Let $I_0$ be a hitting set in $G$, guaranteed by Theorem 10.6. Let $I$ be a maximal independent set containing $I_0$, and let $G' := G - I$. Clearly $\omega(G') = \omega(G) - 1$ and $\Delta(G') \leqslant \Delta(G) - 1$. Note that $\chi(G') \leqslant \lceil \frac{1}{2}(\omega(G')+\Delta(G')+1)\rceil$, since $G'$ is smaller than $G$. We let $\varphi$ be a coloring of $G'$ attaining this bound, and color $I$ with an additional color. So $\chi(G) \leqslant 1+\chi(G') \leqslant 1+\lceil \frac{1}{2}(\omega(G')+\Delta(G')+1)\rceil \leqslant \lceil \frac{1}{2}(\omega(G)+\Delta(G)+1)\rceil$. □

**Lemma 10.8.** *If the Borodin–Kostochka Conjecture is true for all graphs with* $\Delta = 9$*, then it is true for all graphs.*

*Proof.* Assume, to the contrary, that the Borodin–Kostochka Conjecture is true for all graphs with $\Delta = 9$ but is false for some graph with $\Delta$ larger. Among all counterexamples to the conjecture, let $G$ be one for which $\Delta$ is minimum.

Since $G$ is a counterexample, $\omega(G) < \Delta(G)$. We will find a maximal independent set $I$ that intersects every clique in $G$ of size $\Delta(G)-1$. If $\omega(G) < \Delta(G)-1$, then let $I$ be any maximal independent set. If $\omega(G) = \Delta(G)-1$, then $G$ contains a hitting set $I_0$ by Theorem 10.6. Let $I$ be any maximal independent set that contains $I_0$, and let $G' := G - I$.

$I$, $G'$

Since $I$ is maximal, $\Delta(G') < \Delta(G)$. If $\Delta(G') \leqslant \Delta(G)-3$, then greedy coloring shows that $\chi(G') \leqslant \Delta(G')+1 \leqslant \Delta(G)-2$. Using a new color on $I$ yields $\chi(G) \leqslant 1+\chi(G') \leqslant \Delta(G)-1$. If $\Delta(G') = \Delta(G)-2$, then Brooks' Theorem shows that $\chi(G') \leqslant \Delta(G') = \Delta(G)-2$, since $\omega(G') \leqslant \Delta(G)-2$. So, again, $\chi(G) \leqslant \Delta(G)-1$. Finally, assume that $\Delta(G') = \Delta(G)-1 \geqslant 9$. Note that $\omega(G') \leqslant \Delta(G)-2 = \Delta(G')-1$. Recall that $G'$ is not a counterexample to the theorem, since it has smaller maximum degree than $G$. Thus, $\chi(G') \leqslant \Delta(G')-1 = \Delta(G)-2$. Again, using a new color on $I$ shows that $\chi(G) \leqslant \Delta(G)-1$, which contradicts that $G$ is a counterexample. □

To prove Theorem 10.6, we need the following definition. Given a collection $\mathcal{S}$ of sets, the *intersection graph* $X_{\mathcal{S}}$ has one vertex for each set of $\mathcal{S}$ and two vertices are adjacent if their sets intersect. The right of Figure 10.1 shows an example intersection graph. Let $\mathcal{T}$ be the set of all maximum cliques in G. For every component $\mathcal{C}_i$ of $X_{\mathcal{T}}$, let $V_i := \cap_{T\in\mathcal{C}_i} V(T)$ and let $H := G[\cup V_i] \setminus \cup_i E(G[V_i])$; that is, H contains only edges between the $V_i$'s. To prove Theorem 10.6, we will show that H has an independent transversal I, since such an I is clearly a hitting set for G. To this end, we must show that each $V_i$ is large (so that we have many ways to hit it), which motivates the following two lemmas. For brevity, we often write $\bigcup\mathcal{S}$ and $\bigcap\mathcal{S}$ to denote $\bigcup_{S\in\mathcal{S}} S$ and $\bigcap_{S\in\mathcal{S}} S$.

intersection graph

$\mathcal{C}_i, X_{\mathcal{T}}$

$V_i, H$

$\bigcup\mathcal{S}, \bigcap\mathcal{S}$

**Lemma 10.9.** *If $\mathcal{S}$ is a collection of maximum cliques in a graph G, then*

$$\left|\bigcup\mathcal{S}\right| + \left|\bigcap\mathcal{S}\right| \geqslant 2\omega.$$

*Proof.* We use induction on $|\mathcal{S}|$, and the base case $|\mathcal{S}| = 1$ is trivial. So assume $|\mathcal{S}| \geqslant 2$. Choose $S_1 \in \mathcal{S}$ and let $\mathcal{S}' := \mathcal{S} - S_1$. By induction, $|\bigcup\mathcal{S}'| + |\bigcap\mathcal{S}'| \geqslant 2\omega$. So it suffices to show that $|\bigcup\mathcal{S}| + |\bigcap\mathcal{S}| \geqslant |\bigcup\mathcal{S}'| + |\bigcap\mathcal{S}'|$. We rewrite this as

$$\left|\bigcup\mathcal{S}\right| - \left|\bigcup\mathcal{S}'\right| = \left|S_1 \setminus \bigcup\mathcal{S}'\right| \geqslant \left|(\bigcap\mathcal{S}') \setminus S_1\right| = \left|\bigcap\mathcal{S}'\right| - \left|\bigcap\mathcal{S}\right|. \tag{10.1}$$

By adding $|S_1 \cap (\bigcup\mathcal{S}')|$ to each side of (10.1), we see that it suffices to prove the inequality

$$|S_1| \geqslant \left|\left((\bigcap\mathcal{S}') \setminus S_1\right) \cup \left(S_1 \cap (\bigcup\mathcal{S}')\right)\right|.$$

Since $S_1$ is a maximum clique, it is enough to show that $((\bigcap\mathcal{S}') \setminus S_1) \cup (S_1 \cap (\bigcup\mathcal{S}'))$ is also a clique. Clearly, sets $(\bigcap\mathcal{S}') \setminus S_1$ and $S_1 \cap (\bigcup\mathcal{S}')$ are both cliques. Given $v \in S_1 \cap (\bigcup\mathcal{S}')$, there exists $S_2 \in \mathcal{S}'$ such that $v \in S_2$. Now given $w \in (\bigcap\mathcal{S}') \setminus S_1$, clearly $w \in S_2$. Thus, $vw \in E(G)$, as desired. So $((\bigcap\mathcal{S}') \setminus S_1) \cup (S_1 \cap (\bigcup\mathcal{S}'))$ is a clique. Hence, we conclude that $|S_1| \geqslant |((\bigcap\mathcal{S}') \setminus S_1) \cup (S_1 \cap (\bigcup\mathcal{S}'))|$, which proves the lemma. □

**Lemma 10.10.** *Let G be a graph with $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$. If $\mathcal{S}$ is a collection of maximum cliques in G and its intersection graph $X_{\mathcal{S}}$ is connected, then $|\bigcap\mathcal{S}| \geqslant 2\omega(G) - (\Delta(G) + 1)$.*

*Proof.* The key is to show that $|\bigcap\mathcal{S}| > 0$, for then $|\bigcup\mathcal{S}| \leqslant \Delta(G) + 1$, so the lemma follows directly from Lemma 10.9. To prove that $|\bigcap S| > 0$, by Lemma 10.9 it suffices to show that $|\bigcup\mathcal{S}| < 2\omega$. We use induction on $|\mathcal{S}|$, and the base case $|S| = 1$ is trivial.

Now we assume $|S| \geqslant 2$. Let $S_1 \in \mathcal{S}$ be a noncutvertex of $X_{\mathcal{S}}$, and choose $S_2 \in \mathcal{S}$ that intersects $S_1$. Since $|S_1 \cup S_2| \leqslant \Delta(G) + 1$, applying Lemma 10.9 to the set $\{S_1, S_2\}$ shows that $|S_1 \setminus S_2| = |S_1| - |S_1 \cap S_2| \leqslant \omega(G) - (2\omega(G) - (\Delta(G) + 1)) = \Delta(G) + 1 - \omega(G)$. Let $\mathcal{S}' := \mathcal{S} - S_1$. Since $X_{\mathcal{S}'}$ is connected, by induction the lemma holds for $\mathcal{S}'$. Since $\bigcap\mathcal{S}'$ is nonempty, $|\bigcup\mathcal{S}'| \leqslant \Delta(G)+1$. Thus, $|\bigcup\mathcal{S}| \leqslant |\bigcup\mathcal{S}'|+|S_1 \setminus S_2| \leqslant (\Delta(G)+1)+(\Delta(G)+1-\omega(G)) < 2\omega(G)$. By Lemma 10.9, $|\bigcap S| > 0$, so the lemma follows. □

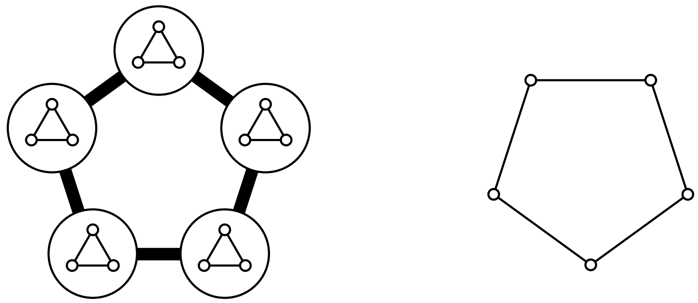


Figure 10.1: Left: A graph G with $\Delta(G) = 8$ and $\omega(G) = \frac{2}{3}(\Delta(G)+1) = 6$, but with no hitting set. Here bold edges indicate copies of $K_{3,3}$ between the vertices in successive triangles. Right: The intersection graph of the maximum cliques of G.

It is interesting to note that the hypothesis $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ in Lemma 10.10 is best possible. That is, weakening the inequality at all yields a statement that is false infinitely often for each value of $\Delta(G)$; see Exercise 9. Figure 10.1 shows an example.

**Lemma 10.11.** *Let* $G$ *be a graph and* $V_1, \cdots, V_s$ *be a partition of* $V(G)$. *Fix an integer* $t$. *If, for every* $i \in [s]$ *and each* $v_i \in V_i$, *we have* $d(v_i) \leqslant \min\{t, |V_i| - t\}$, *then* $G$ *has an independent transversal.*

This essentially follows from Corollary 10.14 with some careful counting.

*Proof.* Suppose G has no IT. By Corollary 10.14, there exists $J \subseteq [s]$ such that $G_J$ has a totally dominating set $D$ of size $2(|J| - 1)$ and $D$ contains a subset $D'$ that is a PIT of $G_J$ of size at least $|J| - 1$. By hypothesis, for each $v_i \in V_i$ we have $d(v_i) \leqslant \min\{t, |V_i| - t\}$. This gives us

$$\begin{aligned}
\sum_{v \in D} d(v) &= \sum_{v \in D \setminus D'} d(v) + \sum_{v \in D'} d(v) \\
&\leqslant \sum_{v \in D \setminus D'} t + \sum_{v \in D'} (|V_i| - t) \\
&\leqslant t(|J| - 1) + \sum_{v \in D'} (|V_i| - t) && (10.2) \\
&\leqslant \sum_{v \in D'} (|V_i| - t + t) \leqslant |V(G_J)|. && (10.3)
\end{aligned}$$

The first inequality in (10.3) holds because $|D'| \geqslant |J| - 1$, and the second inequality in (10.3) holds because $D'$ is a PIT. Since $|D| = 2(|J| - 1)$, we assume $|J| \geqslant 2$. Hence, either $|D \setminus D'| < |J| - 1$ so (10.2) is strict or $|D \setminus D'| > 0$ so (10.3) is strict. That is, the sum of degrees in $D$ is strictly too small to totally dominate $G_J$, a contradiction. □

Now we use Lemmas 10.9–10.11 to prove Theorem 10.6. For convenience, we restate it.

**Theorem 10.6.** *If* $G$ *is a graph with* $\omega(G) > \frac{2}{3}(\Delta(G)+1)$*, then* $G$ *has a hitting set.*

*Proof.* Recall that $\mathcal{T}$ is the collection of maximum cliques in $G$, that $X_\mathcal{T}$ is the intersection graph of $\mathcal{T}$, and that for each component $\mathcal{C}_i$ of $X_\mathcal{T}$ we have $V_i := \cap_{T\in\mathcal{C}_i} V(T)$. Finally, recall that $H := G[\cup V_i] \setminus \cup_i E(G[V_i])$; that is, $H$ contains only edges between the $V_i$'s. By Lemma 10.10, we have $|V_i| \geqslant 2\omega(G) - (\Delta(G)+1)$; in particular, each $V_i$ is nonempty. Thus, to find a hitting set in $G$ it suffices to find an IT in $H$. For this we use Lemma 10.11, with $t := \lfloor \frac{1}{3}(\Delta(G)+1) \rfloor$. $\mathcal{T}$, $X_\mathcal{T}$ $\mathcal{C}_i$, $V_i$ $H$ $t$

Fix an arbitrary part $V_i$ of $H$ and $v \in V_i$. Since $\omega(G) \geqslant \lceil \frac{2}{3}(\Delta(G)+1) \rceil$, clearly $d_H(v) \leqslant \Delta(G)+1-\omega(G) \leqslant \lfloor \frac{1}{3}(\Delta(G)+1) \rfloor = t$. By Lemma 10.9, we have $|\bigcup_{T\in\mathcal{C}_i} V(T)| + |\bigcap_{T\in\mathcal{C}_i} V(T)| \geqslant 2\omega(G) \geqslant \lceil \frac{4}{3}(\Delta(G)+1) \rceil$. By definition, $\bigcup_{T\in\mathcal{C}_i} V(T) \subseteq N_G(v)$. So rearranging terms in the previous inequality gives the new inequality $|V_i| - t = |\bigcap_{T\in\mathcal{C}_i} V(T)| - \lfloor \frac{1}{3}(\Delta(G)+1) \rfloor \geqslant \Delta(G)+1-|\bigcup_{T\in\mathcal{C}_i} V(T)| \geqslant d_H(v)$. That is, $d_H(v) \leqslant \min\{t, |V_i|-t\}$. Because $V_i$ and $v$ were arbitrary, $H$ satisfies the hypotheses of Lemma 10.11 with $t := \lfloor \frac{1}{3}(\Delta(G)+1) \rfloor$. Finally, this IT in $H$ is the desired hitting set in $G$. $\square$

## 10.3 Independent Transversals and Strong Coloring

**Definition 10.12.** Let $G$ be a graph and $V_1, \cdots, V_s$ be a vertex partition of $G$. An *independent transversal (IT)* of $G$ is an independent set that contains exactly onevertex in each $V_j$. A *partial independent transversal (PIT)* is an independent set that contains at most one vertex in each $V_j$. For every $J \subseteq [s]$, let $G_J := G[\cup_{j\in J} V_j]$. A set $W \subseteq V(G)$ *totally dominates* $G$ if each vertex in $V(G)$ has a neighbor in $W$. (This differs from the usual definition of domination, since even each vertex in $W$ must have a neighbor in $W$.) IT: independent transversal PIT: partial independent transversal $G_J$, totally dominates

**Lemma 10.13.** *Let* $G$ *be a graph with a vertex partition* $V_1, \cdots, V_s$*. If, for each* $J \subseteq [s]$*, the graph* $G_J$ *has no totally dominating set of size at most* $2(|J|-1)$*, then* $G$ *has an IT.*

Algorithm 10.1 takes as input a PIT $I$ with $|I| < s$ and outputs either (a) a PIT $I'$ with $|I'| = |I|+1$ or (b) a set $J \subseteq [s]$ where $G_J$ has a totally dominating set of size at most $2(|J|-1)$.

In some part with no vertex of $I$, choose an arbitrary vertex $x_1$. We will transform $I$ into a PIT $I'$ (with $|I'| = |I|$) such that $I' \cup \{x_1\}$ is a PIT. What stops us from adding $x_1$ to $I$? It is precisely $N(x_1) \cap I$. So, if we transform $I$ into $I'$ with $|N(x_1) \cap I'| < |N(x_1) \cap I|$, then we have progressed toward adding $x_1$ to our PIT. To reach such an $I'$ we pick $y \in N(x_1) \cap I$, call its part $V'$, and pick $x_2 \in V' \setminus N(x_1)$. If we can substitute vertex $x_2$ for vertex $y$ in our PIT, then we are closer to adding $x_1$. But now, recursively, we must make progress toward substituting $x_2$ for $y$.

*Proof.* We start with a PIT $I$ that consists of an arbitrary vertex and run Algorithm 10.1, below. By hypothesis, it cannot return a set $J \subseteq [s]$ and a set of size at most $2(|J|-1)$ that totally dominates $G_J$. Thus, it returns a larger PIT. By repeating this process, we eventually reach an IT of $G$. So all that remains is to check the correctness of Algorithm 10.1.

Before analyzing the algorithm in detail, we give a brief overview. For each vertex $w$, let $p(w)$ denote the index such that $w \in V_{p(w)}$. Throughout $F$ is a star forest (each component is $p(w)$

**Algorithm 10.1:** Finding a larger partial independent transversal (PIT)

**Input :** A graph $G$ with $V(G) = V_1, \ldots, V_s$ and a PIT $I$ with $|I| < s$
**Output:** (a) A PIT $I'$ with $|I'| = |I| + 1$ or (b) a set $J \subseteq [s]$ and a set of size at most $2(|J| - 1)$ that totally dominates $G_J$.

1 **let** $i := 1$ and **let** $V(F) := \emptyset$
2 **let** $x_1$ be an arbitrary vertex in $V(G) \setminus \bigcup_{v \in I} V_{p(v)}$, let $J := \{p(x_1)\}$, and **goto** 4
3 **let** $i := i + 1$ and **let** $x_i$ be an arbitrary vertex in $V(G_J) \setminus \cup_{v \in V(F)} N(v)$
4 **if** $(d_I(x_i) = 0)$ and $I \cap V_{p(x_i)} = \emptyset$ **then**
5     **output** $I \cup \{x_i\}$ and **stop**
6 **elseif** $d_I(x_i) = 0$, **then** \\\\ ($I \cap V_{p(x_i)} \neq \emptyset$, so substitute $x_i$ for $y_i$ in $I$)
7     **let** $y_i := I \cap V_{p(x_i)}$ and **let** $I := I - y_i + x_i$
8     **define** $\ell$ such that $y_i \in N(x_\ell)$
9     **let** $F := (F - y_i) \setminus \bigcup_{j > \ell}(x_j \cup N_I(x_j))$ and **let** $i := \ell$ and **let** $J := \bigcup_{v \in F}\{p(v)\}$
10     **goto** 3
11 **else** \\\\ (Grow $F$, by adding star centered at $x_i$ with all leaves in $I$)
12     **let** $V(F) := V(F) \cup \{x_i\} \cup N_I(x_i)$ and **let** $E(F) := E(F) \cup \{x_i y : y \in N_I(x_i)\}$
13     **let** $J := \bigcup_{v \in F}\{p(v)\}$
14     **if** $V(F)$ totally dominates $G_J$, **then**
15         **output** $J$ and $V(F)$ and **stop**
16     **else**
17         **goto** 3

a star), where each center of a star in $F$ is not in $I$ and each leaf of a star is in $I$. Each center $w$ is a vertex that we want to substitute into $I$, replacing the current vertex of $I$ in $V_{p(w)}$. The reason that we cannot currently make this substitution is the vertices of $I$ adjacent to $w$ (its neighbors in $F$). $J$ is the set of indices of parts with vertices in $F$, and $i$ the number of components (stars) in $F$. Figure 10.2 shows an example; there $i$ is the number of finite coordinates of the signature vector, and each coordinate is the degree in $F$ of some center of a star.

Now we analyze the algorithm. The set $I$ is only modified on line 7, so $I$ is always a PIT and $|I|$ never changes. If the algorithm stops on line 5, then we are happy. Now consider $F$. Throughout it is a star forest. This is trivially true on line 1, and is preserved each time that we grow $F$ (line 12) or shrink it (line 9). Furthermore, we always have $|F| \leqslant 2(|J| - 1)$. This is true the first time we visit line 12, since $|F| = |J| = 1 + d_I(v_1) \geqslant 2$. It is preserved each later time that we visit line 12, since we increase $|J|$ by $d_I(v_i) \geqslant 1$ and increase $|F|$ by $1 + d_I(v_i)$. Finally, it is preserved[2] each time we visit line 9. Thus, if the algorithm stops on line 15, then the sets $J$ and $V(F)$ are as desired.

To show the algorithm halts, we introduce a *signature vector* $(d_I(x_1), \ldots, d_I(x_i), \infty)$ to

[2]If line 9 causes $d_I(x_\ell) = 0$, then possibly $|F| = 2|J| - 1$. However, the next time through the loop we swap $x_\ell$ into $I$. Thus, we will have restored the invariant $|F| \leqslant 2(|J| - 1)$ when we next reach line 10.

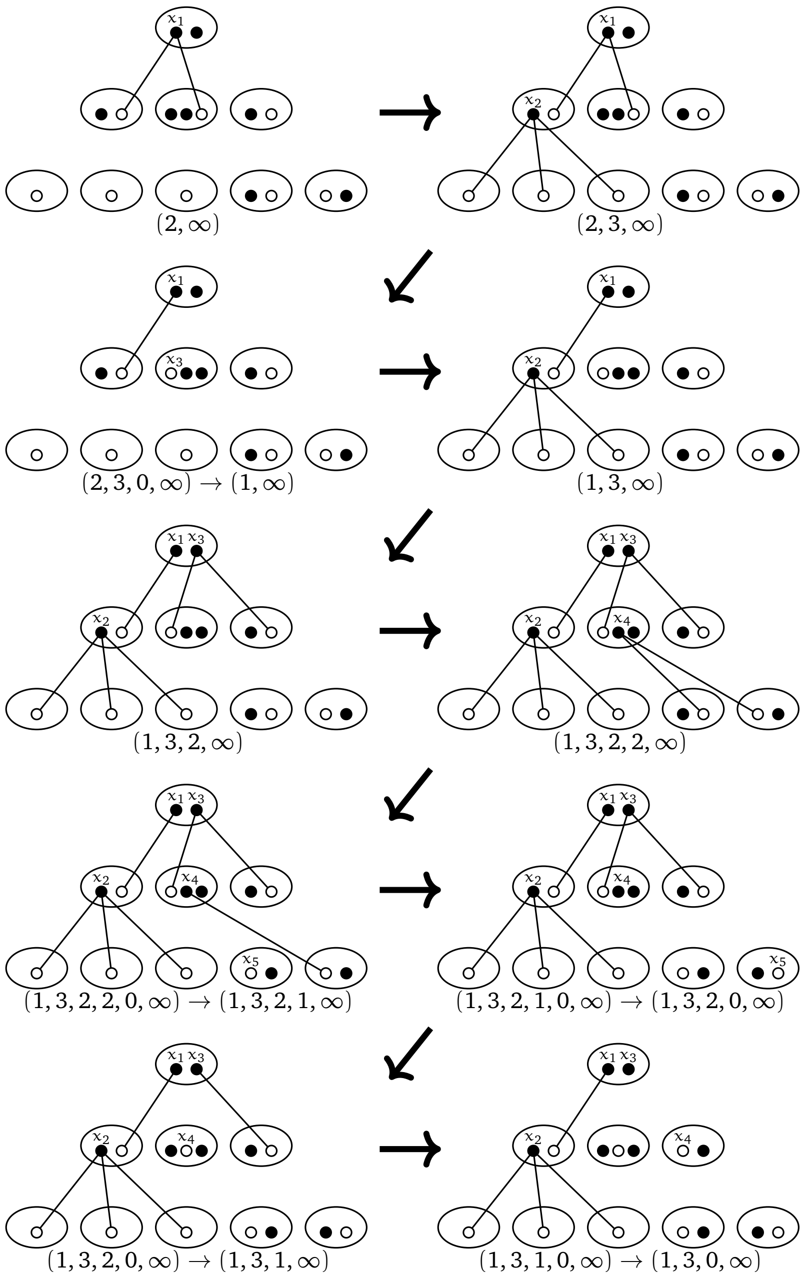



Figure 10.2: An example run of Algorithm 10.1. Each oval depicts a part $V_i$. White vertices are in $I$. All vertices involved in the run are shown at each step, but only edges of $F$ are shown. After the final step, $x_3$ is added to $I$ and the algorithm halts.

measure our progress; denote this vector by $\sigma$. Note that $N_I(x_j) \neq \emptyset$ for all $j \in [i-1]$. Further, these neighborhoods are disjoint. Thus, the finite entries in $\sigma$ sum to at most $|I|$. So $\sigma$ always has length at most $|I| + 1 \leqslant s$. The key observation is that each time we modify $F$ the vector $\sigma$ decreases lexicographically. When we grow $F$, this holds because we replace our infinite coordinate with a finite one. When we shrink $F$, this holds because we decrease $d_I(x_\ell)$. $\square$

By examining the proof of Lemma 10.13, we can strengthen its statement as follows.

**Corollary 10.14.** *Let $G$ be a graph with a vertex partition $V_1, \cdots, V_s$. If $G$ has no IT, then there exists $J \subseteq [s]$ such that $G_J$ has a totally dominating set $D$ of size at most $2(|J|-1)$. Furthermore, $D$ contains a partial independent transversal $D'$ of $G_J$ of size at least $|J|-1$.*

The following lemma is a simple consequence of Lemma 10.13.

**Lemma 10.15.** *Let $G$ be a graph with a vertex partition $V_1, \cdots, V_s$. If $|V_i| \geqslant 2\Delta(G)$, for all $i$, and $v \in V(G)$, then $G$ has an IT that contains $v$.*

*Proof.* By symmetry, assume that $v \in V_1$. Let $G' := G - (N(v) \cup V_1)$. Let $W_2, \cdots, W_s$ be the vertex partition of $G'$ inherited from $G$. Let $\Delta := \Delta(G)$. For all $i \geqslant 2$, note that $|W_i| \geqslant |V_i| - \Delta \geqslant \Delta$; in particular, $W_i \neq \emptyset$. If $G'$ has an IT $I$, then $I \cup \{v\}$ is an IT of $G$. So it suffices to show that $G'$ satisfies the hypotheses of Lemma 10.13.

For each $J \subseteq \{2, \ldots, s\}$, we have $|V(G_J)| = \sum_{j \in J} |W_j| \geqslant (\sum_{j \in J} |V_j|) - \Delta \geqslant 2\Delta|J| - \Delta$. Since each vertex dominates at most $\Delta$ vertices (excluding itself), any set totally dominating $G_J$ has size at least $|V(G_J)|/\Delta \geqslant (2\Delta|J| - \Delta)/\Delta = 2|J| - 1$. So $G'$ has an IT $I$, by Lemma 10.13; thus, $G$ has the IT $I \cup \{v\}$. $\square$

To close this section, we prove a nice upper bound on strong chromatic number.

**Definition 10.16.** A $k$-*partition* of a graph $G$ is a partition of $V(G)$ into parts of size $k$, first adding $\lceil |G|/k \rceil k - |G|$ isolated vertices, so the total number of vertices is a multiple of $k$. We often call these added isolated vertices *fake* vertices. A proper $k$-coloring $\varphi$ of a graph $G$ *respects a* $k$-*partition* of $G$ if each color class of $\varphi$ is an independent transversal of the partition; that is, each color class contains exactly one vertex from each part of the partition. A graph $G$ is *strongly* $k$-*colorable* if, given any $k$-partition of $G$, there exists a $k$-coloring of $G$ that respects the partition. The *strong chromatic number*, $\chi_s(G)$, of a graph $G$ is the minimum $k$ such that $G$ is strongly $k$-colorable. It is true, though not obvious, that if a graph is strongly $k$-colorable, then it is also strongly $(k+1)$-colorable. We include a proof in the appendix; see Theorem A.4.

$\chi_s$

**Theorem 10.17.** *Every graph $G$ has $\chi_s(G) \leqslant 3\Delta(G)$.*

Let $k := 3\Delta(G)$. Given our $k$-partition, we will iteratively grow a partial $k$-coloring that respects the partition. To do this, let $v$ be a vertex that is currently uncolored, and $\alpha$ a color not used on the part containing $v$. By Lemma 10.15, we will find an IT $I$ containing $v$ and use $\alpha$ on each vertex in $I$. Some vertices in $I$ may already be colored, and may have $\alpha$ used elsewhere

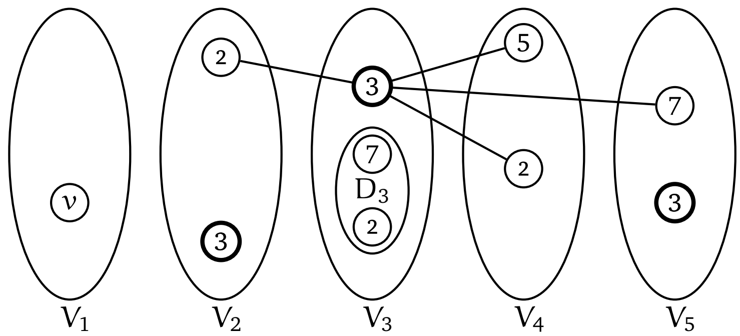



Figure 10.3: A graph $G$, a partition $V_1, \ldots, V_5$ of $V(G)$, and a partial coloring $\varphi$ in the proof of Theorem 10.17; only some vertices and edges are drawn. Here $\alpha = 3$, and each $v_i$ is bold; note that $v_4$ does not exist. Further, $F_3 = \{2, 5, 7\}$, but $|D_3| = 2$ since color 5 is unused on $V_3$ by $\varphi$.

in their parts. To ensure that our new partial $k$-coloring colors more vertices than our current partial $k$-coloring, each such $w \in I$ will "swap colors" with the vertex in its part that currently uses color $\alpha$. To guarantee that this yields a proper coloring, we must exclude certain vertices from appearing in $I$.

*Proof.* Fix a graph $G$, and let $k := 3\Delta(G)$. We can assume that $|G|$ is a multiple of $k$. Fix a vertex partition $V_1, \cdots, V_s$; call it $\mathcal{P}$. We will iteratively grow a partial $k$-coloring $\varphi$ that respects $\mathcal{P}$, ultimately reaching a $k$-coloring of $G$ that respects $\mathcal{P}$. We use induction on the number $t$ of vertices that are uncolored by $\varphi$. When $t = 0$ we are done; so assume $t > 0$. Let $v$ be a vertex that is uncolored by $\varphi$; by symmetry, assume $v \in V_1$. Since $v$ is uncolored, some color $\alpha$ is unused on $V_1$.

$k$ · $\mathcal{P}, \varphi$ · $v$ · $\alpha$

For each $j \in \{2, \ldots, s\}$, let $v_j$ be the vertex (if it exists) in $V_j$ such that $\varphi(v_j) = \alpha$. We will construct an IT $I$ containing $v$, recolor all of $I$ with $\alpha$, and then recolor each vertex $v_j$ previously colored $\alpha$ with the old color of the vertex in its part now colored $\alpha$ (if that vertex was colored). We must ensure that when $v_j$ is recolored, the new color that it gets is not already used in $N(v_j)$. Thus, we let $F_j := \{\varphi(x) : x \in N(v_j)\}$ ($F$ is for forbidden colors). So each vertex in $V_j$ using a color in $F_j$ is unavailable to be in $I$ (it is temporarily "deleted" from $V_j$). Formally, let $D_j := \{w \in V_j : \varphi(w) \in F_j\}$ ($D$ is for deleted vertices). Finally, let $W_j := V_j \setminus D_j$, and let $G' := G[V_1 \cup \bigcup_{j=2}^{s} W_j]$. See Figure 10.3. To show that $G'$ has an IT, note that $|W_j| \geqslant |V_j| - \Delta \geqslant 2\Delta$, for all $j$; thus, $G'$ has an IT $I$ that contains $v$, by Lemma 10.15. Let $w_j := W_j \cap I$. Form $\varphi'$ from $\varphi$ by coloring (or recoloring) each vertex in $I$ with $\alpha$ and recoloring each $v_j$ with $\varphi(w_j)$. Now we show that $\varphi'$ is a partial $k$-coloring that respects $\mathcal{P}$ and $\varphi'$ has fewer uncolored vertices than $\varphi$.

$v_j$ · $F_j$ · $D_j$ · $W_j, G'$ · $w_j, \varphi'$

In $V_1$, $\varphi'$ colors more vertices than $\varphi$ (since it colors $v$), and in each other $V_i$ our new coloring $\varphi'$ colors at least as many vertices as does $\varphi$. In each $V_i$, each color is still used at most once. Finally, we must check that $\varphi'$ is a proper coloring. Since $I$ is an IT, no edge has $\alpha$ used on both endpoints. Now consider some $v_j$ that was recolored with $\varphi(w_j)$. We must check that each vertex in $N(v_j)$ has a color other than $\varphi(w_j)$. Recall that the set of all $v_j$s is independent (since they were all colored with $\alpha$); thus, for each $x \in N(v_j)$ we have $\varphi'(x) = \varphi(x)$. But also $F_j := \{\varphi(x) : x \in N(v_j)\} = \{\varphi'(x) : x \in N(v_j)\}$. Since $W_j := V_j \setminus D_j$, we have $\varphi'(v_j) = \varphi(w_j) \notin F_j$. Thus, $\varphi'$ is proper, as desired. □

Lemma 10.15 has many applications, most of which do not need that the IT contains a prescribed vertex. Exercise 6 considers a pretty example, and in the Notes we provide references to further examples.

## 10.4 Ore Degree and a Strengthening of Brooks' Theorem

Ore-degree $\theta$
low, high
vertex-critical

**Definition 10.18.** The *Ore-degree*, denoted $\theta(G)$, of a graph $G$ is $\max_{vw \in E(G)} d(v) + d(w)$. For brevity, we often write $\theta$, rather than $\theta(G)$; similarly for $\chi$, $\omega$, $\Delta$, and $\delta$. A vertex $v$ is *low* if $d(v) = \chi - 1$ and *high* otherwise. A graph $G$ is *vertex-critical* if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

Greedy coloring proves the trivial bound $\chi \leqslant \Delta + 1$. And Brooks' Theorem characterizes the graphs $G$ where equality holds: If $\Delta \geqslant 3$, then equality holds precisely when $G$ contains $K_{\Delta+1}$. For Ore-degree, the analogous statement is $\chi \leqslant \lfloor \frac{\theta}{2} \rfloor + 1$. Again we color greedily, now starting with all vertices of degree at least $\lfloor \frac{\theta}{2} \rfloor + 1$. These form an independent set; thus, $\chi \leqslant \lfloor \frac{\theta}{2} \rfloor + 1$. Our main result in this section characterizes graphs for which equality holds, when $\chi \geqslant 7$.

**Theorem 10.19.** *If* $\chi(G) = \left\lfloor \frac{\theta(G)}{2} \right\rfloor + 1$ *and* $\chi(G) \geqslant 7$*, then* $G$ *contains* $K_{\chi(G)}$.

Theorem 10.19 remains true [339] if we weaken the hypothesis to $\chi \geqslant 6$. The proof when $\chi = 6$ follows the same outline as for larger $\chi$, but it has many more technical details, so we omit it. In contrast, the statement becomes false when we weaken the hypothesis to $\chi \geqslant 5$. And for $\chi \geqslant 4$, it has an infinite family of counterexamples (no one containing another as a subgraph); see Exercise 11.

To prove Theorem 10.19, we can assume that it is false, and that $G$ is vertex-critical counterexample; otherwise, we apply the result to a vertex-critical subgraph. Let $t := \Delta - \delta$. By hypothesis, $\chi - 1 = \lfloor \frac{\theta}{2} \rfloor \geqslant \lfloor \frac{\Delta+\delta}{2} \rfloor = \lfloor \frac{2\Delta - t}{2} \rfloor \geqslant \Delta - \frac{t+1}{2}$. Since $G$ is vertex-critical, $\Delta - t = \delta \geqslant \chi - 1$. Combining these inequalities, gives $t \leqslant \frac{t+1}{2}$, which yields $t \leqslant 1$. Thus, if $G$ is vertex critical, then $\theta \geqslant 2\Delta - 1$. If $\theta = 2\Delta$, then Theorem 10.19 simplifies to Brooks' Theorem. So the interesting case is when $\theta = 2\Delta - 1$. Now *low* means degree $\Delta - 1$ and *high* means degree $\Delta$. Hence, we can restate Theorem 10.19 as follows.

t

**Theorem 10.20.** *Let* $G$ *be a graph in which the vertices of degree* $\Delta(G)$ *form an independent set. If* $\chi(G) = \Delta(G) \geqslant 7$*, then* $G$ *contains* $K_{\chi(G)}$.

Since $G$ is vertex-critical, we start with a low vertex $x_0$ and a $(\Delta-1)$-coloring $\varphi_0$ of $G-x_0$. Our idea is to modify $\varphi_0$ by "stealing" a color for $x_0$ from one of its low neighbors. This yields a new uncolored vertex $x_1$ and a new $(\Delta-1)$-coloring $\varphi_1$ of $G-x_1$. If we are unable to extend $\varphi_i$ to a $(\Delta-1)$-coloring of $G$, then by Brooks' Theorem we conclude that $x_i$ must lie in big cliques with many of its neighbors. By repeatedly modifying our coloring, eventually we find two disjoint cliques with orders summing to $\Delta$, and two vertices in one joined to the other. Since $G$ contains no copy of $K_\Delta$, the subgraph $J$ induced by the union of their vertices is not complete. Since $G$ is vertex-critical, $G-J$ has a $(\Delta-1)$-coloring $\varphi$. Since high vertices form an independent set, most vertices in $J$ are low in $G$. Thus, we are able to extend $\varphi$ to a $(\Delta-1)$-coloring of $G$, a contradiction.

To formalize the approach sketched in the previous paragraph, we use a tool called Mozhan Partitions. In the proof below of Theorem 10.20, we will only use a very simple instance of Mozhan Partitions. However, we define things more generally, since the more general version has proved useful in related problems.

**Definition 10.21.** Let $G$ be a graph with a vertex $v$ such that $\chi(G-v)<\chi(G)$. Fix positive integers $s$ and $k_1,\dots,k_s$ such that $1+\sum_{i=1}^{s}k_s=\chi(G)$. Consider a $\chi(G)$-coloring $\varphi$ of $G$ such that color $\chi(G)$ is used only on a single vertex, call it $x$. Let $a_0:=0$, for each $i\in[s]$ let $a_i:=\sum_{j=1}^{i}k_j$, and let $V_i:=\cup_{j=1+a_{i-1}}^{a_i}\varphi^{-1}(j)$. That is, $V_1$ consists of the first $k_1$ color classes, $V_2$ consists of the next $k_2$ color classes, etc. For example, we might have $\Delta=13$, $s=4$, $k_i=3$ for all $i\in\{1,\dots,4\}$ and $a_i=3i$ for all $i\in\{0,\dots,4\}$. Now $V_1:=\varphi^{-1}(1)\cup\varphi^{-1}(2)\cup\varphi^{-1}(3),\dots,V_4:=\varphi^{-1}(10)\cup\varphi^{-1}(11)\cup\varphi^{-1}(12)$.

$s, k_1,\dots,k_s$

$x, a_i, V_i$

Among all such colorings $\varphi$, choose one that minimizes the total number of edges within parts. That is, it minimizes $\sum_{i=1}^{s}\|G[V_i]\|$. Such a coloring is called *minimal*, and its induced (ordered) partition $V_1,\cdots,V_s\cup\{x\}$ is called a *Mozhan Partition*. Given a graph $G$, a Mozhan Partition of $G$, and a vertex $w\in N_{V_i}(x)$, for some $i$, to *swap* $x$ *and* $w$, we swap the colors on $x$ and $w$, move $x$ to $V_i$, and make $w$ the new singleton color class. (Swapping $x$ and $w$ will not necessarily give a proper coloring, but it will when $d_{V_i}(x)=k_i$.)

minimal

Mozhan Partition

swap

**Lemma 10.22.** *Fix a graph* $G$. *Let* $\varphi$ *be a minimal* $\chi(G)$*-coloring of* $G$ *with a Mozhan Partition* $V_1,\cdots,V_s,\{x\}$. *If* $d_{V_i}(x)=k_i\geqslant 3$ *for some* $i\in[s]$, *then* $N_{V_i}(x)\cup\{x\}$ *induces* $K_{k_i+1}$. *Now swapping* $x$ *with any of its neighbors in* $V_i$ *yields another minimal coloring.*

*Proof.* Let $H_i$ denote the component of $G[V_i\cup\{x\}]$ that contains $x$. First suppose that $\Delta(H_i)>k_i$. Let $w$ be a vertex in $H_i$ with $d_{H_i}(w)>k_i$; among all such vertices, choose one that is closest to $x$ in $H_i$. Denote a shortest path in $H_i$ from $x$ to $w$ by $x_0,x_1,\dots,x_\ell$, with $x_0=x$ and $x_\ell=w$. If there exists $j\in[\ell-1]$ such that $d_{H_i}(x_j)<k_i$, then we can recolor $x$ with $\varphi(x_1)$, recolor $x_1$ with $\varphi(x_2)$, etc., and lastly recolor $x_j$. This gives a $(\chi(G)-1)$-coloring of $G$, a contradiction. So assume instead that $d_{H_i}(x_j)=k_i$ for all $i\in[\ell-1]$. Similar to the previous case, recolor $x$ with $\varphi(x_1)$, recolor $x_1$ with $\varphi(x_2)$, etc. Finally, recolor $w$ with $\varphi(x)$. We have decreased $\|G[V_i]\|$ by $d_{V_i}(w)-d_{V_i}(x)>0$. This contradicts the minimality in our definition of minimal coloring. Thus, $\Delta(H_i)=k_i$.

If $\chi(H_i) \leqslant k_i$, then we recolor $H_i$ to get a $(\chi(G)-1)$-coloring of G, a contradiction. So $\chi(H_i) > k_i = \Delta(H_i)$. Since $H_i$ is connected, by Brooks' Theorem, $H_i \cong K_{k_i+1}$.

Suppose we swap $x$ with a neighbor in $V_i$. Since $H_i \cong K_{k_i+1}$, this yields a proper coloring. And $|E(G[V_i])|$ remains unchanged. So this new coloring is also minimal. □

Now we can prove Theorem 10.20.

*Proof of Theorem 10.20.* We can assume that G is vertex-critical. If $\omega \geqslant \Delta$, then we are done; so assume $\omega < \Delta$. Let $k_1 := \lfloor \frac{\Delta-1}{2} \rfloor$ and $k_2 := \lceil \frac{\Delta-1}{2} \rceil$. Since $\Delta \geqslant 7$, we know $k_2 \geqslant k_1 \geqslant 3$. If $x$ is high, then all of its neighbors are low. By Lemma 10.22, swapping $x$ with any neighbor yields another minimal coloring. So we assume $x$ is low.

Now we apply Algorithm 10.2 below to find a reducible subgraph (this is the algorithm's output). We first prove that the algorithm gives the desired output; we call this subgraph J. In the final two paragraphs of the proof, we show how to extend a $(\Delta-1)$-coloring of $G-J$ to all of G. (Also, $p(i)$ denotes the part of $x_i$, with always $p(i) \in \{1,2\}$.)

By line 3, vertex $x_i$ is always low, so $d_{V_j}(x_i) = k_j$ for each $j \in [2]$. In particular, $d_{V_{p(i)}}(x_i) = k_{p(i)}$ for each $i$. Hence, by Lemma 10.22, $N_{V_{p(i)}}(x_i) \cup \{x_i\}$ induces $K_{k_{p(i)}+1}$ and swapping $x_i$ with some low neighbor in $V_{p(i)}$ yields another minimal coloring. Thus, $\varphi_i$ is always minimal. (We could formalize all this with induction on $i$, but we omit the details.)

**Algorithm 10.2:** Repeatedly swapping $x_i$ for a low neighbor $x_{i+1}$ that has not moved

**Input :** A graph G and a minimal coloring $\varphi$, with Mozhan Partition $V_1, V_2, \{x\}$
**Output:** A minimal coloring $\varphi_i$ (with its vertex $x_i$) and a part $V_j$ such that $x_i \cup N_{V_j}(x_i)$ induces $K_{k_j+1}$ and $x_i$ has two low neighbors $y_1, y_2$ in $V_j$ such that $y_\ell \cup N_{V_{3-j}}(y_\ell)$ induces $K_{k_{3-j}+1}$ for each $\ell \in [2]$

1 **let** $x_1 := x$, $\varphi_1 := \varphi$, $i := 1$, $p(i) := 1$
2 **while** $x_i$ has no low neighbor in $V_{p(i)}$ that has already moved
3     **let** $x_{i+1}$ be a low vertex in $V_{p(i)} \cap N(x_i)$
4     **swap** $x_i$ and $x_{i+1}$; call this new coloring $\varphi_{i+1}$; mark $x_i$ as having moved
5     **let** $i := i+1$
6     **let** $p(i) := 1$ if $i$ is odd and $p(i) := 2$ if $i$ is even.
7 **output** $\varphi_i$, $x_i$, and $V_{p(i)}$

By lines 3 and 4, only low vertices move. Further, a low vertex $w$ only moves by being swapped with $x_i$. But by line 2, this only happens if $w$ has not already moved. Thus, each vertex moves at most once.

Now we assume the algorithm outputs $\varphi_i$, $x_i$, and $V_{p(i)}$. We must show that these satisfy the desired properties of the output (in the algorithm's description). By symmetry, assume that $w, y$ $V_{p(i)}$ is $V_1$. Let $w$ be a low neighbor of $x_i$ in $V_1$ that has already moved; see Figure 10.4. Let $y$ be the low neighbor of $w$ that was swapped with $w$ when $w$ moved into $V_1$; so $y \in V_2$.

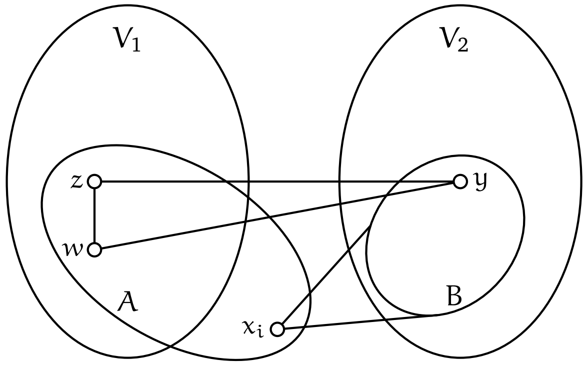



Figure 10.4: The output of Algorithm 10.2: two vertex-disjoint cliques, $A$ and $B$, with specified edges between them.

Let $A := N_{V_1}(x_i) \cup \{x_i\}$. By Lemma 10.22, $A$ induces $K_{k_1+1}$. Since high vertices form an independent set, at most one vertex in $A$ is high. Since $k_1 \geqslant \lfloor \frac{7-1}{2} \rfloor = 3$, $x_i$ has at least two low neighbors in $A$. Let $z$ be one of these other than $w$. By definition, $w \leftrightarrow y$. Also, $w \leftrightarrow z$, since $A$ induces a clique. It is easy to check that $z$ has never moved. Thus, $z \leftrightarrow y$. Let $B := N_{V_2}(w)$. By Lemma 10.22, swapping $x_i$ with either $w$ or $z$ creates another minimal coloring. So vertices $w$ and $z$ are both joined to $B$. That is, the algorithm gives the desired output.

$A$ $z$ $B$

By criticality, we can $(\Delta-1)$-color $G-(A\cup B)$. Now we extend this coloring to all of $G$. For each $v \in A\cup B$, form $L(v)$ from $[\Delta-1]$ by removing each color used on a neighbor of $v$. To extend our coloring to $G$, it suffices to $L$-color $G[A\cup B]$. Note that $|L(v)| \geqslant d_{A\cup B}(v)$ if $v$ is low and $|L(v)| \geqslant d_{A\cup B}(v)-1$ if $v$ is high. Clearly, $A\cup B$ has a pair $v_1$, $v_2$ of nonadjacent vertices, since $|A\cup B| = \Delta$, but $G$ has no copy of $K_\Delta$. By symmetry, assume $v_1 \in A\setminus\{w,z\}$ and $v_2 \in B$.

$v_1, v_2$

Our idea is to color $v_1$ and $v_2$ with a common color, and then greedily $L$-color the rest of $A\cup B$, ending with $w$ and $z$. Coloring greedily works, since $w$ and $z$ each dominate $A\cup B$. So all that remains is to show that $L(v_1)\cap L(v_2) \neq \emptyset$. Since $L(v_1)\cup L(v_2) \subseteq [\Delta-1]$, it suffices to show that $|L(v_1)|+|L(v_2)| \geqslant \Delta$. Let $\mathrm{low}(v_1,v_2)$ denote the number of low vertices in $\{v_1,v_2\}$.

$$\begin{aligned}
|L(v_1)|+|L(v_2)| &\geqslant (d_{A\cup B}(v_1)-1)+(d_{A\cup B}(v_2)-1)+\mathrm{low}(v_1,v_2)\\
&\geqslant (|A|-1+d_B(v_1)-1)+(|B|-1+|\{w,z\}|-1)+\mathrm{low}(v_1,v_2)\\
&= |A|+|B|-2+d_B(v_1)+\mathrm{low}(v_1,v_2)\\
&= \Delta-2+d_B(v_1)+\mathrm{low}(v_1,v_2).
\end{aligned}$$

So, if $d_B(v_1)+\mathrm{low}(v_1,v_2) \geqslant 2$, then we are done. If $x_i \not\leftrightarrow y$, then we are happy, since $\mathrm{low}(x_i,y)=2$. So assume $x_i \leftrightarrow y$, which implies that $d_B(x_i) \geqslant 1$. Since $x_i$ is low, we are done unless $x_i$ is joined to all of $A$; so assume it is. But now we can strengthen by 1 the inequality in line 2 (to account for edge $x_iv_2$), since $d_{A\cup B}(v_2) \geqslant |B|-1+|\{w,z,x_i\}|$. Again we are done unless every low vertex $v_2 \in B$ is joined to $A$. But in that case, $d_B(v_1) \geqslant 2$ for all $v_1 \in A$, which concludes the proof. □

## 10.5 Clustered Coloring

In this section we revisit the defective (possibly improper) colorings, we saw at the end of Section 10.1. When $r := 2$, Theorem 10.3 shows that $G$ is $\lfloor \frac{2}{3}\text{mad}(G) + 1 \rfloor$-choosable with defect 2. But the promised coloring might have monochromatic paths and cycles that are arbitrarily long. In Theorem 10.23 we require slightly larger lists, but in exchange we get that each monochromatic component has size at most 9. A coloring has *clustering* $r$ if each monochromatic component has order at most $r$. For a defective coloring $\varphi$ of a graph $G$, let $G[\varphi]$ denote the subgraph of $G$ induced by edges that are monochromatic under $\varphi$. Our main result is the following theorem.

clustering

$G[\varphi]$

**Theorem 10.23.** *Every graph $G$ is $\lfloor \frac{7}{10}\mathit{mad}(G) + 1 \rfloor$-choosable with clustering 9.*

This section's outline mirrors the proof of Theorem 10.3 (via Lemmas 10.4 and 10.5). The main difference is that proving Lemma 10.26 is much more involved than proving Lemma 10.4. For this, we start with an L-coloring $\varphi$ of $G$ with defect 2. We then find a big independent set $W$ such that $G[\varphi] - W$ has only small components. Finally, we show how to L-color $W$ so that each monochromatic component has order at most 9. Our next lemma will help with this final step.

The following remark should be obvious from context, but for clarity we emphasize it.

**Remark 10.24.** Throughout this section, an L-coloring need not be proper.

**Lemma 10.25.** *Let $H$ be a bipartite graph with parts $W$ and $X$. Let $L$ be a list assignment such that $|L(w)| = 2$ for all $w \in W$ and $|L(x)| = 1$ for all $x \in X$. If every L-coloring has defect at most 2, then $H$ has an L-coloring where each monochromatic component has at most 2 vertices in $W$.*

For each $\alpha \in L(H)$, let $H_\alpha$ denote the subgraph induced by all vertices $v$ where $\alpha \in L(v)$. For each $\alpha$, we orient $E(H_\alpha)$ so that $\Delta^+(H_\alpha) \leqslant 1$ and $\Delta^-(H_\alpha) \leqslant 1$. We color the vertices in arbitrary order, choosing a color for each vertex arbitrarily from its list, subject to the following constraint: whenever our choice of color creates the possibility (but not yet realization) of a directed monochromatic $P_3$, we avoid this possibility by coloring the uncolored vertex with some other color. We call a directed path a *dipath*.

dipath

$L(H)$

$H_\alpha$

*Proof.* Let $L(H)$ denote $\cup_{v \in V(H)} L(v)$. For each $\alpha \in L(H)$, let $H_\alpha$ denote the subgraph induced by all vertices $v$ where $\alpha \in L(v)$. Since $|L(x)| = 1$ for all $x \in X$, each edge of $H$ appears in $H_\alpha$ for at most one $\alpha$. By hypothesis, every L-coloring has defect 2. Thus, $\Delta(H_\alpha) \leqslant 2$ for every $\alpha \in L(H)$. So we orient the edges of every $H_\alpha$ such that each vertex has indegree at most 1 and has outdegree at most 1 (though its indegree and outdegree in $H$ may be as large as 2). Orient every other edge of $H$ arbitrarily. We construct an L-coloring $\varphi$ by Algorithm 10.3.

Clearly, Algorithm 10.3 outputs an L-coloring $\varphi$. We must simply check that $\varphi$ has the desired property. Assume not. Recall that $H$ is directed and each vertex has indegree at most 1 and outdegree at most 1 in the directed subgraph $H_\alpha$, for each $\alpha \in L(H)$. So there exists a color $\alpha$ and a dipath $w_1x_1w_2x_2w_3$ with $w_i \in W$ and $y_i \in Y$ such that all 5 vertices on the path use color $\alpha$.

If $w_1$ is colored before $w_2$, then $w_2$ is colored immediately after $w_1$ and $\varphi(w_2) \neq \varphi(w_1)$, by lines 4–5. Similarly, if $w_2$ is colored before $w_3$, then $w_3$ is colored immediately after $w_2$ and $\varphi(w_3) \neq \varphi(w_2)$, again by lines 4–5. So assume that $w_3$ is colored before $w_2$ and that $w_2$ is colored before $w_1$. Now $w_1$ is colored immediately after $w_2$ and $\varphi(w_1) \neq \varphi(w_2)$, by lines 8–9. Hence, there cannot exist such a monochromatic dipath $w_1x_1w_2x_2w_3$. □

The proof of Lemma 10.25 above is complete as written, but we offer a bit more intuition about the correctness of Algorithm 10.3. The reader may find it disconcerting that coloring a single vertex can create both an instance on the left of Figure 10.5 and an instance in the center of Figure 10.5. When this happens, we immediately address the former, on line 4, but doing so may create further instances (in other colors), and it is unclear when we will be able to address the latter, on line 8. In fact, we do *not* need to handle all instances in the center of Figure 10.5. It is only essential that we handle them when they progress to an instance shown on the right

---

**Algorithm 10.3:** L-coloring H such that each monochromatic component has at most 2 vertices in $W$.

---

**Input** : A bipartite graph H with parts $W$ and X, a list assignment L such that $|L(w)| = 2$ for all $w \in W$ and $|L(x)| = 1$ for all $x \in X$ and each L-coloring has defect 2

**Output:** An L-coloring such that each monochromatic component contains at most 2 vertices of $W$

1 **color** each vertex in X with the unique color in its list and **let** $i := 1$
2 **let** $w_i$ be an arbitrary uncolored vertex of $W$
3 **choose** $\varphi(w_i)$ arbitrarily from $L(w_i)$
4 **if** there is uncolored $w' \in W$ and dipath $w_ixw'$ with $\varphi(w_i) = \varphi(x)$ and $\varphi(w_i) \in L(w')$, **then**
5     **choose** $\varphi(w')$ arbitrarily from $L(w') \setminus \{\varphi(w_i)\}$
6     **let** $i := i + 1$ and **let** $w_i := w'$
7     **goto** 4
8 **if** there is uncolored $w' \in W$ and dipath $w'xw_i$ with $\varphi(w_i) = \varphi(x)$ and $\varphi(w_i) \in L(w')$, **then**
9     **choose** $\varphi(w')$ arbitrarily from $L(w') \setminus \{\varphi(w_i)\}$
10     **let** $i := i + 1$ and **let** $w_i := w'$
11     **goto** 4
12 **if** $W$ has some uncolored vertex, **then**
13     **let** $i := i + 1$
14     **goto** 2
15 **output** $\varphi$

---

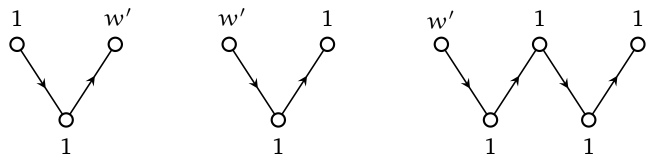


Figure 10.5: Left: Line 4 in the algorithm. Center: Line 8 in the algorithm. Right: An extended instance of that in the center, which will be fixed immediately.

of Figure 10.5. If this happens, then we have just colored the center vertex. But in doing so we have not created a new instance on the left of Figure 10.5, so we can immediately handle the instance on the right, on line 8.

Now we prove our analogue of Lemma 10.4.

**Lemma 10.26.** *Let* $I$ *be an independent set in a graph* $G$*, and let* $L$ *be a list assignment for* $G$*. If* $5|L(v)| > 2d(v) + 1$ *for all* $v \in V(G) \setminus I$ *and* $5|L(v)| > 2d(v)$ *for all* $v \in I$*, then* $G$ *has an* $L$*-coloring with clustering 9.*

We consider the set $\mathcal{C}$ of all L-colorings that minimize the number of monochromatic edges. For each $\varphi \in \mathcal{C}$, we show that $\Delta(G[\varphi]) \leqslant 2$. Starting from some $\varphi_0 \in \mathcal{C}$, we pick a set $W \subset V(G)$ and $L'(w) \subseteq L(w)$ with $|L'(w)| = 2$ for each $w \in W$. A key step is showing that recoloring any vertices in $W$ with colors in their lists $L'$ yields another L-coloring $\varphi'$ such that $\varphi' \in \mathcal{C}$. To finish, we find one such $\varphi'$ with clustering 9. (The set $I$ is not too important. On a first reading, the reader can focus on the case $I = \emptyset$.)

$\mathcal{C}$ *Proof.* Let $\mathcal{C}$ denote the set of all L-colorings that minimize the number of monochromatic
$L(\varphi, v)$ edges. When $\varphi \in \mathcal{C}$ and $v \in V(G)$, let $L(\varphi, v)$ denote the set of all colors $\alpha \in L(v)$ such that starting from $\varphi$ and recoloring $v$ with $\alpha$ yields an L-coloring $\varphi'$ for which $\varphi' \in \mathcal{C}$. In particular, $\varphi(v) \in L(\varphi, v)$.

**Claim 1.** *If* $\varphi \in \mathcal{C}$*, then* $\Delta(G[\varphi]) \leqslant 2$.

Proof. Suppose, to the contrary, that $\varphi \in \mathcal{C}$ and there exists $v \in V(G)$ with $d_{G[\varphi]}(v) \geqslant 3$. Since $\frac{5}{2}|L(v)| > d(v)$, there exists $\beta \in L(v)$ such that $\varphi$ uses $\beta$ on at most 2 neighbors of $v$. So recoloring $v$ with $\beta$ yields an L-coloring with fewer monochromatic edges, which contradicts that $\varphi \in \mathcal{C}$. ♢

**Claim 2.** *If* $\varphi, \varphi' \in \mathcal{C}$ *and* $v \in V(G) \setminus I$ *and* $d_{G[\varphi]}(v) = d_{G[\varphi']}(v) = 2$*, then* $|L(\varphi, v) \cap L(\varphi', v)| \geqslant 2$.

Proof. Suppose the claim is false for $\varphi, \varphi'$, and $v$. So $|L(\varphi, v) \cap L(\varphi', v)| \leqslant 1$. Now $|L(\varphi, v)| + |L(\varphi', v)| = |L(\varphi, v) \cup L(\varphi', v)| + |L(\varphi, v) \cap L(\varphi', v)| \leqslant |L(v)| + 1$. By symmetry between $\varphi$ and $\varphi'$, we assume that $|L(\varphi, v)| \leqslant \frac{1}{2}(|L(v)| + 1)$. For each $\alpha \in L(\varphi, v)$, color $\alpha$ is used by $\varphi$ on 2 neighbors of $v$. For each $\alpha \notin L(\varphi, v)$, color $\alpha$ is used by $\varphi$ on at least 3 neighbors of $v$.

So $d(v) \geqslant 2|L(\varphi, v)| + 3(|L(v)| - |L(\varphi, v)|) \geqslant 2(\frac{1}{2}(|L(v)| + 1) + 3(\frac{1}{2}(|L(v)| - 1) = \frac{5}{2}|L(v)| - \frac{1}{2}$, which contradicts the hypothesis. $\diamondsuit$

Choose $\varphi_0 \in \mathcal{C}$ and $W \subseteq V(G) \setminus I$ such that $W$ is independent in $G[\varphi_0]$ and $d_{G[\varphi_0]}(w) = 2$ for all $w \in W$; subject to this, choose $W$ to be maximum.[3] Denote $W$ by $\{w_1, \ldots, w_t\}$. For all $i \in [t]$, we recursively define $L$-colorings $\varphi_i$ as follows. Form $\varphi_i$ from $\varphi_{i-1}$ by recoloring $w_i$ with some color in $(L(\varphi_0, w_i) \cap L(\varphi_{i-1}, w_i)) \setminus \{\varphi_0(w_i)\}$. Formally, we prove that $\varphi_i$ exists and that $\varphi_i \in \mathcal{C}$ by induction on $i$. For $i = 0$ this is trivial. For $i > 0$, $\varphi_{i-1} \in \mathcal{C}$ by hypothesis. Now $|L(\varphi_0, v_i) \cap L(\varphi_{i-1}, v_i)| \geqslant 2$ by Claim 2, so $\varphi_i$ exists and $\varphi_i \in \mathcal{C}$. Finally, let $L'(v) := \{\varphi_0(v), \varphi_t(v)\}$ for all $v \in V(G)$. So $|L'(v)| = 2$ when $v \in W$ and $|L'(v)| = 1$ otherwise. Next, we prove that every $L'$-coloring is in $\mathcal{C}$.

$W$ $w_1, \ldots, w_t$ $\varphi_i$ $L'$

**Claim 3.** *$\varphi' \in \mathcal{C}$ for every $L'$-coloring $\varphi'$.*

We will show that $W$ is an independent set in $G[\varphi']$ and $d_{G[\varphi']}(w) = 2$ for all $w \in W$. Thus, $\|G[\varphi']\| = \|G[\varphi'] - W\| + 2\|W\| = \|G[\varphi_0] - W\| + 2\|W\| = \|G[\varphi_0]\|$, so $\varphi' \in \mathcal{C}$.
<u>Proof.</u> For all $w \in W$ and all $i \in \{0, \ldots, t\}$, we show that $W$ is an independent set in $G[\varphi_i]$ and $d_{G[\varphi_i]}(w) = 2$; we use induction on $i$. The base case, $i = 0$, holds by the definition of $W$. So assume $i > 0$. Suppose there exist $w_i, w_j \in W$ and $w_i w_j \in E(G)$ such that $\varphi_i(w_i) = \varphi_i(w_j)$. By hypothesis, $2 = d_{G[\varphi_{i-1}]}(w_j) = |N_{G[\varphi_{i-1}]}(w_j)| = |N_{G[\varphi_{i-1}]}(w_j) \setminus \{w_i\}|$. Thus, $d_{G[\varphi_i]}(w_j) = 1 + |N_{G[\varphi_i]}(w_j) \setminus \{w_i\}| = 1 + |N_{G[\varphi_{i-1}]}(w_j) \setminus \{w_i\}| = 1 + 2 = 3$, contradicting Claim 1. Hence, $W$ is an independent set in $G[\varphi_i]$. Since $\varphi_{i-1}, \varphi_i \in \mathcal{C}$, as shown above, and $\|G[\varphi_i]\| = \|G[\varphi_{i-1}]\| - d_{G[\varphi_{i-1}]}(w_i) + d_{G[\varphi_i]}(w_i)$, we see that $d_{G[\varphi_i]}(w_i) = d_{G[\varphi_{i-1}]}(w_i) = 2$, by hypothesis.

Again consider $w_i, w_j \in W$ with $w_i w_j \in E(G)$; assume also that $i < j$. In each $\varphi_i$, $W$ is independent, so $\varphi_0(w_i) \neq \varphi_0(w_j)$ and $\varphi_t(w_i) \neq \varphi_0(w_j)$ and $\varphi_t(w_i) \neq \varphi_t(w_j)$. Finally, suppose that $\varphi_0(w_i) = \varphi_t(w_j)$. Form $\varphi'$ from $\varphi_0$ by recoloring $w_j$ with $\varphi_t(w_j)$. Since $\varphi_t(w_j) \in L(\varphi_0, w_j)$, we have $\varphi' \in \mathcal{C}$. If $\varphi_t(w_j) = \varphi_0(w_i)$, then $d_{G[\varphi']}(w_i) = 3$ (as above), which contradicts Claim 1. Thus, $\varphi'(w_i) \neq \varphi'(w_j)$. This implies that $W$ is an independent set in $G[\varphi']$. Now for an arbitrary $L'$-coloring $\varphi''$, similarly $W$ is an independent set. So $d_{G[\varphi'']}(w_j) = d_{G[\varphi_0]}(w_j) = d_{G[\varphi_t]}(w_j) = 2$. Hence, $\|G[\varphi'']\| = \|G[\varphi''] - W\| + 2\|W\| = \|G[\varphi_0] - W\| + 2\|W\| = \|G[\varphi_0]\|$, which implies that $\varphi'' \in \mathcal{C}$. $\diamondsuit$

Let $H$ be a bipartite graph with parts $W$ and $X$, where each $x_i \in X$ represents a monochromatic component $C_i$ of $G[\varphi_0] - W$ and $x_i w_j \in E(H)$ if $\varphi_0(C_i) \in L'(w_j)$. Let $L'_H(x_i) := \{\varphi_0(C_i)\}$ and $L'_H(w_j) := L'(w_j)$. Recall, from Claim 3, that $\varphi' \in \mathcal{C}$ for every $L'$-coloring $\varphi'$ of $G$. So, by Claim 1, $\Delta(G[\varphi']) \leqslant 2$. Thus, $\Delta(H[\varphi'_H]) \leqslant 2$ for every $L'_H$-coloring $\varphi'_H$ of $H$. By Lemma 10.25, $H$ has an $L'_H$-coloring $\varphi'_H$ such that each monochromatic component contains at most 2 vertices of $W$. Clearly, $\varphi'_H$ gives rise to an $L'$-coloring $\varphi'$ of $G$. If each component of $G[\varphi']$ has order at most 9, then we are done. $H$

[3] This choice of $W$ may seem arbitrary. Intuitively, we want to be able to recolor vertices somewhat independently, so we take $W$ to be independent. But we also want as much flexibility as possible to recolor, so we take $W$ to be maximum. The requirement that $d_{G[\varphi_0]}(w) = 2$ is more technical, but its value will become apparent.

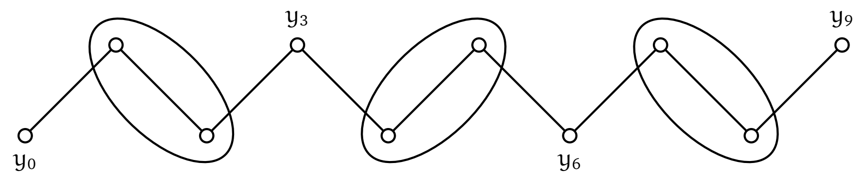



Figure 10.6: If $\varphi'_H$ has a monochromatic path P on at least 10 vertices, then let $W'$ be an independent set in $G \setminus I$ that contains one vertex from each circled pair on P. Replacing $W \cap V(P)$ in $W$ with $W'$ gives a set larger than $W$, contradicting our choice of $W$.

Assume instead that some component $C_0$ has order at least 10. Since $\Delta(G[\varphi']) \leqslant 2$, $C_0$ contains an induced path $P := y_1 \cdots y_8$ such that $d_{G[\varphi']}(y_i) = 2$ for all $i$; see Figure 10.6. Since $I$ is an independent set, it has at most one vertex in each of pairs $\{y_1, y_2\}$, $\{y_4, y_5\}$, and $\{y_7, y_8\}$. So $P \setminus I$ contains an independent set $W'$ with $|W'| = 3$. By our choice of $\varphi'_H$, we know $|W \cap V(P)| \leqslant 2$. Thus, $|(W \setminus V(P)) \cup W'| > |W|$, and $d_{G[\varphi']}(w) = 2$ for all $w \in W'$. This contradicts our choice of $\varphi_0$ and $W$ to maximize $|W|$. □

**Lemma 10.27.** *Let* $G$ *be a graph with vertex partition* $V(G) = A, B$*, let* $I \subseteq B$ *be an independent set, and let* $L$ *be a list assignment for* $B$*. If* $5|L(v)| > 5d_A(v) + 2d_B(v)$ *for all* $v \in I$ *and* $5|L(v)| > 5d_A(v) + 2d_B(v) + 1$ *for all* $v \in B \setminus I$*, then for every coloring* $\varphi$ *of* $G[A]$ *with clustering 9 there is an* $L$*-coloring* $\varphi'$ *of* $G[B]$ *such that* $\varphi \cup \varphi'$ *is a coloring of* $G$ *with clustering 9.*

*Proof.* Form $L'(v)$ from $L(v)$ by removing each color used on $N_A(v)$, for each $v \in B$. Now $|L'(v)| \geqslant |L(v)| - d_A(v)$, so $5|L'(v)| > 2d_B(v)$ for each $v \in I$ and $5|L'(v)| > 2d_B(v) + 1$ for each $v \in B \setminus I$. Thus, $G[B]$ has an $L'$-coloring $\varphi'$ with clustering 9, by Lemma 10.26. By construction, no edge from $A$ to $B$ is monochromatic. So $\varphi \cup \varphi'$ is a coloring of $G$ with clustering 9. □

**Theorem 10.24.** *Every graph* $G$ *is* $\lfloor \frac{7}{10} mad(G) + 1 \rfloor$*-choosable with clustering 9.*

We show that we can color some proper subgraph by induction and extend the coloring to the whole graph $G$ by Lemma 10.27. If no such proper subgraph exists, then a counting argument shows that $2\|G\| > |G| mad(G)$, a contradiction. This is similar to the proof of Theorem 10.3.

$k, L$ *Proof.* Let $k := \lfloor \frac{7}{10} mad(G) + 1 \rfloor$, and let $L$ be a $k$-assignment for $G$. We use induction on $|G|$.
$X_1, \ldots, X_t$ Let $X_1, \ldots, X_t$ be a maximal sequence of disjoint subsets of $V(G)$ such that, with $A_i := \cup_{j=1}^{i-1} X_j$
$A_i, B_i$ and $B_i := V(G) \setminus A_i$, for each $i \in [t]$ either

(a) $X_i = \{x_i\}$ and $5|L(x_i)| \leqslant 5d_{A_i}(x_i) + 2d_{B_i}(x_i)$ or

(b) $X_i = \{x_i^1, x_i^2\}$ and $x_i^1 x_i^2 \in E(G)$ and $5|L(x_i^j)| = 5d_{A_i}(x_i^j) + 2d_{B_i}(x_i^j) + 1$ for each $j \in [2]$.

$A, B$ Let $A := \cup_{j=1}^{t} X_i$ and $B := V(G) \setminus A$. First suppose that $B \neq \emptyset$. By induction, $G[A]$ has an $L$-clustering $\varphi$, since $|G[A]| < |G|$. Because $X_1, \ldots, X_t$ is maximal, $5|L(v)| > 5d_A(v) + 2d_B(v)$

for all $v \in B$. Let $I$ be the set of vertices $v$ in $B$ such that $5|L(v)| = 5d_A(v) + 2d_B(v) + 1$. Since $X_1, \ldots, X_t$ is maximal, no adjacent vertices in $B$ satisfy (b). Thus, $I$ is an independent set. Hence, $\varphi$ can be extended to all of $G$, by Lemma 10.27. I

Instead assume $B = \emptyset$. Let $R \subseteq [t]$ be the set of indices $i$ such that $|X_i| = 1$ and let $S := [t] \setminus R$. Note that $5d_{A_i}(z) + 2d_{B_i}(z) = 3d_{A_i}(z) + 2d_G(z)$ for all $i$ and all $z \in V(G)$. As a result we have R S

$$\begin{aligned}
5k|G| &= 5 \sum_{v \in V(G)} |L(v)| \\
&\leqslant \sum_{i \in R} 3d_{A_i}(x_i) + 2d_G(x_i) + \sum_{i \in S} \big((3d_{A_i}(x_i^1) + 2d_G(x_i^1) + 1) + (3d_{A_i}(x_i^2) + 2d_G(x_i^2) + 1)\big) \\
&= 2 \sum_{v \in V(G)} d_G(v) + 3 \sum_{i \in R} d_{A_i}(x_i) + 3 \sum_{i \in S} \left( d_{A_i}(x_i^1) + d_{A_i}(x_i^2) + \frac{2}{3} \right) \\
&\leqslant 4\|G\| + 3\|G\|.
\end{aligned}$$

So $5k|G| \leqslant 7\|G\|$. Thus, $2\frac{\|G\|}{|G|} \geqslant \frac{10}{7}k = \frac{10}{7}\lfloor \frac{7}{10}\text{mad}(G) + 1 \rfloor > \text{mad}(G)$, a contradiction. □

It is worth noting that the set $I$ is not too important; that is, it only slightly reduces the size of the lists needed. We explore this idea in Exercise 12.

## 10.6 Equitable Coloring

A $k$-coloring $\varphi$ is *equitable* if $\big||\varphi^{-1}(i)| - |\varphi^{-1}(j)|\big| \leqslant 1$ for all $i, j \in [k]$; that is, every two color classes differ in size by at most one. Our main result in this section is stated next. equitable

**Theorem 10.28.** *If $G$ is a graph with $\Delta(G) \leqslant k$, then $G$ has an equitable $(k+1)$-coloring.*

It is easy to check that $K_{k,k}$ has no equitable $k$-coloring when $k$ is odd. Thus, Theorem 10.28 is best possible. Clearly, $K_{k,k}$ has an equitable 2-coloring. So, in general, having an equitable $k$-coloring need not imply having an equitable $(k+1)$-coloring.

**Definition 10.29.** A $k$-coloring of a graph $G$ is *nearly equitable* if $|G| = ks$ for some $s \in \mathbb{Z}^+$ and each color class has size $s$ except for color classes $V^-$ and $V^+$, where $|V^-| = s - 1$ and $|V^+| = s + 1$. For a nearly equitable coloring $\varphi$ of a graph $G$, we build a directed graph $H(\varphi)$, or simply $H$, that has the color classes of $\varphi$ as its vertices. For color classes $W$ and $X$ in $V(H)$, we add the directed edge $WX$ if there exists a vertex $w$ of $W$ with no neighbor in $X$; so we could move $w$ to $X$ and still have a proper (though perhaps not equitable and not nearly equitable) coloring. Such a vertex $w$ *witnesses* the edge $WX$ in $H$. nearly equitable $V^-, V^+$ H witnesses

Let $\mathcal{A}$ be the vertices of $H$ with a dipath to $V^-$ and let $\mathcal{B} := V(H) \setminus \mathcal{A}$. Let $\mathcal{B}'$ be the vertices with a dipath from $V^+$. Finally, let $\mathcal{A}'$ be the subset of $\mathcal{A}$ such that for every $W \in \mathcal{A}'$, every other vertex $X \in \mathcal{A}$ has a dipath to $V^-$ in $H[\mathcal{A} - W]$. Each vertex of $\mathcal{A}'$ is a *terminal* vertex. $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$ terminal

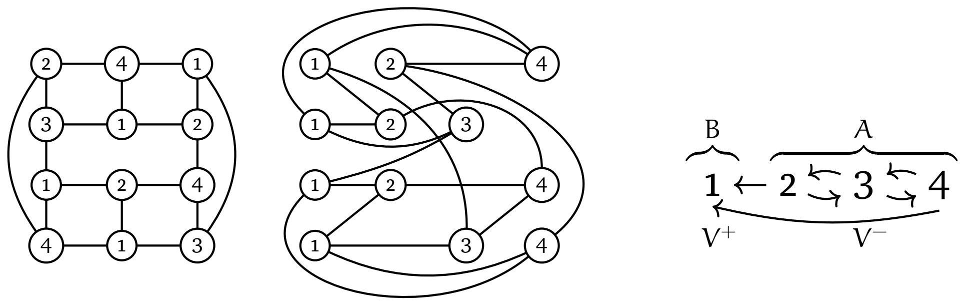



Figure 10.7: Left and center: A nearly-equitable 4-coloring of a 3-regular graph drawn (a) in the plane and (b) with vertices grouped vertically into independent sets (keeping each vertex in the same row). Right: The graph $H$ arising from this 4-coloring of $G$. Here $\mathcal{A}' = \{2, 4\}$ and $\mathcal{B}' = \{1\}$, so $a = 3$, $a' = 2$, $b = 1$, and $b' = 1$.

$A, B, A', B'$
$a, b, a', b'$
solo edge/vertex

Let $A := \bigcup \mathcal{A}$, let $B := \bigcup \mathcal{B}$, let $A' := \bigcup \mathcal{A}'$, let $B' := \bigcup \mathcal{B}'$, let $a := |\mathcal{A}|$, let $a' := |\mathcal{A}'|$, let $b := |\mathcal{B}|$, and let $b' := |\mathcal{B}'|$. For $w \in W \in \mathcal{A}'$ and $x \in X \in \mathcal{B}$, an edge $wx$ in $G$ is a *solo edge* if $w$ is the only neighbor of $x$ in $W$; each endpoint of a solo edge is a *solo vertex*. Figure 10.7 shows an example of $G$ (left and center) and the graph $H$ arising from $G$ (right).

When proving Theorem 10.28, it is easy to reduce to the case where $k+1$ divides $|G|$. Now suppose $G$ has a nearly equitable $(k+1)$-coloring. Intuitively, when $H$ has many edges, we have many options to "improve" $\varphi$, which we make precise below. For example, if $H$ has a dipath from $V^+$ to $V^-$, then we move each witness of an edge on the path to the next color class on the path. This yields an equitable $(k+1)$-coloring, so we are done. The main idea when proving Theorem 10.28 is that if certain pairs of vertex subsets in $H$ have few edges between them, then in $G$ their corresponding sets of color classes must have many edges between them. By counting carefully, we find so many edges that $G$ violates the hypothesis $\Delta(G) \leqslant k$. Most of the work in proving Theorem 10.28 goes into proving Lemma 10.30 below, which enables us to transform a nearly equitable coloring into an equitable one. We first prove Theorem 10.28, assuming Lemma 10.30. After that, we prove Lemma 10.30.

*Proof of Theorem 10.28, assuming Lemma 10.30.* We assume $|G| = (k+1)s$, for some integer $s$. If not, then let $\tilde{G}$ denote the disjoint union of $G$ and $K_t$, where $t := \lceil |G|/(k+1) \rceil (k+1) - |G|$. Since all vertices of the $K_t$ must get distinct colors, proving the theorem for $\tilde{G}$ proves it for $G$.

We use induction on $\|G\|$. Let $G' := G - xy$ for an arbitrary edge $xy$ of $G$. By hypothesis, $G'$ has an equitable $(k+1)$-coloring $\varphi$. We are done unless $\varphi(x) = \varphi(y)$, so assume this is true. Since $d(x) \leqslant k$, some color class $Z$ contains no neighbors of $x$. Moving $x$ to $Z$ yields a nearly equitable $(k+1)$-coloring of $G$. By Lemma 10.30, $G$ has an equitable $(k+1)$-coloring. □

**Lemma 10.30.** *Let $\varphi$ be a nearly equitable $(k+1)$-coloring of a graph $G$. If $d(v) \leqslant k$ for all $v \in A' \cup B$, as in Definition 10.29, then $G$ has an equitable $(k+1)$-coloring.*

*Proof.* Recall that $\mathcal{A}$ is the vertices of $H$ with a dipath to $V^-$, that $\mathcal{B} := V(H) \setminus \mathcal{A}$, and that $b := |\mathcal{B}|$. (Other variables are as in Definition 10.29.) Our proof is by induction on $b$. $\mathcal{A}, \mathcal{B}, b$

**Claim 1.** *Let $G$ be a graph with $\Delta(G) \leqslant k$ and let $\varphi$ be a nearly equitable $(k+1)$-coloring of $G$. The following conditions hold.*

*(i)* $b \geqslant 1$ *and* $a' < a$.

*(ii) For all* $W \in \mathcal{A}$ *and* $y \in B$*, we have* $d_W(y) \geqslant 1$.

*(iii) For all* $W \in \mathcal{B} \setminus \mathcal{B}'$ *and* $z \in B'$ *we have* $d_W(z) \geqslant 1$.

*(iv) For all* $w \in A'$*, if* $wz$ *is a solo edge, then* $d_A(w) \geqslant a - 1$ *and* $d_{B-z}(w) \leqslant b - 1$.

Proof. (i) As we mentioned above, if $H$ has a dipath from $V^+$ to $V^-$, then we can move each witness of an edge on this path to the next color class on the path. This yields an equitable $k$-coloring of $G$, and we are done. So instead we assume that $V^+ \notin \mathcal{A}$. Thus, $b \geqslant 1$ and $|B| = bs + 1$ and $|A| = as - 1$. If $a = 1$, then $\mathcal{A} = V^-$, so $|A| = s - 1$. Also, $b = k$ and $|B| = ks + 1$. Each $w \in B$ has a neighbor in $V^-$, so $ks - k = |A|k \geqslant |E(A, B)| \geqslant |B| = ks + 1$, a contradiction. Thus $a > 1$. Now $V^- \in \mathcal{A} \setminus \mathcal{A}'$. Hence, $a' < a$.

(ii) Suppose, to the contrary, that there exist such $W$ and $y$, but $d_W(y) = 0$. Let $Y$ denote the color class of $y$. Now $y$ witnesses the edge $YW$. Since $W \in A$, there exists a dipath $P$ from $W$ to $V^-$. Extending $P$ by $YW$ yields a dipath from $Y$ to $V^-$. So $y \in Y \in \mathcal{A}$, which contradicts our assumption that $y \in B$.

(iii) Suppose, to the contrary, that there exist such $W$ and $z$, but $d_W(z) = 0$. Let $Z$ denote the class of $z$. Similar to above, we can extend the dipath in $H$ from $V^+$ to $Z$ one edge further to $W$. So $W \in \mathcal{B}'$, which contradicts our assumption.

(iv) Fix $w \in W \in \mathcal{A}'$ and $z \in B$ such that $wz$ is a solo edge. We will show that $w$ cannot witness any edge of $H[\mathcal{A}]$. Given this assumption, we have $d_X(w) \geqslant 1$ for all $X \in \mathcal{A} - W$, so $d_A(w) \geqslant a - 1$. Recall that $d_G(w) \leqslant \Delta(G) \leqslant k = a + b - 1$. So $d_{B-z}(w) = d_G(w) - d_A(w) - |\{z\}| \leqslant (a + b - 1) - (a - 1) - 1 = b - 1$. So we must show that $w$ cannot witness any edge of $H[\mathcal{A}]$.

Suppose, to the contrary, that $w$ witnesses the edge $WW' \in H[\mathcal{A}]$. Move $z$ into $W$ and $w$ into $W'$. Since $W$ is terminal, there exists a dipath in $H[\mathcal{A} - W]$ from $W'$ to $V^-$. We move each witness of an edge on this path to the next color class on the path. This yields an equitable $a$-coloring of $G[A + z]$. Furthermore, $\varphi$ gives a nearly equitable (or equitable) $b$-coloring of $G[B - z]$. By (ii) above, each $y \in B$ has $d_A(y) \geqslant a$, so $d_B(y) \leqslant k - a = b - 1$. Thus, by the induction hypothesis, $G[B - z]$ has an equitable $b$-coloring. Combining this with the equitable $a$-coloring of $G[A + z]$ yields an equitable $(a + b)$-coloring of $G$. $\diamondsuit$

We order the classes of $\mathcal{A}$ as $V^-, X_1, \ldots, X_{a-1}$ such that each class $X_i$ has an out-edge in $H$ to a class earlier in the order; we also require that all terminal classes come after all non-terminal classes. Let $X_\ell$ be the last non-terminal class. (Such an $X_\ell$ exists because $a' < a$, by Claim 1(i).) $X_1, \ldots, X_{a-1}$ $X_\ell$

So some terminal class $X_j$ has no edges in $H$ to classes before $X_\ell$. Thus, $d^+_{\mathcal{A}}(X_j) < a - \ell$. (1) If $a - \ell \leqslant b$, then $d^+_{\mathcal{A}}(X_j) < b$. (2) Otherwise, $a' = a - (\ell + 1) = a - \ell - 1 \geqslant b$.

S **Case 1: $d^+_{\mathcal{A}}(W) < b$ for some $W \in \mathcal{A}'$.** This includes (1) above. Let $S$ be the solo vertices D in $W$, and let $D := W \setminus S$. See Figure 10.8. Intuitively, $|E(D, B \setminus N(S))|$ is small (since each vertex of $D$ has many neighbors in $A$ because $d^+_{\mathcal{A}}(W) < b$, by our case), so $|E(S, N_B(S))|$ is large, so $|E(S, A)|$ is small, and some vertex of $S$ has too few neighbors in $A$, contradicting Claim 1(iv). We formalize this as follows.

For every $w \in W$, $d_A(w) \geqslant |\mathcal{A}| - (1 + d^+_{\mathcal{A}}(W)) \geqslant a - b$, so $d_B(w) \leqslant k - d_A(w) < 2b$. By definition, every vertex in $B \setminus N(S)$ has at least two neighbors in $D$. Since $d_B(w) < 2b$ for all $w \in D$, we get $2|B \setminus N(S)| \leqslant |E(D, B \setminus N(S))| < 2b|D|$; so, $|B \setminus N(S)| < b|D|$. Thus, $|N_B(S)| = |B| - |B \setminus N(S)| > bs + 1 - b|D|$. Counting all edges incident to $S$ gives $k|S| \geqslant |N_B(S)| + |E(S, A)|$. Combining these 2 inequalities, and using $|S| + |D| = s$, gives

$$\begin{aligned}
|E(S, A)| &\leqslant k|S| - |N_B(S)| \\
&< k|S| - bs - 1 + b|D| \\
&= k|S| - b|S| - 1 \\
&< (a - 1)|S|.
\end{aligned}$$

By Pigeonhole, some vertex $w \in S$ has $d_A(w) \leqslant a - 2$. But this contradicts Claim 1(iv).

**Case 2: $a' \geqslant b$.** We will find a solo vertex $w \in W \in \mathcal{A}'$ with solo neighbors $z_1, z_2 \in B$ that are nonadjacent; see Figure 10.9. To find such $w, z_1, z_2$ we use a counting argument, which we defer to the end of this case. Choose $w' \in W$ that witnesses some first edge on a path from $W$ to $V^-$. As in the proof of Claim 1(iv), we move each witness of an edge on this path to the next class along the path. This gives an equitable $(a-1)$-coloring of $G[(A \setminus W) + w']$. Let $G' := G[B \cup W - w']$. We must find an equitable $(b+1)$-coloring of $G'$. Let $G'' := G[B - z_1]$, and note that $\varphi$ restricts to a nearly equitable $b$-coloring of $G''$. By Claim 1(ii), we have $d_B(x) \leqslant b - 1$ for all $x \in B$. Thus, by induction $G''$ has an equitable $b$-coloring $\varphi''$. Let $Z$ be the color class of $\varphi''$ containing $z_2$. Recall that $d_A(w) \geqslant a - 1$.

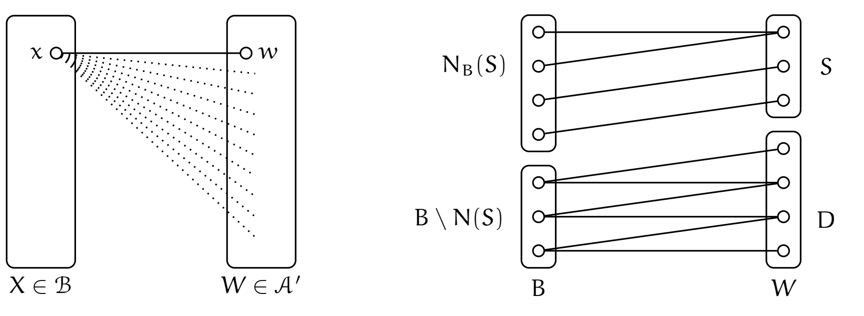



Figure 10.8: Left: A solo edge. Right: The induced bipartite subgraph with parts $B$ and $W$, which we consider in Case 1.

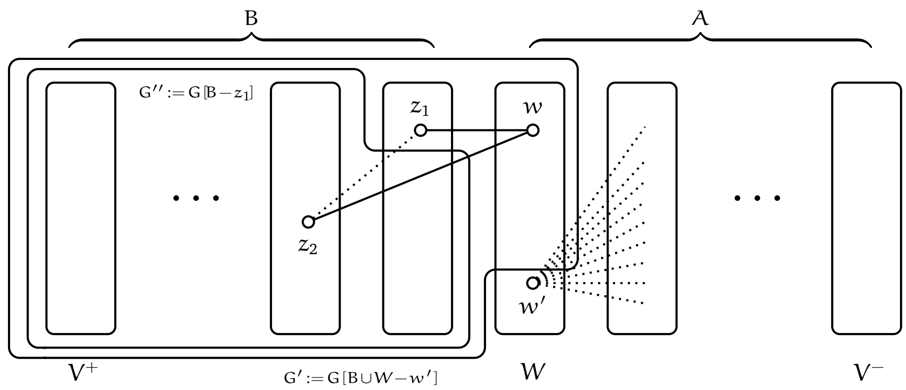



Figure 10.9: The solo vertices $w, z_1, z_2$ and the subgraphs $G'$ and $G''$ in Case 2.

So $d_{B-z_1-z_2}(w) \leqslant a+b-1-(a-1)-2 = b-2$. Thus, there exists a color class $U$ of $\varphi''$ with $d_U(w) = 0$. Moving $w$ into $U$ and moving $z_2$ into $W$ (along with $z_1$) gives a nearly equitable $(b+1)$-coloring of $G'$. For each $v \in V(G')$ and $W' \in \mathcal{A} - W$, we have $d_{W'}(v) \geqslant 1$, by Claim 1(ii). Thus, $\Delta(G'') \leqslant b$. Now $V^- := Z - z_2$, so $d_{V^-}(z_2) = 0$; thus, $a(G'') \geqslant 2$. Hence, $b(G'') < b(G)$; so $G''$ has an equitable $(b+1)$-coloring, by induction. Along with the equitable $(a-1)$-coloring of $G[(A \setminus W) + w]$, this forms an equitable $(a+b)$-coloring of $G$.

Now we do the counting to show that the desired vertices $w, z_1, z_2$ exist. For each $z \in B'$, let $\sigma(z)$ denote the number of solo neighbors of $z$. By Claim 1(ii,iii), we have

$$\begin{aligned} k \geqslant d_A(z) + d_B(z) &\geqslant (a + a' - \sigma(z)) + (b - b' + d_{B'}(z)) \\ &= k + 1 + d_{B'}(z) + a' - b' - \sigma(z). \end{aligned}$$

So $\sigma(z) \geqslant 1 + d_{B'}(z) + a' - b'$. Let $I$ be a maximal independent set in $G[B']$ that contains $V^+$. By maximality, $\sum_{v \in I}(d_{B'}(v) + 1) \geqslant |B'| = b's + 1$. Since $a' \geqslant b$, by our case, and $|I| \geqslant |V^+| \geqslant s + 1$, we get

$$\begin{aligned} \sum_{z \in I} \sigma(z) &\geqslant \sum_{z \in I} (a' - b' + d_{B'}(z) + 1) \\ &\geqslant (s+1)(a' - b') + b's + 1 \\ &= a' - b' + a's + 1 \\ &> a's \\ &= \left|A'\right|. \end{aligned}$$

Thus, by Pigeonhole, there exists $w \in W \in \mathcal{A}'$ with solo neighbors $z_1, z_2 \in I$. This completes Case 2, and finishes the proof. □

## Notes

Theorem 10.1 is due to Lovász [286]. For an efficient algorithm, we start with an arbitrary partition of $V(G)$ into parts $V_1, \dots, V_s$ and move any vertex out of its part if its degree there is too large. After each move, $f(\mathcal{P})$ decreases by at least 1. For every partition $\mathcal{P}$ we have $0 \leqslant f(\mathcal{P}) \leqslant \|G\|$. So we finish after at most $\|G\|$ moves. The same idea yields efficient algorithms for many of the results in this chapter. Theorem 10.2 is due to Catlin [79] and Borodin and Kostochka [65]. Theorem 10.3 (as well as Lemmas 10.4 and 10.5) were proved by Hendrey and Wood [216].

Theorem 10.6 is due to King [258], who also mentioned Lemma 10.7 as an application. Building on this work, Christofides, Edwards, and King characterized the graphs for which Theorem 10.6 is sharp (see Exercise 9). Prior to [258], Rabern [338] proved a weaker version of Theorem 10.6; it gave the same conclusion, but required the stronger hypothesis $\omega \geqslant \frac{3}{4}(\Delta+1)$. Lemma 10.8 was first proved by Kostochka [269]. His hitting set lemma required the much stronger hypothesis $\omega \geqslant \Delta + \frac{3}{2} - \sqrt{\Delta}$, but the outline of the proof was similar to what we presented here, drawing on Lemma 10.9 (proved by Hajnal [196]) and Lemma 10.10 (first proved by Kostochka in [269]).

Lemmas 10.13 and 10.15 were proved by Haxell [207, 206]. Our proof follows [183, Sketch of Proof of Theorem 2], which implies Corollary 10.14. See [210, Theorems 2.2 and 3.1] for a nice presentation of the earlier, nonconstructive proofs. Fleischner and Stiebitz [165] and Szabo and Tardos [368, Construction 3.3] gave examples (for each value of $\Delta$) of graphs and partitions that admit no independent transversal, but for which $|V_i| = 2\Delta - 1$ for all $i$. Thus, the hypothesis in Lemma 10.15 of $|V_i| \geqslant 2\Delta$ is best possible. Graf and Haxell [183] list many applications of Lemma 10.15; in her dissertation, Graf [182, Chapter 4] gives efficient algorithms for some of these. Strong coloring was introduced by Fellows [160] (under a different name) and Alon [10]. Fellows showed[4] that if $G$ is strongly $k$-colorable, then $G$ is also strongly $(k+1)$-colorable; see Theorem A.4. Haxell proved [208] that $\chi_s(G) \leqslant 3\Delta(G) - 1$ for all $G$. The proof we present is a simplification of this original proof, due to Aharoni, Berger, and Ziv [4]; see also [210]. Haxell [209] also showed that $\chi_s(G) \leqslant (1 + o(1))\frac{11}{4}\Delta(G)$. It is natural to conjecture that $\chi_s(G) \leqslant 2\Delta(G)$ for all $G$, and this problem remains open.

In Chapter 8 we proved the Cycle-Plus-Triangle (CPT) Theorem: If $G$ is the edge-disjoint union of a cycle $C_{3t}$ and $t$ vertex disjoint triangles on the same vertex set, then $AT(G) = 3$. This implies that $\chi(G) = 3$, which is equivalent[5] to the statement that $\chi_s(C_{3t}) = 3$ for all $t$. It is natural to ask whether the CPT Theorem extends to other 2-regular graphs $G$, even for coloring. The easy answer is "No", since if $G$ includes a 4-cycle, then the union of $G$ and the triangles can include a $K_4$. Erdős asked the same question [165] if we require that $G$ contains no 4-cycle. The answer is still "No", as witnessed by infinitely many graphs. The simplest of

[4]He only proved this for countably infinite graphs, but it is straightforward to adapt the proof to finite graphs.

[5]In the CPT Theorem we require that no triangle use an edge of the long cycle, and for strong coloring we have no such requirement. But we can overcome this obstacle by subdividing such a cycle edge by 6 new vertices and adding 2 new triangles on these new vertices.

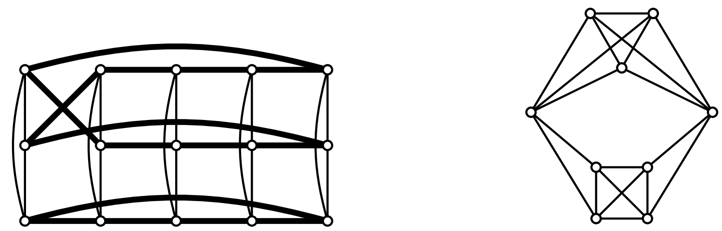


Figure 10.10: Left: A 4-regular graph that is the edge-disjoint union of a 2-factor (a 5-cycle and a 10-cycle) and a triangle factor, but is not 3-colorable. Right: The graph $O_5$ is the single exceptional graph when we weaken the hypothesis on Theorem 10.19 to $\Delta \geqslant 5$.

these is formed from the cartesian product $C_5 \square K_3$ by removing edges $(v_1, w_1)(v_2, w_1)$ and $(v_1, w_2)(v_2, w_2)$, with $v_1 v_2 \in E(C_5)$ and $w_1 w_2 \in E(K_3)$, and adding edges $(v_1, w_1)(v_2, w_2)$ and $(v_1, w_2)(v_2, w_1)$; see Figure 10.10.

Theorem 10.19 is due to Kierstead and Kostochka [249], but we follow Rabern [339], where the result is extended to the case $\chi(G) = 6$. Kostochka, Rabern, and Stiebitz [270] further extended the result to $\chi(G) = 5$; but in that case there is a single exceptional graph, called $O_5$; see Figure 10.10. We can form $O_5$ from $C_5$ by "blowing up" each of two adjacent vertices into copies of $K_2$ and a third nonadjacent vertex into $K_3$, with new vertices inheriting neighbors from old ones. Thus, if $\chi(G) = \lfloor \frac{\theta(G)}{2} \rfloor + 1$ and $\chi(G) \geqslant 5$, then either G contains $K_{\chi(G)}$ or G is $O_5$. In his dissertation [340, Section 3.1], Rabern generalized these results, using a version of the vertex shuffle due to Catlin (rather than what we presented here, which is due to Mozhan). Cranston and Rabern [100] used Mozhan partitions to show that if $\chi = \Delta \geqslant 13$, then $\omega \geqslant \Delta - 3$. This strengthened a result from Mozhan's dissertation (unavailable, unfortunately) which proved the same conclusion when $\chi = \Delta \geqslant 31$.

Theorem 10.24, along with its supporting lemmas, is due to Hendrey and Wood [216]. For more on defective and clustered coloring, see Wood's excellent dynamic survey [416].

Theorem 10.28 is called the Hajnal-Szemerédi Theorem. It was conjectured by Erdős in 1964 [150] and proved by Hajnal and Szemerédi in 1970 [197]. However, their proof was long and difficult. Looking for a simpler proof, Seymour [359] asked whether every graph G with $\delta(G) \geqslant \frac{s}{s+1}|G|$ contains as a subgraph the sth power of a hamiltonian cycle. If Seymour's Conjecture is true, then for a graph G with $|G| = (k+1)(s+1)$ and $\Delta(G) \leqslant k$ we have $\delta(\overline{G}) \geqslant |G| - (k+1) = s(k+1) = \frac{s}{s+1}|G|$. Now any $s+1$ consecutive vertices on the hamiltonian cycle in $\overline{G}$ form an independent set in G. So G has an equitable $(k+1)$-coloring.

The case $s = 1$ of Seymour's Conjecture is called Dirac's Theorem [119] and the case $s = 2$ is called Posa's Conjecture [150]. The latter was proved by Fan and Kierstead with hamiltonian cycle replaced by hamiltonian path [158]. For each choice of s, Seymour's Conjecture was proved when $|G|$ is sufficiently large in terms of s [267].

It is natural to ask which graphs with $\Delta = k$ have no equitable k-coloring. Clearly these include $K_{k+1}$, $C_{2t+1}$ (when $k = 2$), and $K_{k,k}$ when k is odd. Chen, Lih, and Wu conjectured [80] that these are the only such connected graphs, and proved this conjecture when $k = 3$. Kierstead and Kostochka later proved the case $k = 4$ [250].

The proof by Hajnal and Szemerédi of their eponymous theorem did not give a polynomial time algorithm. The first proof yielding an efficient algorithm is due to Mydlarz and Szemerédi (unpublished manuscript). Shortly thereafter, Kierstead and Kostochka [248] found a short proof that also yields an efficient algorithm. Ultimately, the two groups combined their efforts [252], yielding an algorithm that runs in $O(k|G|^2)$ time, when $\Delta \leqslant k$. It is this proof that we presented in Section 10.6.

The Hajnal–Szemerédi Theorem has been generalized in various ways. For example, Kierstead and Kostochka [251] and others have considered the list-coloring analogue of the problem. Another variation considers the analogous statement for Ore-degree. Kierstead and Kostochka proved the following. If G is a graph with $\theta(G) \leqslant 2k+1$, then G has an equitable $(k+1)$-coloring [247]. In the same paper, they extended the Chen–Lih–Wu Conjecture above as follows: If G is a connected graph, $k \geqslant 3$, and $\theta(G) \leqslant 2k$, then G has an equitable k-coloring unless either $G \cong K_{k+1}$ or $G \cong K_{m,2k-m}$, where m is odd. Further work in this direction is in [249].

## Exercises

10.1. Show that Theorem 10.1 is best possible. For every choice of integers $s \geqslant 1$ and $r_1, \dots, r_s$ with each $r_i \geqslant 0$, construct a graph G with $\Delta = \sum_{i=1}^{s}(r_i - 1)$ and with no partition of $V(G)$ into parts $V_1, \dots, V_s$ and $\Delta(G[V_i]) \leqslant r_i$.

10.2. Show that every graph G has a bipartite subgraph H with $d_H(v) \geqslant \frac{1}{2}d_G(v)$ for all $v \in V(G)$.

10.3. Use Theorem 10.1 to show if $\chi(G) \geqslant \Delta(G) \geqslant 7$, then G contains $K_{\lfloor \frac{1}{2}(\Delta(G)+1) \rfloor}$. [65]

10.4. Extend Lemma 10.4 to the case where each color $\alpha$ has its own bound $r_\alpha$ on the maximum degree of a subgraph colored $\alpha$. The resulting version truly generalizes Theorem 10.1.

10.5. Rewrite the final counting argument in the proof of Theorem 10.3 to use discharging.

10.6. Use Lemma 10.15 to show that if a graph G is k-colorable with defect 2, then G is $(k+1)$-colorable with clustering $6\Delta(G)$. [14, Proof of Theorem 4.1]

10.7. Extend Theorem 10.3 to correspondence coloring (as in Definition 4.27). [216]

10.8. Prove that $\chi_s(mK_n) = n$ for all $n \geqslant 1$ and $m \geqslant n-1$. [160]

10.9. For each integer $k \geqslant 2$ such that $k \equiv 2 \pmod 3$, construct an infinite family of examples (no one a subgraph of another) showing that Theorem 10.6 is best possible. More precisely, for each such k each example should have $\Delta = k$ and $\omega = \frac{2}{3}(k+1)$ but should not have any hitting set. [258, 85] These examples also prove the bound $\omega > \frac{2}{3}(\Delta+1)$ in Lemma 10.10 is best possible.

10.10. For each integer $k \geqslant 3$, let $f(k)$ be the infimum of $\Delta(G) - \chi_f(G)$, where the infimum is taken over all graphs $G$ with $\Delta(G) = k$ and $\omega(G) < \Delta(G)$. Recall that $\chi_f(G)$ denote the fractional chromatic number of $G$. Molloy and Reed proved that Reed's Conjecture holds fractionally. [307, Section 21.3] That is, $\chi_f(G) \leqslant (\Delta(G) + \omega(G) + 1)/2$ for all graphs $G$. For each $k \geqslant 7$, use this result to prove that $f(k) \geqslant \min\{f(k-1), \frac{1}{2}\}$. [259]

10.11. Construct an infinite family of 4-critical graphs with $\Theta = 7$. These graphs (excluding $K_4$) necessarily have $\omega(G) \leqslant 3$. Thus, they serve as counterexamples if we try to weaken the hypothesis in Theorem 10.19 to $\chi(G) \geqslant 4$. [270]

10.12. Modify the proof of Theorem 10.24 to require slightly larger lists, but to guarantee clustering 6. [216]

# Chapter 11

# Precoloring Extension

If only I had the theorems! Then I should find the proofs easily enough.
—Bernhard Riemann

By far the most common method for proving coloring bounds for planar graphs is reducibility and discharging. We use a counting argument to show that our graph $G$ has some subgraph $H$ with low degrees, we color $G - H$ by induction, and we can extend the coloring to $H$ precisely because its degrees are low. In this chapter, we still use induction, but our choice of $H$ is quite different. Here we prove various list-coloring results for planar graphs. Typically, we delete a small number of vertices on the outer face; unlike in previous examples, the degrees of these vertices may be arbitrarily large! However, our use of *list assignments* allows us to reserve a color $\alpha$ for each deleted vertex $v$ by deleting $\alpha$ from the lists of neighbors of $v$ in the smaller graph $G - H$ that we color inductively.

Of course when we delete $\alpha$ from these lists for $G - H$, the list sizes may decrease. So we ultimately must prove a statement that is stronger than what we initially aimed for, by prescribing smaller list sizes for specific vertices. The stronger this statement is, the more powerful types of reductions we will have available when coloring our smaller graph by induction. However, as we strengthen our statement, it is increasingly likely to become false, or possibly to have some well-understood exceptions, which must be excluded from our eventual theorem. Finding precisely the correct statement to prove often requires significant trial and error. But once we find it, the proofs can appear almost magical.

## 11.1 5-Choosability of Planar Graphs

In this section, we prove the following beautiful result.

**Theorem 11.1.** *Every planar graph is* 5*-choosable.*

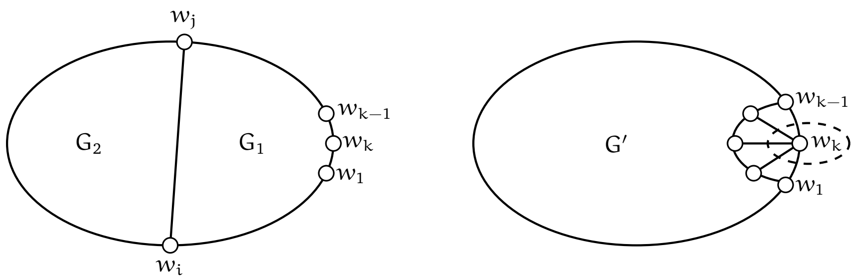



Figure 11.1: Left: C has a chord. Right: C has no chord.

near-triangulation

This theorem is implied by a stronger statement, Lemma 11.2, below. A plane graph is a *near-triangulation* if every face has length 3, except for possibly the outer face. If G is not a near-triangulation, then we can add edges until it becomes one, without making the coloring problem easier. Thus, it suffices to consider only near-triangulations.

**Lemma 11.2.** *Let* $G$ *be a near-triangulation. Label the vertices of the boundary cycle*[1] $C$ *of the outer face as* $w_1, \dots, w_k$ *in clockwise order. Now* $G$ *is* $L$*-colorable whenever* $L$ *is a list assignment satisfying the following* 3 *properties:*

1. $|L(w_1)| = 1$, $|L(w_2)| = 1$, *and* $L(w_1) \neq L(w_2)$*;*
2. $|L(w_i)| = 3$ *for every other vertex* $w_i$ *on* $C$*; and*
3. $|L(w)| = 5$ *for every vertex* $w$ *not on* $C$.

*Proof.* We use induction on $|G|$; the base case, $|G| = 3$, is easy. For the induction step we have two possibilities: either (1) C has a chord $w_iw_j$ with $j - i \geqslant 2$, as on the left of Figure 11.1, or (2) C has no chord, as on the right of Figure 11.1.

**Case 1: C has a chord $w_iw_j$.** Let $G_1$ be the subgraph induced by $w_1, \dots w_i, w_j, \dots, w_k$ and the vertices interior to this cycle. Define $G_2$ analogously for $w_i, \dots, w_j$. By induction, $G_1$ has an L-coloring $\varphi_1$ (recall that $w_1$ and $w_2$ have distinct lists of size 1). To color $G_2$, we again use induction, now with $w_i$ and $w_j$ having the lists $\{\varphi_1(w_i)\}$ and $\{\varphi_1(w_j)\}$. Formally, let $L'(w_i) := \{\varphi_1(w_i)\}$, $L'(w_j) := \{\varphi_1(w_j)\}$, and $L'(w) := L(w)$ for all other $w \in V(G_2)$. By induction, $G_2$ has an $L'$-coloring $\varphi_2$. Since $\varphi_1$ and $\varphi_2$ agree on $w_i$ and $w_j$, together they give an L-coloring of G.

**Case 2: C has no chord.** Recall that $w_k$ is the neighbor of $w_1$ on C other than $w_2$. Choose colors $\alpha, \beta \in L(w_k) \setminus L(w_1)$. Let $L'(v) := L(v) \setminus \{\alpha, \beta\}$ for each neighbor $v$ of $w_k$ that is not on C, let $L'(v) := L(v)$ for all other vertices, and let $G' := G - w_k$. Since C has no chord, each neighbor $v$ of $w_k$ other than $w_{k-1}$ and $w_1$ has $|L(v)| = 5$, so $|L'(v)| \geqslant 3$. In $G'$, each such

[1] Not every near-triangulation has an outer cycle! Its boundary walk could contain cut-vertices. But we can add edges so that it is a near triangulation that does have an outer cycle, without changing the boundary vertices.

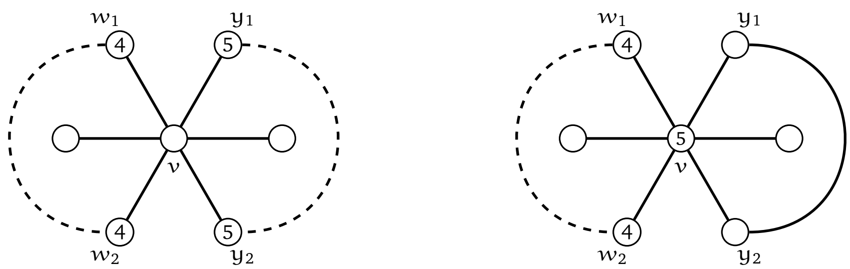



Figure 11.2: Examples of Lemma 11.3. Left: If $w_1 \not\leftrightarrow w_2$ and $y_1 \not\leftrightarrow y_2$, then we can guarantee that $\varphi(w_1) = \varphi(w_2)$ and also $\varphi(y_1) = \varphi(y_2)$. Right: If only $w_1 \not\leftrightarrow w_2$, then we can still guarantee that $\varphi(w_1) \neq \varphi(w_2)$.

vertex is now on the outer face. So, by induction, $G'$ has an $L'$-coloring $\varphi'$. To extend $\varphi'$ to G, we simply give $w_k$ a color in $\{\alpha, \beta\} \setminus \{\varphi'(w_{k-1})\}$. The resulting coloring is a proper L-coloring, since each other neighbor of $w_k$ uses a color not in $\{\alpha, \beta\}$. □

In Theorem 2.3, we construct planar graphs that are not 4-choosable. Thus, Theorem 11.1 is best possible. To conclude this short first section, we prove an easy corollary of Lemma 11.2 that is useful for us in Section 3.3.

**Lemma 11.3.** *Let* $G$ *be a planar graph containing a* $6^-$*-vertex* $v$ *with four neighbors* $w_1, w_2, y_1, y_2$. *If* $w_1 \not\leftrightarrow w_2$, *then* $G$ *has a* 5*-coloring* $\varphi$ *such that* $\varphi(w_1) = \varphi(w_2)$. *If also* $y_1 \not\leftrightarrow y_2$, *then we can further require that* $\varphi(y_1) = \varphi(y_2)$. *See Figure 11.2.*

*Proof.* Let $G$, $v$, and $w_1, w_2, y_1, y_2$ satisfy the hypotheses. Let $S := \{v, w_1, w_2, y_1, y_2\}$. First suppose that both $w_1 \not\leftrightarrow w_2$ and $y_1 \not\leftrightarrow y_2$. See the left of Figure 11.2. We assign lists to $V(G - S)$ as follows. For each $x \in N(S)$, let $L(x) := [3]$ and otherwise let $L(x) := [5]$. Since $G - S$ can be embedded as a near-triangulation (possibly after adding some edges), Lemma 11.2 implies that $G - S$ has an L-coloring $\varphi$. To extend $\varphi$ to G, we let $\varphi(w_1) = \varphi(w_2) := 4$ and $\varphi(y_1) = \varphi(y_2) := 5$. Finally, since at most four colors appear on neighbors of $v$, we color $v$ with some available color.

The case when $y_1 \leftrightarrow y_2$ is even simpler. See the right of Figure 11.2. Now we let $L(x) := [3]$ if $x \in N(\{w_1, w_2, v\})$ and $L(x) := [5]$ otherwise. By Lemma 11.2, $G - \{w_1, w_2, v\}$ has an L-coloring $\varphi$. To extend $\varphi$ to G, we color $w_1$ and $w_2$ with 4 and color $v$ with 5. □

## 11.2 $(3,2)$-Decomposability of Planar Graphs

The goal of this section is to prove the following result.

**Theorem 11.4.** *Every planar graph* $G$ *decomposes into graphs* $G_1$ *and* $G_2$*, where* $G_1$ *is* 3*-degenerate and* $\Delta(G_2) \leqslant 2$.

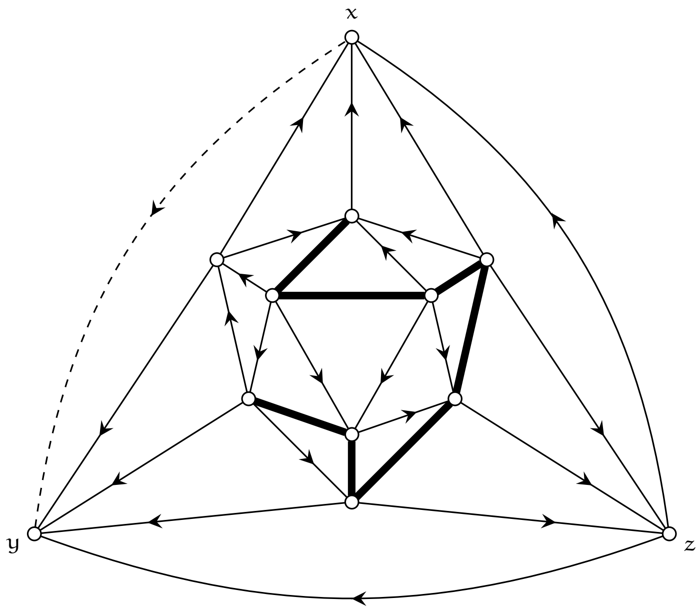



Figure 11.3: A $(3, 2)$-decomposition of the icosahedron.

decomposes
k-degenerate

Recall that G *decomposes* into $G_1$ and $G_2$ if $V(G_1) = V(G_2) = V(G)$ and $E(G_1) \cup E(G_2) = E(G)$ and $E(G_1) \cap E(G_2) = \emptyset$. We also recall that a graph H is $k$-*degenerate* if and only if H has an acyclic orientation D such that $\Delta^+(D) \leqslant k$. (Each such orientation D has a source $v$, with $d_D^-(v) = 0$, that can appear first in the degeneracy order; the rest of the order can be computed recursively.) A graph G is $(k, \ell)$-*decomposable* if G decomposes into graphs $G_1$ and $G_2$ such that $G_1$ is k-degenerate and $\Delta(G_2) \leqslant \ell$. So we can rephrase Theorem 11.4 as follows: Every planar graph is $(3, 2)$-decomposable.

$(k, \ell)$-decomposable

**Example 11.5.** Let G be a planar graph with $\delta(G) = 5$, e.g., G could be the icosahedron; see Figure 11.3, but ignore its $(3, 2)$-decomposition. Now clearly G is not $(3, 1)$-decomposable: for every matching M we have $\delta(G - M) \geqslant 4$, so no vertex can be first in a hypothetical 3-degeneracy order of $G - M$. Similarly, G is not $(2, 2)$-decomposable: if H is a spanning subgraph with $\Delta(H) \leqslant 2$, then $\delta(G - E(H)) \geqslant 3$. Hence, in both parameters Theorem 11.4 is the best possible. ♢

boundary vertex/edge
interior vertex

At a high level, our proof of Theorem 11.4 is similar to the proof in the previous section that planar graphs are 5-choosable. We prove a stronger statement that facilitates a proof by induction. Fix a near-triangulation G. If a vertex $v$ or edge $e$ lies on the boundary of the outer face of G, then $v$ is a *boundary vertex* or $e$ is a *boundary edge*. If a vertex of G is not a boundary vertex, then $v$ is an *interior vertex*. Theorem 11.4 follows immediately from our next theorem, which is this section's main result.

**Theorem 11.6.** *Let $G$ be a near triangulation, let $xy$ be a boundary edge of $G$, and let $z$ be a boundary vertex, with $z \notin \{x, y\}$, that is incident with no chord of the boundary cycle. If neither $x$ nor $y$ is a boundary neighbor of $z$, then let $z'$ be a boundary neighbor of $z$. Now there exist a subgraph $H$ and an acyclic orientation $D$ of $G - E(H)$ such that the following 3 properties all hold:* $x, y, z$ $z'$ $H$, $D$

*(i) For every interior vertex $w$, we have $d^+_D(w) \leqslant 3$ and $d_H(w) \leqslant 2$.*

*(ii) For every boundary vertex $w$, we have $d^+_D(w) \leqslant 2$ and $d_H(w) \leqslant 2$. Furthermore, if $w \neq z'$, then $d^+_D(w) + d_H(w) \leqslant 3$.*

*(iii) We have $d^+_D(y) = d_H(x) = d_H(y) = 0$ and $N^+_D(x) = \{y\}$ and $d^+_D(z) + d_H(z) \leqslant 2$.*

*We call $(D, H)$ a $(3, 2)$-*decomposition *of $G$ with respect to (w.r.t.) $(x, y, z)$ or $(x, y, z, z')$.* $(3, 2)$-decomposition

As an example, Figure 11.3 shows a $(3, 2)$-decomposition of the icosahedron.

Before starting the proof, we provide some intuition. The desired result of this section, that $G$ decomposes into a 3-degenerate graph $G_1$ and a graph $G_2$ with $\Delta(G_2) \leqslant 2$, follows immediately from (i) and the first half of (ii). The remaining hypotheses are refinements that help facilitate our proof by induction. Like the proof of 5-choosability, we have two main cases: (1) the boundary cycle has a chord and (2) the boundary cycle has no chord.

Hypothesis (iii) is designed to help handle case (1). Our chord $vw$ splits the graph into two induced subgraphs, $G_1$ and $G_2$, intersecting only in $\{v, w\}$. We first get the desired partition $(D_1, H_1)$ for the subgraph $G_1$ containing $xy$, with $x_1 = x$ and $y_1 = y$; afterwards we get the desired partition $(D_2, H_2)$ for $G_2$, with $x_2 = v$ and $y_2 = w$. It is straightforward to check that the desired partition $(D, H)$ for $G$ is $(D_1 \cup (D_2 - xy), H_1 \cup H_2)$. Precisely because of (iii), we see that taking this union does not create problems at $v$ and $w$. In case (2), we let $G' := G - z$, get the desired partition $(D', H')$ for $G'$ and then extend it to $G$. Here, the first half of (ii) is crucial. For each neighbor $v$ of $z$ (in $G$), we add to $D'$ the edge $\overrightarrow{vz}$. So all that remains is to handle the two boundary edges incident with $z$. For this, the hypothesis $d^+_D(z) + d_H(z) \leqslant 2$ is invaluable (we defer further details to the actual proof).

*Proof.* We use induction on $|G|$. The base case, $|G| = 3$, is easy: let $E(H) = \emptyset$ and $E(D) = \{\overrightarrow{xy}, \overrightarrow{zx}, \overrightarrow{zy}\}$. Now assume that $|G| \geqslant 4$. Let $C$ be the boundary of the outer face.

**Case 0: $C = xyz$.** Let $G' := G - z$, let $x' := x$, let $y' := y$, and choose $z'$ to be some boundary vertex of $G'$ (other than $x'$ and $y'$) that is not incident with a chord of its boundary.[2] By induction, $G'$ has a $(3, 2)$-decomposition $(D', H')$ w.r.t. $(x', y', z')$. Let $H := H'$ and $D := D' \cup \{\overrightarrow{zx}, \overrightarrow{zy}\} \cup \{\overrightarrow{vz} : v \in N_G(z) \setminus \{x, y\}\}$. It is easy to check that $(D, H)$ is the desired $(3, 2)$-decomposition of $G$. (This uses that $N^+_{D'}(\{x, y\}) = \emptyset$.)

**Case 1: $C$ has a chord $vw$.** *Case 1a: $C$ has a chord $vw$ such that $x, y, z$ all lie in $G_1$ (or all lie in $G_2$), where $G_1$ and $G_2$ are the induced subgraphs of $G$ separated by chord $vw$.* See the left of Figure 11.4. Choose the chord $vw$ so that $G_2$ is as small as possible; thus, $G_2$ has no chord of its

[2]Every triangulated cycle $J$ contains at least two vertices of degree 2; when $|J| \geqslant 4$, these 2 vertices can be chosen to be nonadjacent. The proof is easy by induction. We take as $z'$ one of these degree 2 vertices in $V(G') \setminus \{x', y'\}$.

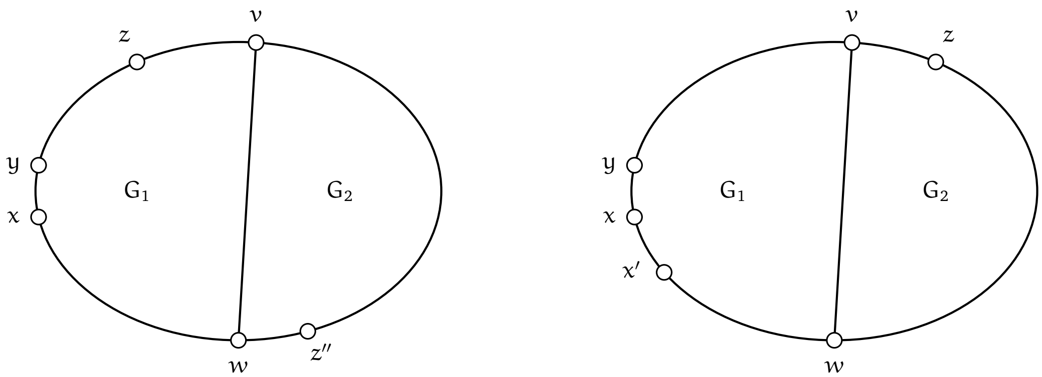



Figure 11.4: Left: In Case 1a, cycle C has a chord $vw$ such that vertices $x, y, z$ all lie on the same side of the chord. Right: In Case 1b, cycle C has a chord $vw$ but for every such chord vertex $z$ lies on the side opposite of vertices $x$ and $y$.

boundary cycle $C_2$. By induction, we get a $(3,2)$-decomposition $(D_1, H_1)$ for $G_1$ w.r.t. $(x, y, z)$, and possibly $z'$. Let $z''$ be a boundary neighbor of $w$ in $G_2$, with $z'' \neq v$. By induction, $G_2$ has a $(3,2)$-decomposition $(D_2, H_2)$ w.r.t. $(v, w, z'')$. Note that $z''$ is not incident to a chord of $C_2$, since no such chord exists, by our choice of $vw$; this is important to satisfy the hypotheses of the theorem, so that we can finish by induction.

Now the desired $(3,2)$-decomposition of G is $(D_1 \cup (D_2 - \overrightarrow{vw}), H_1 \cup H_2)$. All three of properties (i), (ii), (iii) are easy to verify. In particular, the digraph is acyclic, since all outneighbors of $\{v, w\}$ lie in $V(G_1)$.

*Case 1b: For every chord $vw$ of* C*, vertices $x$ and $y$ lie in one subgraph, say* $G_1$*, and vertex $z$ lies in the other,* $G_2$. See the right of Figure 11.4. Choose the chord $vw$ of C so that $G_1$ is as small as possible. So the boundary cycle $C_1$ of $G_1$ has no chord. Let $x'$ be a boundary neighbor of $x$ in $G_1$, with $x' \neq y$. By induction, $G_1$ has a $(3,2)$-decomposition $(D_1, H_1)$ w.r.t. $(x, y, x')$. Note that $x'$ is not incident to a chord of $C_1$, since no such chord exists; as above, this is important to satisfy the hypotheses of the theorem. Since $D_1$ is acyclic, we assume by symmetry that $D_1$ has no directed $w, v$-path. By induction, $G_2$ has a $(3,2)$-decomposition $(D_2, H_2)$ either w.r.t. $(v, w, z)$ or w.r.t. $(v, w, z, z')$. As above, the desired $(3,2)$-decomposition of G is $(D_1 \cup (D_2 - \overrightarrow{vw}), H_1 \cup H_2)$. Again all three of properties (i), (ii), (iii) are easy to verify.

$w, w^*$ **Case 2: C has no chord (and C is not a 3-cycle).** Let $w$ and $w^*$ be the boundary neighbors
$G', C'$ of $z$, in G, where $w \notin \{x, y, z'\}$ and $w^* \in \{x, y, z'\}$; see Figure 11.5. Let $G' := G - z$ and let $C'$ be the boundary of $G'$.

*Case 2a:* $C'$ *has a chord* $vw$. See the left of Figure 11.5. Now $wvz$ is a separating 3-cycle in G; let $V_1$ and $V_2$ be the vertex sets of its components. Let $G_i := G[V_i \cup \{v, w, z\}]$ for each $i \in [2]$. We assume $x, y \in V(G_1)$. Note that each $G_i$ is a near-triangulation. By induction, $G_1$ has a $(3,2)$-decomposition $(D_1, H_1)$ w.r.t. $(x, y, z)$ or $(x, y, z, z')$. Similarly, $G_2$ has a $(3,2)$-decomposition $(D_2, H_2)$ w.r.t. $(v, w, z)$. Now our desired $(3,2)$-decomposition of G is $(D_1 \cup (D_2 - \{\overrightarrow{vw}, \overrightarrow{zv}, \overrightarrow{zw}\}), H_1 \cup H_2)$.

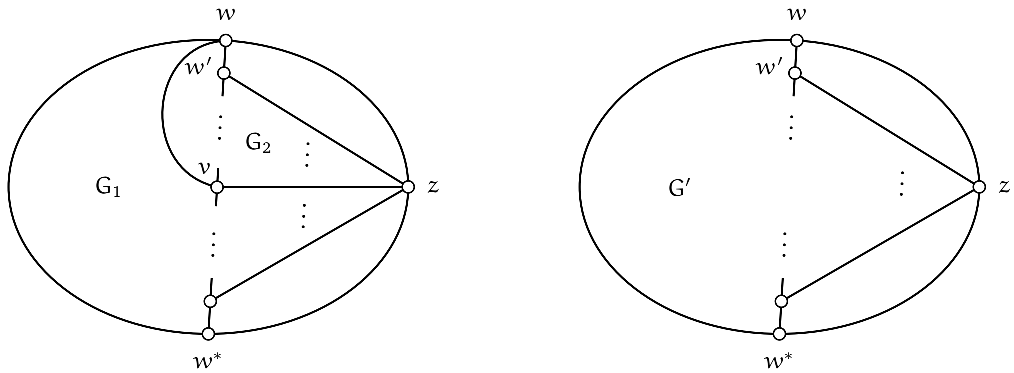



Figure 11.5: Left: In case 2a, cycle $C'$, the boundary cycle of $G - z$, has a chord $vw$ at $w$. Here $G_2$ has boundary cycle $vwz$. Right: In Case 2b, cycle $C'$ has no chord at $w$.

*Case 2b:* $C'$ *has no chord at* $w$. See the right of Figure 11.5. Let $I := N_G(z) \setminus \{w, w^*\}$. Let $w'$ be the boundary neighbor of $w$ in $G'$ that is an interior vertex of $G$; note that $w'$ exists, because $C$ has no chord, so $d_G(z) \geqslant 3$. By induction we will find a $(3,2)$-decomposition $(D', H')$ of $G'$ w.r.t. $(x, y, w, w')$ or simply w.r.t. $(x, y, w)$, and extend $D'$ to $G$ by making each vertex in $I$ be an inneighbor of $z$. Let $\tilde{D} := D' \cup \{\overrightarrow{vz} : v \in I\}$. To finish, we must add each of edges $wz$ and $w^*z$ to either $H'$ or $\tilde{D}$ (suitably oriented). Since $d_{H'}(z) = d^+_{\tilde{D}}(z) = 0$, each such assignment will create no problems at $z$. So we must simply ensure that we create no problems at $w$ and $w^*$. Note that, because $(D', H')$ is a $(3,2)$-decomposition w.r.t. $(x, y, w, w')$, by induction we also have $d^+_{D'}(w) + d_{H'}(w) \leqslant 2$. $\tilde{D}$

First suppose that $w^* \in \{x, y\}$; now we add $\overrightarrow{zw^*}$ to $\tilde{D}$. If $d_{H'}(w) \leqslant 1$, then we add $wz$ to $H'$; otherwise, we add $\overrightarrow{wz}$ to $\tilde{D}$. In the latter case, we have $d^+_{\tilde{D}}(w) \leqslant 1$, since by hypothesis $d^+_{D'}(w) + d_{H'}(w) \leqslant 3$. It is easy to verify that (i)–(iii) hold. In particular, the digraph is acyclic, since $w^* \in \{x, y\}$ and this vertex subset has no outneighbors.

Now assume instead that $w^* = z'$. By (iii), we have $d^+_D(w) + d_{H'}(w) \leqslant 2$; so we can either add $wz$ to $H'$ or else add $\overrightarrow{wz}$ to $\tilde{D}$. And by (ii), we have $d^+_D(w^*) + d_H(w^*) \leqslant 3$; but this inequality is no longer required in the $(3,2)$-decomposition of $G$, since $w^* = z'$. So again we can either add $w^*z$ to $H$ or else add $\overrightarrow{w^*z}$ to $\tilde{D}$. □

## 11.3 $(4,2)$-Choosability of Planar Graphs: Lists of Size 4 with Lists on Neighbors Sharing at Most 2 Colors

In this section we consider an arbitrary planar graph $G$ and a 4-assignment $L$ for $G$ such that $|L(v) \cap L(w)| \leqslant 2$ for all $vw \in E(G)$. For every such $G$ and $L$, we show that $G$ has an $L$-coloring.[3]

[3]The hypothesis $|L(v) \cap L(w)| \leqslant 2$ is sharp; Mirzakhani constructed planar graphs and 4-assignments $L$ with $|L(v) \cap L(w)| \leqslant 3$ for each edge $vw$ that admit no $L$-coloring (see Exercise 4).

The proof follows the same general idea as that for 5-choosability of planar graphs: we allow smaller lists on certain vertices, to be more amenable to a proof by induction.

More specifically, we aim to color some vertex $v$ on the outer face, delete $v$, and delete $v$'s color from the lists of its neighbors, which are now themselves on the outer face in this smaller graph $G'$; finally, we color $G'$ by induction. This outline suggests that vertices on the outer face should be allowed smaller lists. As before, if the boundary $C$ of the outer face has a chord $xy$, then coloring one endpoint, $x$, may decrease the size of the list for the other, $y$, too much.

So we prefer to allow the endpoints of a single edge on $C$ to be precolored. In this way we color the subgraph on one side of the chord by induction, and then color (also by induction) the subgraph on the other side. In the second subgraph, we take as precolored the chord's endpoints, so the colorings agree there, giving a coloring of the whole graph.

This approach nearly works. But when we color some vertex $w$ on $C$, the color used on $w$ may also reduce the sizes of lists for its neighbors on $C$. With a little work, we can choose our color for $w$ so that we decrease the list size for at most *one* neighbor of $w$ on $C$. However, by allowing even a single vertex $z$ on $C$, besides the endpoints of our precolored edge, with list size at most 2, we make the theorem false. (A simple counterexample is when $G$ is a 3-cycle.) To patch this case, we require that $z$ has a "good neighbor" satisfying stronger conditions on how its list interacts with that of $z$. We also require that the precolored edge on $C$ be prescribed *two* options for the colors of its endpoints, rather than just one.

To make all of these ideas precise, we need a number of definitions. To help the reader build intuition, we illustrate many of these terms in Figure 11.6 and Example 11.9.

$(\star,\ell)$-list assignment
rooted plane graph
root edge
boundary vertices
(primary) boundary neighbors

**Definition 11.7.** Fix a positive integer $\ell$. A $(\star,\ell)$-*list assignment* for $G$ is a list assignment $L$ such that $|L(x)\cap L(y)|\leqslant \ell$ for every edge $xy\in E(G)$. A *rooted plane graph* $(G,w_1w_2)$ is a plane embedding of a planar graph $G$ together with a specified *root edge* $w_1w_2$ on the boundary of the outer face. Each endpoint of the root edge is a root vertex. The *boundary vertices* of $G$ are the vertices on the boundary $C$ of its outer face. Fix a non-root boundary vertex $v$ in a rooted plane graph $(G,w_1w_2)$. The *boundary neighbors* of $v$ in $(G,w_1w_2)$ are all neighbors of $v$ on the boundary $C$ of the outer face. When the boundary neighbors are denoted $x_1,\dots,x_s$ in order around $C$, the *primary boundary neighbors* are $x_i$ and $x_{i+1}$ such that the root edge $w_1w_2$ lies on the portion of $C$ joining $x_i$ and $x_{i+1}$ but containing no other boundary neighbors.

$\tilde{L}$
$(\star,2)$-list assignment
good neighbor

A list assignment $L$ for a rooted plane graph $(G,w_1w_2)$ assigns a set $L(v)$ of colors to each vertex $v\in V(G)\setminus\{w_1,w_2\}$ and assigns to $(w_1,w_2)$ a set $L(w_1,w_2)$ of ordered pairs of distinct colors. An $L$-*coloring of* $(G,w_1w_2)$ is a proper coloring $\varphi$ of $G$ such that $\varphi(v)\in L(v)$ for all $v\in V(G)\setminus\{w_1,w_2\}$ and $(\varphi(w_1),\varphi(w_2))\in L(w_1,w_2)$. Fix a list assignment $L$ of $(G,w_1w_2)$. Now the list assignment $\tilde{L}$ of $G$ associated with $L$ is given by $\tilde{L}(v):=L(v)$ for all $v\in V(G)\setminus\{w_1,w_2\}$ and $\tilde{L}(w_1):=\{\alpha:\exists\beta,(\alpha,\beta)\in L(w_1,w_2)\}$ and $\tilde{L}(w_2):=\{\beta:\exists\alpha,(\alpha,\beta)\in L(w_1,w_2)\}$. Finally, $L$ is a $(\star,2)$-*list assignment of* $(G,w_1w_2)$ if $\tilde{L}$ is a $(\star,2)$-list assignment of $G$. Let $L$ be a $(\star,2)$-list assignment of $(G,w_1w_2)$. Fix $x,y\in V(C)$ where $x$ is a non-root vertex and $y$ is a primary boundary neighbor of $x$ (possibly a root vertex). Now $y$ is a *good neighbor* of $x$ if at least one of the following conditions holds: $|\tilde{L}(x)\cap\tilde{L}(y)|\leqslant 1$ or $|\tilde{L}(y)|=4$.

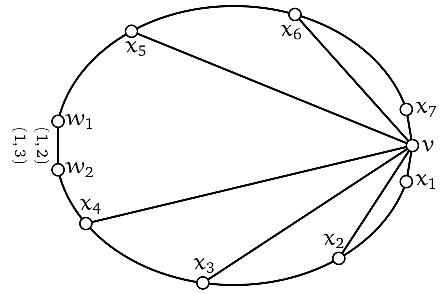



Figure 11.6: An instance illustrating many of the terms in Definitions 11.7 and 11.8. Example 11.9 provides the details.

**Definition 11.8.** A $(\star, 2)$-list assignment L for a rooted plane graph $(G, w_1w_2)$ is *valid* if $|L(w)| = 4$ for each interior vertex $w$, and (at least) one of the following holds: valid

(a) $|L(w_1, w_2)| \geqslant 1$ and $|L(w)| \geqslant 3$ for each non-root boundary vertex $w$.

(b) $|L(w_1, w_2)| \geqslant 2$ and there exists a unique non-root boundary vertex $w^*$ such that $|L(w)| \geqslant 3$ for all $w \in C \setminus \{w_1, w_2, w^*\}$ and $|L(w^*)| = 2$ and $w^*$ has a good neighbor.

When the root edge $w_1w_2$ is clear from context, we will often write "valid for G" to mean "valid for $(G, w_1w_2)$".

**Example 11.9.** $(G, w_1w_2)$ is a rooted plane graph with root edge $w_1w_2$. The boundary neighbors of $v$ are $x_1, \ldots, x_7$ and its primary boundary neighbors are $x_4$ and $x_5$. Suppose that L is a $(\star, 2)$ assignment for G with $|L(v)| = 2$, $L(w_1, w_2) = \{(1, 2), (1, 3)\}$, and $|L(y)| \geqslant 3$ for all other boundary vertices, and $|L(z)| = 4$ whenever $z$ is either $x_4$ or an interior vertex. Now $\tilde{L}(w_1) = \{1\}$ and $\tilde{L}(w_2) = \{2, 3\}$. Note that L is valid for $(G, w_1w_2)$ because it satisfies Definition 11.8(b), with $w^* := v$ and with $x_4$ as the good neighbor of $v$. $\diamondsuit$

**Remark 11.10.** When L is valid and (b) holds, we can assume $|\tilde{L}(x) \cap L(w^*)| \leqslant 1$ where $x$ is a good neighbor of $w^*$. Otherwise, $|\tilde{L}(x)| = 4$, so we pick a color $\alpha \in \tilde{L}(x) \cap \tilde{L}(w^*)$; now let $L'(x) := L(x) \setminus \{\alpha\}$ and $L'(v) := L(v)$ when $v \neq x$. Since $\tilde{L}$ is a valid $(\star, 2)$-list assignment, so is $\tilde{L'}$, but also $|L'(x) \cap L'(w^*)| \leqslant 1$. So we assume $|\tilde{L}(x) \cap \tilde{L}(w^*)| \leqslant 1$; but to prove that $x$ is a good neighbor of $w^*$, we can prove that either $|\tilde{L}(x) \cap \tilde{L}(w^*)| \leqslant 1$ or $|L(x)| \geqslant 4$.

**Theorem 11.11.** *If* L *is a valid list assignment for a rooted plane graph* $(G, w_1w_2)$*, where* G *is a near-triangulation, then* $(G, w_1w_2)$ *has an* L*-coloring.*

This theorem immediately implies the following, which motivates this section. (As usual, if G is not a near-triangulation, then we can add edges to make it one. And proving the theorem for the resulting graph also proves it for G.)

**Corollary 11.12.** *Let* $G$ *be a planar graph and* $L$ *a list assignment for* $G$*. If* $|L(v)| = 4$ *for all* $v \in V(G)$ *and* $|L(v) \cap L(w)| \leqslant 2$ *whenever* $vw \in E(G)$*, then* $G$ *has an* $L$*-coloring.*

*Proof of Theorem 11.11.* Our proof is by induction on $|G|$. (Throughout this proof, whenever we say that "(a) holds" or "(b) holds", we mean in Definition 11.8.) Our base case is $|G| = 3$. If (a) holds, then we color $w_1$ and $w_2$ from their lists, and afterward we color $w_3$ (from its list) to avoid the colors on $w_1$ and $w_2$. Assume instead that (b) holds. First suppose that $|\tilde{L}(w_1)| = 1$ or $|\tilde{L}(w_2)| = 1$; by symmetry, say $|\tilde{L}(w_1)| = 1$. Because (b) holds, $|\tilde{L}(w_2)| \geqslant 2$. Because $\tilde{L}(w_1) \cap \tilde{L}(w_2) = \emptyset$, we can color greedily in the order $w_1, w_3, w_2$. Suppose instead that $|\tilde{L}(w_1)| = |\tilde{L}(w_2)| = 2$. By symmetry, assume that $w_1$ is a good neighbor of $w_3$. So we color $w_1$ with a color not in $L(w_3)$, and then color $w_2$ and $w_3$ greedily.

**Claim 1.** $G$ *has no separating* 3*-cycle* $z_1z_2z_3$*.*

Proof. Suppose the contrary. Denote by $G_{\text{in}}$ and $G_{\text{out}}$ the subgraphs induced by $\{z_1, z_2, z_3\}$ and all of the vertices inside and outside, respectively, of the cycle $z_1z_2z_3$. By induction, $G_{\text{out}}$ (with root edge $w_1w_2$) has an L-coloring, $\varphi_{\text{out}}$. Consider $(G_{\text{in}}, z_1z_2)$, where $L'(z_1, z_2) = \{(\varphi(z_1), \varphi(z_2))\}$ and $L'(z_3) = \{\varphi(z_1), \varphi(z_2), \varphi(z_3)\}$, and $L'(v) := L(v)$ for all $v$ strictly inside the cycle $z_1z_2z_3$. By induction, $G_{\text{in}}$ has an $L'$-coloring $\varphi_{\text{in}}$. Now colorings $\varphi_{\text{out}}$ and $\varphi_{\text{in}}$ agree on $z_1z_2z_3$, so their union is an L-coloring of G. ♢

**Case 1a: C has no chord and (a) holds.** Assume $L(w_1, w_2) = \{(\alpha_1, \alpha_2)\}$. Let $w_3$ be the other boundary neighbor of $w_2$. Since C has no chord and G has no separating cycle (and G is a near-triangulation), vertices $w_1$ and $w_2$ have a unique common neighbor $y$, not on C; similarly, $w_2$ and $w_3$ have a unique common neighbor $z$, not on C (possibly $y = z$, a distinction that we consider near the end of this case). See the left of Figure 11.7.

$G', L'$ Let $G' := G - w_2$, let $L'(v) := L(v) - \alpha_2$ for all $v \in N_G(w_2) \setminus \{w_1, y\}$, and $L'(v) := L(v)$ for all $v \in V(G) \setminus N_G(w_2)$. Finally, let $L'(w_1, y) := \{(\alpha_1, \alpha_3), (\alpha_1, \alpha_4)\}$, where $\alpha_3, \alpha_4 \in L(y) \setminus \{\alpha_1, \alpha_2\}$. If $|L(y) \setminus \{\alpha_1, \alpha_2\}| \geqslant 3$, then we choose $\alpha_3, \alpha_4 \in L(y) \setminus \{\alpha_1, \alpha_2\}$ arbitrarily, with a single exception; if $y = z$ and $\alpha_2 \notin L(y)$ and $L(y) \cap L(w_3) \neq \emptyset$, then we choose $\alpha_3, \alpha_4$ such that $|\tilde{L}'(y) \cap L'(w_3)| = |\{\alpha_3, \alpha_4\} \cap L'(w_3)| \leqslant 1$.

Now we show that $L'$ is a valid $(\star, 2)$-list assignment for $(G', w_1y)$. If $\alpha_2 \notin L(w_3)$, then $|L'(w_3)| = |L(w_3)| \geqslant 3$. Thus, $|L'(x)| \geqslant 3$ for all $x \in V(C') \setminus \{w_1, y\}$, and (a) holds. By induction $G'$ has an $L'$-coloring $\varphi'$, and coloring $w_2$ with $\alpha_2$ yields an L-coloring of G.

Instead assume that $\alpha_2 \in L(w_3)$, so $|L'(w_3)| = 2$. We show that $z$ is a good boundary neighbor of $w_3$ in $G'$. Suppose that $y \neq z$. If $\alpha_2 \notin L(z)$, then $|L'(z)| = 4$. Otherwise, $|L'(w_3) \cap L'(z)| = |L(w_3) \cap L(z)| - |\{\alpha_2\}| \leqslant 2 - 1 = 1$. So always (b) holds. Again, by induction, $G'$ has an $L'$-coloring $\varphi'$, and coloring $w_2$ with $\alpha_2$ yields an L-coloring of G. Now assume instead that $y = z$. By our choice of $\alpha_3, \alpha_4$, we have $|\tilde{L}'(y) \cap L'(w_3)| \leqslant 1$. Thus, $z$ is a good boundary neighbor of $w_3$, so (b) holds, and we again finish by induction.

**Case 1b: C has no chord and (b) holds.** So we have $w^* \in V(C) \setminus \{w_1, w_2\}$ such that $|L(w^*)| = 2$; denote a good neighbor of $w^*$ by $x$. If $w^*$ has two good neighbors, then we choose $x$ arbitrarily among them, unless $w^*$ is adjacent to a root vertex $w_i$, for some $i \in [2]$ such that

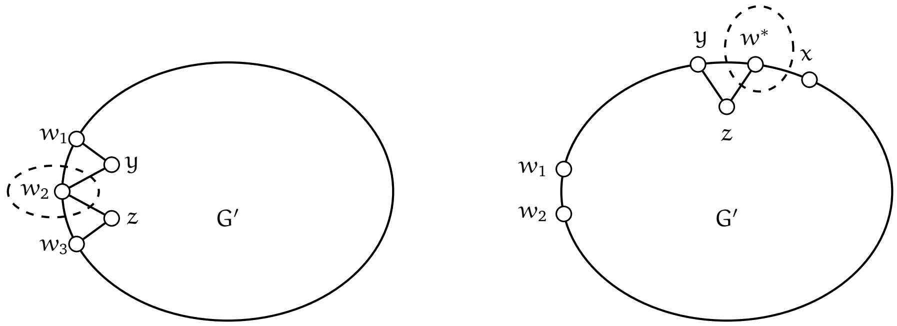



Figure 11.7: C has no chord. Left: In Case 1a, hypothesis (a) holds. Right: In Case 1b, hypothesis (b) holds. (Recall that hypotheses (a) and (b) are as in Definition 11.8.)

$|\tilde{L}(w_i)| = 1$. In this case, let $x := w_i$. Let $y$ denote the other boundary neighbor of $w^*$. Let $z$ denote a common neighbor of $w^*$ and $y$; note that $z$ exists, $z$ is unique, and $z$ lies inside C, because C has no chords and G has no separating 3-cycles.

First suppose that $y$ is not a root vertex. Fix $\alpha \in L(w^*) \setminus L(x)$; recall from Remark 11.10 that such an $\alpha$ exists. Let $G' := G - w^*$ and form $L'$ from L by deleting $\alpha$ from the lists of all neighbors of $w^*$; see the right of Figure 11.7. We will show by induction that $G'$ has an $L'$-coloring, and extend it to G by coloring $w^*$ with $\alpha$. Our choice of $\alpha$ ensures that $|L'(x)| \geqslant 3$. So it suffices to show that either $|L'(y)| \geqslant 3$, so (a) holds for $G'$, or else $y$ has a good neighbor (recall that $|L(w_1, w_2)| \geqslant 2$ since (b) holds for G), so (b) holds for $G'$. Suppose $|L'(y)| = 2$. If $|L'(z)| = 4$, then $z$ is a good neighbor for $y$ in $G'$. Otherwise, $\alpha \in L(y) \cap L(z)$, so $|L'(y) \cap L'(z)| \leqslant 2 - |\{\alpha\}| = 1$; again, $z$ is a good neighbor for $y$.

Now assume instead that $y$ is a root vertex; by symmetry, say $y = w_1$. If $\alpha \notin \tilde{L}(w_1)$, then the argument above still works. So assume that $\alpha \in \tilde{L}(w_1)$. If $|\tilde{L}(w_1)| = 1$, then we would have instead chosen $w_1$ as our good neighbor of $w^*$. So we assume $|\tilde{L}(w_1)| \geqslant 2$. Thus, we can color $w_1$ and $w_2$ from their lists such that $w_1$ avoids $\alpha$. But now $|L'(w)| \geqslant 3$ for all $w \in V(C') \setminus \{w_1, w_2\}$. So (a) holds for $G'$ and we are done by induction.

**Case 2: C has a chord $xy$.** *Case 2a: C has a chord $xy$ such that $|L(v)| \geqslant 3$ for all $v \in V(C_2) \setminus \{x, y\}$.* Here $xy$ splits G into subgraphs $G_1$ and $G_2$, and $C_2$ is the boundary cycle of $G_2$. (This includes the case that (a) holds.) Let $L_1$ be the restriction of L to $G_1$; note that $L_1$ is valid for $G_1$. By induction, $(G_1, w_1w_2)$ has an $L_1$-coloring $\varphi_1$. Let $L_2$ be the restriction of L to $G_2$, except that now $xy$ is the root edge and $L_2(x, y) = \{(\varphi_1(x), \varphi_1(y))\}$. Since $L_2$ is valid for $(G_2, xy)$, by induction $(G_2, xy)$ has an $L_2$-coloring $\varphi_2$. Since $\varphi_1$ and $\varphi_2$ agree on $\{x, y\}$, their union is an L-coloring of G.

*Case 2b: There exists $w^* \in V(C) \setminus \{w_1, w_2\}$ with $|L(w^*)| = 2$ and every chord $xy$ of C separates $w^*$ from $w_1w_2$. That is, $w_1w_2 \in V(G_1)$ and $w^* \in V(G_2) \setminus \{x, y\}$.* We choose the chord $xy$ so that $G_1$ is as small as possible; thus, $C_1$ has no chord. Since $|L(w^*)| = 2$, and L is valid, we must have $|L(w_1, w_2)| \geqslant 2$. Let $L_1$ be the restriction of L to $G_1$. By induction, $G_1$ has an $L_1$-coloring $\varphi_1$. To continue we need the following key claim.

**Claim 2.** $G_1$ *has another* $L_1$*-coloring* $\varphi_1'$ *such that* $(\varphi_1'(x), \varphi_1'(y)) \neq (\varphi_1(x), \varphi_1(y))$.

Assume that Claim 2 is true. Let $L_2$ denote the restriction of $L$ to $G_2$, except that $xy$ is the new root edge and $L(x, y) = \{(\varphi_1(x), \varphi_1(y)), (\varphi_1'(x), \varphi_1'(y))\}$. Note that the primary neighbors of $w^*$ in $G_2$ are the same as the primary neighbors of $w^*$ in $G$. Thus, $w^*$ has a good neighbor in $G_2$ (since $w^*$ has a good neighbor in $G$). By induction, $G_2$ has an $L_2$-coloring $\varphi_2$. By construction, $\varphi_2$ agrees on $\{x, y\}$ with either $\varphi_1$ or $\varphi_1'$. So either $\varphi_2 \cup \varphi_1$ or $\varphi_2 \cup \varphi_1'$ is an $L$-coloring of $G$. Thus, all that remains is to prove Claim 2.

To prove Claim 2, we restrict $L$ to $G_1$. We first try deleting $\varphi(y)$ from $L(y)$, and then try deleting $\varphi(x)$ from $L(x)$. If either of the resulting list assignments is valid, then we are done. So we assume not and deduce significant structure of $L$. Ultimately, we find adjacent vertices with lists sharing at least 3 colors, which contradicts the lemma's hypothesis.

<u>Proof.</u> [Proof of Claim 2] By symmetry, between $x$ and $y$, we assume $y \notin \{w_1, w_2\}$. So $y$ is not a root vertex. Recall from our choice of $G_1$ that $C_1$ has no chord. Denote $L(y)$ by $\{\varphi(y), \alpha_1, \alpha_2\}$. Let $x'$ and $y'$ denote, respectively, the primary boundary neighbors[4] in $G_1$ of $x$ and $y$ other than $y$ and $x$; see the left of Figure 11.8. We first try deleting $\varphi(y)$ from $L(y)$. Recall that we are in case 2b, so (b) holds; we are thus done if $y$ has a good neighbor, so we assume not. Thus $\alpha_1, \alpha_2 \in \tilde{L}(x) \cap \tilde{L}(y')$. If $x$ is also not a root vertex, then we repeat the same argument, but with $x$ in place of $y$, deleting $\varphi(x)$ from $L(x)$. Hence,

$\alpha_1, \alpha_2, x', y'$

$$\alpha_1, \alpha_2 \in L(x) \cap L(y) \cap \tilde{L}(x') \cap \tilde{L}(y'). \tag{11.1}$$

This case when $x$ is not a root vertex will require most of our work. But before continuing with it, we must handle the possibility that $x$ is a root vertex. Suppose it is; say $x = w_1$. Since $\alpha_1, \alpha_2 \in L(x)$, we see that $L(w_1, w_2) = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$; possibly, we have $\beta_1 = \beta_2$. By symmetry, we assume that $\varphi(x) \neq \alpha_1$. Now we modify $L_1$ by coloring $w_1w_2$ as $(\alpha_1, \beta_1)$. Since $|L_1(v)| \geqslant 3$ for all $v \in V(C_1) \setminus \{w_1, w_2\}$, the resulting list assignment $L_1'$ is valid, so $G_1$ admits an $L_1'$-coloring $\varphi'$. Since $\varphi'(y) \neq \varphi(y)$, we are done.

$\alpha_3$ So we now assume that statement (11.1) holds. We denote $L(x')$ by $\{\alpha_1, \alpha_2, \alpha_3\}$. Since $G$ is a near-triangulation with no separating 3-cycle, by Claim 1, vertices $x$ and $y$ have a unique common neighbor $z$ in $G_1$. Furthermore, since we chose $G_1$ to have no chords, we know $z$ is in the interior of $G_1$. By planarity, we know either (i) $N_{G_1}(x') \cap N_{G_1}(y) \subseteq \{x, z\}$ or (ii) $N_{G_1}(y') \cap N_{G_1}(x) \subseteq \{y, z\}$. By symmetry, we assume the former.

$z$

Now we try to color $x$ and $y$ and finish by induction. Recall that $L(y) = \{\alpha_1, \alpha_2, \varphi(y)\}$. By hypothesis, $|L(z) \cap L(y)| \leqslant 2$; thus, there exists $i \in [2]$ such that $|L(z) \cap \{\alpha_i, \varphi(y)\}| \leqslant 1$. By symmetry, we assume $i = 1$. Now we color $y$ with $\varphi(y)$, delete $\varphi(y)$ from the list of each neighbor of $y$, and finally delete $y$. We treat $x$ similarly, but with $\alpha_1$ in place of $\varphi(y)$. Call the resulting graph $G'$ and list-assignment $L'$. More formally, let $G_1' := G_1 - \{x, y\}$. (To be precise, we may also need to update the list for the root edge, as we clarify below.)

$G_1'$

[4]To be precise, we must first show that $x$ is not a root vertex before we can refer to its primary boundary neighbors. We will do this soon.

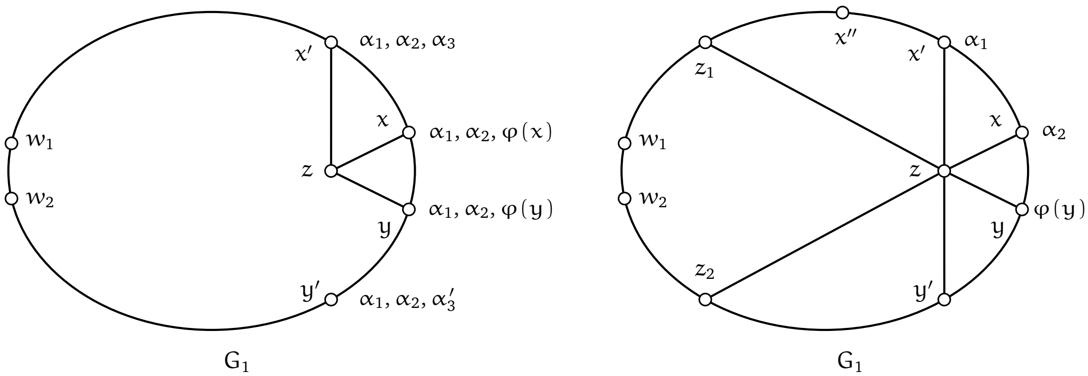



Figure 11.8: Part of the induced subgraph $G_1$, together with lists for some of its vertices. Left: The lists for $x', x, y, y'$. Right: Additional vertices $z_1, z_2, x''$ and edges $zz_1, zz_2, zy'$.

If $L'$ is valid for $G_1'$, then we can $L'$-color $G_1'$ by induction, and we are done. So we assume $L'$ is not valid for $G_1'$. By assumption, $|L(z) \cap \{\alpha_i, \varphi(y)\}| \leqslant 1$, so $|L'(z)| \geqslant 3$. And since $\alpha_1, \alpha_2 \in L(y')$, we know $\varphi(y) \notin L(y')$. Thus, $|L'(y')| = |L(y')| = 3$. Furthermore, for all $v \in V(C_1') \setminus V(C)$, we have $|L'(v)| \geqslant 3$. If $x'$ is a root vertex, then recall that $\alpha_1, \alpha_2 \in \tilde{L}(x')$. Thus, we can color the root edge (restrict its list) so that $x'$ avoids $\alpha_1$. In the resulting list assignment $L'$, we have $|L'(v)| \geqslant 3$ for all $v \in V(C_1') \setminus \{w_1, w_2\}$, so $L'$ is indeed valid. A simpler version of the same argument works if $x'$ is not a root vertex, but $\alpha_1 \notin L(x')$ or if $x'$ has a good neighbor. So instead we assume that $x'$ is not a root vertex, but $\alpha_1 \in L(x')$ and $x'$ has no good neighbor (in $G_1'$ w.r.t. $L'$).

Since $G$ is a near-triangulation with no separating cycles, $x$ and $x'$ have a unique common neighbor in $G_1'$; call it $z'$. Suppose that $z' \neq z$. Clearly, $z'$ is a primary boundary neighbor of $x'$ in $G_1'$. Since $L'$ is not valid for $G_1'$, we know that $z'$ is not a good neighbor for $x'$. Thus $|L'(z')| = 3$. However, this implies that $\alpha_1 \in L(z')$. Since also $\alpha_1 \in L(x')$, we have $|L'(x') \cap L'(z')| \leqslant 2 - |\{\alpha_1\}| = 1$. Thus, $z'$ is indeed a good neighbor for $x'$, a contradiction. Hence, in fact $z' = z$. Furthermore, $\alpha_2, \alpha_3, \varphi(y) \in L(z)$ and $\alpha_1 \notin L(z)$; also $\alpha_3 \neq \varphi(y)$. (If any of these is false, then either $z$ is a good neighbor for $x'$ and we are done or else $|L(z) \cap L(x')| = 3$, which contradicts the hypothesis.)

Now the edge $x'z$ implies that (ii) above also holds: $N_{G_1}(y') \cap N_{G_1}(x) \subseteq \{y, z\}$. So we can repeat the same argument with $x, x'$ replaced by $y, y'$. This implies that $y'$ is not a root vertex, that $zy' \in E(G_1)$, and for some color $\alpha_3'$ we have both $L(y') = \{\alpha_1, \alpha_2, \alpha_3'\}$ and $\alpha_2, \alpha_3', \varphi(x) \in L(z)$. As a result, $\{\alpha_2, \alpha_3, \alpha_3', \varphi(x), \varphi(y)\} \subseteq L(z)$. So $\alpha_3' = \alpha_3$, since $|L(z)| = 4$ and the other colors are pairwise distinct. Thus, $L(z) = \{\alpha_2, \alpha_3, \varphi(x), \varphi(y)\}$.

Recall that $L'(x') = \{\alpha_2, \alpha_3\}$. Let $x''$ be the primary boundary neighbor of $x'$ in $G_1$ other $x''$
than $x$; see the right of Figure 11.8. Since $L'$ is not valid, we know that $x''$ is not a good neighbor for $x'$ w.r.t. $G_1'$ and $L'$. Thus, $L'(x') \subseteq L'(x'')$. Since $L$ is a $(\star, 2)$-list assignment, this implies $\alpha_1 \notin \tilde{L}(x'')$. So now we instead try to color $x, y, x'$ and finish by induction. More formally, we

color $x'$ with $\alpha_1$, color $x$ with $\alpha_2$, and color $y$ with $\varphi(y)$. Afterwards, we delete $\alpha_1$ from the
$G''_1$ list of each neighbor of $x'$, we call the resulting lists $L''_1$, and we let $G''_1 := G_1 - \{x', x, y\}$. (We might try to handle $\alpha_2$ and $\varphi(y)$ similarly to $\alpha_1$, but this is not needed since $N_{G_1}(x) = \{x', y, z\}$ and $N_{G_1}(y) = \{x, y', z\}$.)

Now we try to show that $L''_1$ is valid for $G''_1$. Recall that $\alpha_1 \notin \tilde{L}(x'')$ and $\varphi(y) \notin L(y')$. Thus, $|L''(v)| \geqslant 3$ for all $v \in V(C'') \setminus \{w_1, w_2, z\}$. Since $|L(w_1, w_2)| = 2$, we are done if $z$ has a good neighbor; so we assume not. Recall that $L''(z) = L(z) \setminus \{\alpha_2, \varphi(y)\} = \{\alpha_3, \varphi(x)\}$.
$z_1, z_2$ Denote by $z_1$ and $z_2$ the primary boundary neighbors of $z$ in $(G''_1, w_1w_2)$. Since $z$ has no good neighbor, we have $\alpha_3, \varphi(x) \in \tilde{L}(z_1) \cap \tilde{L}(z_2)$. In particular, $y'$ is a boundary neighbor of $z$ in $(G''_1, w_1w_2)$, but $y'$ is not a primary boundary neighbor of $z$.

Now we repeat the same argument as above, but swapping the roles of $x, x'$ with those of $y, y'$. We let $z'_1, z'_2$ be the primary boundary neighbors of $z$ in $(G_1 - \{x, y, y'\}, w_1w_2)$. If $z$ has a good neighbor, then we are done; so we assume not. Since $z$ has no good neighbor, we have $\alpha_3, \varphi(y) \in \tilde{L}(z'_1) \cap \tilde{L}(z'_2)$. In particular, $x'$ is a boundary neighbor of $z$ here, but $x'$ is not a primary boundary neighbor of $z$. But this means that $z_1 = z'_1$ and $z_2 = z'_2$; that is, when repeating the argument, $z$ has the same primary boundary neighbors. As a result, we have $\alpha_3, \varphi(x), \varphi(y) \in \tilde{L}(z_i) \cap L(z)$ for each $i \in [2]$, contradicting that $L$ is a $(\star, 2)$-list assignment of $(G, w_1w_2)$. This contradiction finishes the proof of the claim. $\diamondsuit$

Claim 2 finishes the proof of the theorem. $\square$

## 11.4 3-Choosability of Girth 5 Planar Graphs

In this section, our goal is prove that $\chi_\ell(G) \leqslant 3$ for every planar graph $G$ with girth at least 5. This bound on girth is best possible: in Section 2.4 we construct planar graphs with girth 4 that are not 3-choosable. For more context, note that our desired result also gives a short proof of Grötzsch's Theorem, as follows. Suppose $G$ is planar and triangle-free. By the Folding Lemma (Lemma 4.20) and Corollary 4.21, we assume $G$ has girth 5. Now let $L(v) := [3]$ for all $v \in V(G)$. By assumption $G$ has an $L$-coloring, which is a 3-coloring.

As in the previous sections, we in fact prove something stronger, which helps to facilitate our proof by induction.

$\varphi$ **Theorem 11.13.** *Let $G$ be a plane graph with girth at least 5. Let $\varphi$ be a proper coloring of a path*
$P, C$ *or cycle $P$, with order at most 6 and with all its vertices on the boundary $C$ of the outer face of $G$.*
$L$ *Let $L$ be a list-assignment for $V(G)$ such that (a) $L(v) = \varphi(v)$ for all $v \in V(P)$ and (b) $|L(v)| \geqslant 2$ for all $v \in V(C) \setminus V(P)$ and (c) $|L(v)| = 3$ otherwise. If no edge joins two vertices $v, w$ with $|L(v)| \leqslant 2$ and $|L(w)| \leqslant 2$ unless $v, w \in V(P)$, then $\varphi$ can be extended to an $L$-coloring of $G$.*

Our proof is by induction on $|G|$. Our plan is to either (a) split $G$ into two subgraphs and color each inductively, so that the colorings agree on the vertices common to both subgraphs or (b) color some vertices on the boundary of the outer face, delete them, and delete their colors from the lists of their neighbors. In the hypotheses of Theorem 11.13, it is the precolored path

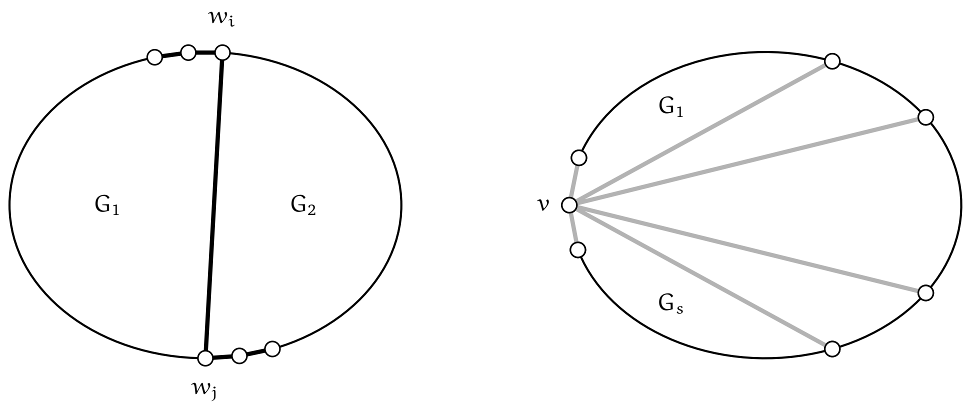



Figure 11.9: The proof of Claim 1. Left: P contains a chord $w_i w_j$ of C. Right: $v$ is a cut-vertex in an endblock B of G. Throughout this section, gray denotes edges that are not part of the precolored path for G, but are part of the path for one or more subgraphs that we L-color inductively.

P that allows us to proceed in (a) and the set of vertices with lists of size 2, together with the girth constraint, that helps in (b). However, many details remain.

*Proof.* We denote the vertices of C by $w_1, \ldots, w_k$ and the vertices of P by $w_1, \ldots, w_q$. Our proof is by induction on $|G|$, and the base case $|G| \leqslant 3$ is trivial. $w_i$, $k$, $q$

In each claim below, for brevity we omit the introductory phrase "We may assume". In each case we suppose the claim is false and show how we can L-color G by induction.

**Claim 1.** G *is* 2*-connected and no edge of* P *is a chord of* C.

Proof. If G is disconnected, then we can L-color each component by induction. So suppose instead that some vertex $v$ of P is a cut-vertex. Let $G_1$ and $G_2$ be subgraphs of G with $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$. By induction, we L-color both $G_1$ and $G_2$. Since these colorings agree on $v$, their union is an L-coloring of G. If some edge $w_i w_j$ of P is a chord of C (as on the left in Figure 11.9), then G again splits into $G_1$ and $G_2$, this time with $V(G_1) \cap V(G_2) = \{w_i, w_j\}$. By induction, we L-color $G_1$ and $G_2$. Since these colorings agree on $\{w_i, w_j\}$, their union is again an L-coloring of G.

Suppose instead that G has a cut-vertex, but has no cut-vertex in P. Let B be an endblock of G (containing no vertices of P), and let $v$ be a cut-vertex of B; see the right of Figure 11.9. By induction, we L-color all of G except $V(B) - v$. Now for each neighbor $w$ of $v$ in B with $|L(w)| = 2$, color $w$. Recall that each such $w$ is on C.

As a result, B can be split into parts (with $v$ repeated in each part) and other vertices appearing in at most two parts, such that in each part the precolored vertices induce a path of order at most 3. Now we L-color each part by induction. Since these L-colorings agree on their pairwise common vertices, their union gives an L-coloring of G. ♢

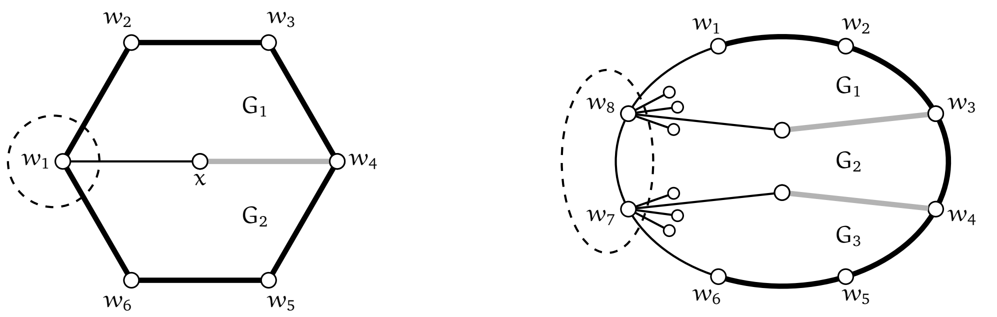



Figure 11.10: The proof of Claim 2. Left: P is a cycle. Right: One possibility when P is a path but $|V(P)| + 3 > |V(C)|$.

**Claim 2.** P *is a path and* $|V(P)| + 3 \leqslant |V(C)|$.

Proof. If P is a cycle, then P is chordless by the girth constraint. So we delete some vertex $w_1$ of P and delete its color from the lists of all of its neighbors. To see that we can finish by induction, note that in the smaller graph $G'$ each vertex with a list of size 2 was a neighbor of $v$; thus these vertices form an independent set and lie on the outer face in $G'$. If no vertex $x$ with a list of size 2 is adjacent to a precolored vertex of $G'$, then we can L-color $G'$ by induction. Suppose instead that $w_1$ had some neighbor $x$ that is also adjacent to $w_i$ for some $i \neq 1$. By hypothesis, $q \leqslant 6$. Since G has girth at least 5, we have $i = 4$; furthermore, vertex $x$ is unique. See the left of Figure 11.10. Now we split $G'$ into $G_1$, with precolored path $w_2w_3w_4x$, and $G_2$, with precolored path $xw_4w_5w_6$. By induction, we L-color both $G_1$ and $G_2$; the union of their L-colorings is an L-coloring of G.

Assume instead that P is a path but $|V(P)| + 3 > |V(C)|$. See the right of Figure 11.10. By assumption $|L(w_i)| = 3$ for each $w_i \in V(C) \setminus V(P)$, so we can extend $\varphi$ to $V(C)$. Now we delete all vertices of $V(C) \setminus V(P)$, and also delete the colors of each of these (deleted) vertices from the lists of their neighbors. Since G has girth at least 5, the vertices with lists of size 2 in the resulting graph form an independent set. And since $|V(C)| \leqslant 8$, every vertex $x$ in G has at most two neighbors on C. But possibly some neighbor $x$ of a deleted vertex also has a second neighbor $w_i$ on C. In this case, we color each such $x$, but do not delete it. (By the girth constraint, G has at most three such vertices $x$. For example, in addition to those shown, G could contain a common neighbor of $w_3$ and $w_7$.) As above, we split G into subgraphs each of which we can L-color by induction. Since the boundary of each subgraph is precolored, these L-colorings agree on their pairwise common vertices; so their union is an L-coloring of G. $\diamondsuit$

**Claim 3.** C *has no chord.*

Proof. Assume to the contrary, that G has a chord $w_iw_j$; again see the left side of Figure 11.9. So G has subgraphs $G_1$ and $G_2$ with $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{w_i, w_j\}$. We assume that $G_1$ has at least as many vertices of P as $G_2$ does and, subject to this, that $G_2$ is as small as possible. We first L-color $G_1$ by induction. Now $w_i$ and $w_j$ have colors. By the minimality of $G_2$, the outer face of $G_2$ is chordless; so at most two vertices of $G_2$ have lists of size 2 and are adjacent to $w_i$ or $w_j$. We color all such vertices. To finish on $G_2$ by induction, it suffices to observe that the colored vertices in $G_2$ induce a path and the number of them is at most $2 + |V(G_2) \cap V(P)| \leqslant 2 + (|V(P)| + |\{w_i, w_j\}|)/2 \leqslant 2 + 4 = 6$. $\diamondsuit$

We denote by *int*$(C)$ the vertices that lie inside $C$; that is, let $\mathrm{int}(C) := V(G) \setminus V(C)$. int(C)

**Claim 4.** G *has no path* $w_i v w_j$ *where* $v$ *is in int*$(C)$, *except possibly when* $q = 6$ *and the path is of the form* $w_4 v w_7$ *or* $w_3 v w_k$. *In particular,* $v$ *has only two neighbors on* $C$.

Proof. Assume the contrary. So $G$ has subgraphs $G_1$ and $G_2$ with $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{w_i, w_j, v\}$, as on the left of Figure 11.11. As in the proof of Claim 3, we assume that $G_1$ has at least as many vertices of $P$ as $G_2$ does and, subject to this, that $G_2$ is as small as possible. Vertex $v$ may have many neighbors with only two colors; but by the minimality of $G_2$, none of these neighbors lie in $V(G_2) \setminus \{w_i, w_j, v\}$.

As above, by induction we L-color $G_1$; this gives colors to $w_i$, $w_j$, and $v$. Again, we would like to L-color $G_2$ by induction, so that these L-colorings agree on the path $w_i v w_j$ (and both agree with the precoloring of $P$). So in $G_2$ we take as precolored all vertices of $P$, as well as $w_i v w_j$ and also any neighbors of $w_i$ and or $w_j$ that (initially) have lists of size 2. We show below that this is at most 6 precolored vertices in $G_2$.

If neither $w_i$ nor $w_j$ is on $P$, then the number of precolored vertices when coloring $G_2$ is at most $|\{w_i, w_j, v\}| + 2 = 5$, since each of $w_i$ and $w_j$ has at most one neighbor with its list of size 2, by the minimality of $G_2$. If both $w_i$ and $w_j$ are on $P$, then neither has a neighbor with a list of size 2. So the number of precolored vertices when coloring $G_2$ is at most $|\{v\}| + (|V(P)| + 2)/2 = 5$. Lastly, suppose that exactly one of $w_i$ and $w_j$ is on $P$; say $w_j$. Now the number of precolored vertices when coloring $G_2$ is at most $\lfloor (|V(P)|+1)/2 \rfloor + |\{v\}| + |N[\{w_i, w_j\} \setminus V(P)]| \leqslant 3+1+2 = 6$. If $7 \in \{i, j\}$ and $|L(w_8)| = 2$, then we must choose a color for $w_8$, different from the color on $w_7$ when we color $G_1$, and include $w_8$ in the precolored path when we color $G_2$ inductively; see the right of Figure 11.11. (We handle similarly the case when $k \in \{i, j\}$ and $|L(w_{k-1})| = 2$.)

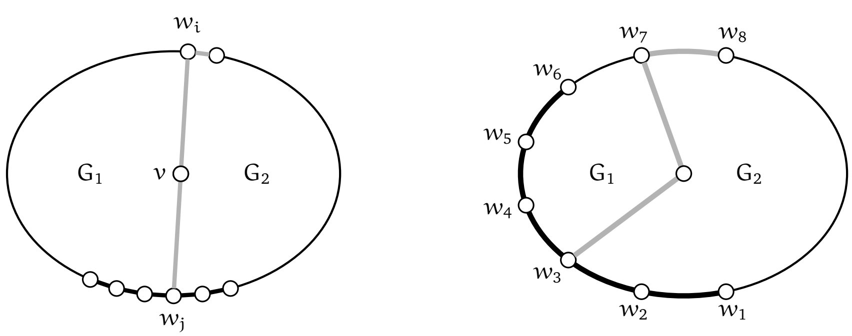



Figure 11.11: The proof of Claim 4. Left: The general case. Right: A specific instance of this case that must be handled carefully.

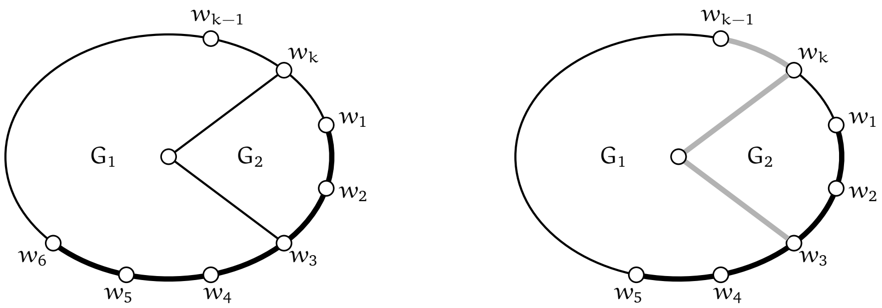



Figure 11.12: In the proof of Claim 4, we must verify that the precolorings on $G_1$ and $G_2$ are valid. Left: The exceptional case when $q = 6$ and $(x, y) = (w_k, w_1)$. Right: A case when $q = 5$ that must be handled specially.

Finally, we must ensure this precoloring of $G_2$ is valid. The only way it is invalid is if some vertex $x$ of $G_1$ has a neighbor $y$ in $G_2$ that is not in $G_1$, and $y$ is precolored. Since $y$ is precolored, it must be a vertex of $P$; and since $x$ is not precolored it is not a vertex of $P$. Since $G_2$ is as small as possible, the only neighbors of $v$ in $G_2$ are $w_i$ and $w_j$; so $x \in \{w_i, w_j\}$. By symmetry, we assume $x = w_i$. Since $C$ has no chord, by Claim 3, the edge $w_iy$ lies on $C$. So either $(x, y) = (w_k, w_1)$ or else $(x, y) = (w_{q+1}, w_q)$; by symmetry, we assume the former. See the left of Figure 11.12. The latter case is nearly identical.

Now $j \geqslant 3$, since otherwise the cycle $w_1 \cdots w_j v w_k$ has length at most 4, contradicting that $G$ has girth at least 5. But also $j \leqslant 3$, since otherwise $G_1$ has fewer vertices of $P$ than $G_2$ does, contrary to our assumption. Further, $q \geqslant 5$, since $G_1$ contains at least as many vertices of $P$ as $G_2$ does. If $q = 6$, then we are in an exceptional case, as claimed. (The other exceptional case comes, by symmetry, from our assumption above that $(x, y) = (w_k, w_1)$.) So assume instead that $q = 5$; see the right of Figure 11.12. Before coloring $G_1$ by induction, we first color $w_k$, to avoid the color on $w_1$, and $v$; if $|L(w_{k-1})| = 2$, then we also color $w_{k-1}$. Thus, when we $L$-color $G_1$ by induction, we have always precolored at most 6 vertices on its boundary, and they induce either a path or a cycle. $\diamondsuit$

**Claim 5.** *(a)* $G$ *has no path* $w_ixyw_j$ *where* $x, y \in$ *int*$(C)$ *and* $|L(w_i)| = 2$. *(b)* $G$ *has no path* $w_ixyw_j$ *where* $x, y \in$ *int*$(C)$ *and* $|L(w_i)| = 3$ *and* $j \in \{1, q\}$. *(See the left of Figure 11.13.)*

<u>Proof.</u> For (a), note that $w_i \notin V(P)$. So splitting into $G_1$ and $G_2$ along this path, and assuming $|V(G_2) \cap V(P)| \leqslant |V(G_1) \cap V(P)|$, gives $|V(G_2) \cap V(P)| \leqslant 3$. Now we $L$-color $G_1$ by induction. Then in $G_2$ we get at most 3 colored vertices from $P$ and at most 3 more colored vertices on the path. So we can $L$-color $G_2$ by induction. For (b) the proof is similar. Suppose we have such a path, say with $j = 1$. Now $w_i \notin V(P)$, since $|L(w_i)| = 3$; so all of $P$ lies in $G_1$ and only $w_1$ lies in $G_2$. Again we $L$-color $G_1$ by induction. Before $L$-coloring $G_2$ by induction, we may need to also color at most 1 neighbor of $w_i$ that initially has a list of size 2. But this is fine, since $G_2$ still has at most 5 precolored vertices. $\diamondsuit$

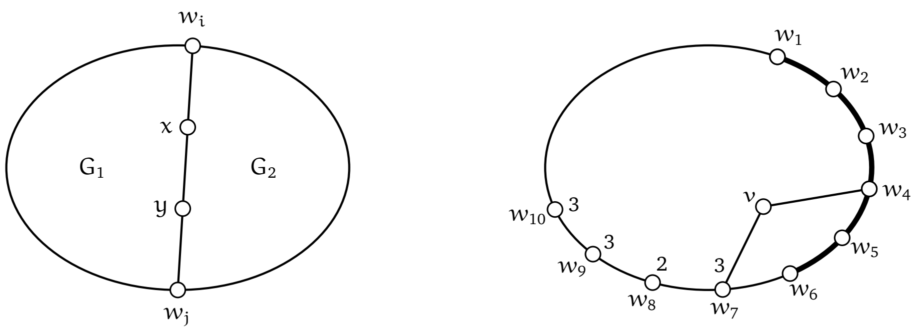



Figure 11.13: Left: The proof of Claim 5: forbidding many short separating paths in G. Right: The case in the proof of Claim 7 when $|L(w_{q+2})| = 2$ but $|L(w_{q+4})| = 3$.

**Claim 6.** G *has no separating cycle* C *of length* 5 *or* 6*.*

Proof. Suppose it does. We L-color C and its outside by induction. Now, also by induction, we L-color C and its inside, taking the colors on $V(C)$ from the first coloring. ♢

**Claim 7.** *(a)* $|L(w_{q+2})| = 2$*. (b) If* $k \geqslant q+4$*, then* $|L(w_{q+4})| = 2$*.*

Proof. (a) If $|L(w_{q+2})| = 3$, then we simply delete $w_q$ and delete its color from the list for each of its neighbors. By Claim 4, no neighbor of $w_q$ has another neighbor on C, so no vertex with a deleted color has a neighbor with list size less than 3. Thus, we finish by induction.

(b) Now assume $|L(w_{q+2})| = 2$. So $|L(w_{q+3})| = 3$; suppose also that $|L(w_{q+4})| = 3$. Now we simply color $w_{q+1}$ and $w_{q+2}$, delete them, and delete the color used on $w_i$ from the list of every neighbor of $w_i$, for each $i \in \{q+1, q+2\}$; finally, if possible we finish by induction on the resulting graph. By Claim 3, this is possible unless some neighbor of $w_{q+1}$ or $w_{q+2}$ has another neighbor on P. So by Claim 4, we succeed unless $q = 6$ and there is a path of the form $w_4vw_7$ (the other case listed in Claim 4 is excluded, since $w_{q+4}$ exists, so $k \notin \{q+1, q+2\}$); see the right of Figure 11.13. So assume this is the case. Now by Claim 6, the interior of the cycle $w_4w_5w_6w_7v$ is empty. So we can color $v$ to avoid the colors on $w_4$ and $w_7$, delete vertices $w_5$ and $w_6$ (in addition to vertices $w_7$ and $w_8$, which we already colored and deleted), and color the resulting graph by induction. ♢

**Claim 8.** G *has an* L*-coloring.*

Proof. Since $|L(w_{q+4})| = 2$, by Claim 7(b), we know $|L(w_{q+3})| = 3$, so we can color $w_{q+3}$ from $L(w_{q+3}) \setminus L(w_{q+4})$. Next, we color $w_{q+2}$ and $w_{q+1}$, delete $\{w_{q+1}, w_{q+2}, w_{q+3}\}$, and delete the color used on $w_i$ from the list of every neighbor of $w_i$, for each $i \in \{q+1, q+2, q+3\}$. Finally, we finish by induction on the resulting graph, as we explain shortly.

As in the proof of the previous claim, if $q = 6$ and G has a vertex $v$ adjacent to both $w_4$ and $w_7$, then we color $v$, and delete $w_5$ and $w_6$, before finishing by induction. Similarly, if $q = 6$,

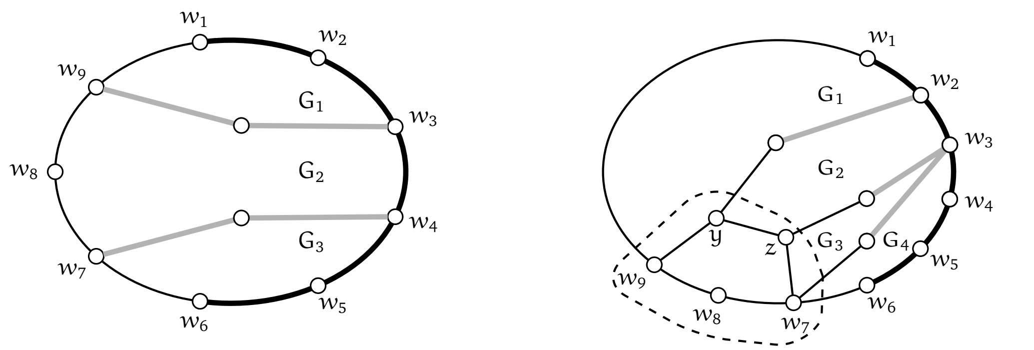



Figure 11.14: Left: The case $k = q + 3$ in the proof of Claim 8. Right: The more general case in the proof of Claim 8.

$k = q + 3$, and G has a vertex $v'$ adjacent to $w_3$ and $w_k$, then before finishing by induction we color $v'$ and delete $w_1$ and $w_2$; see the left of Figure 11.14.

There may also be a path $w_{q+3}yzw_{q+1}$, but in this case Claim 6 shows that $y, z$ are unique and that the interior of cycle $w_{q+1}w_{q+2}w_{q+3}yz$ is empty; see the right of Figure 11.14. In this case, we color and delete $y$ and $z$, and delete the color of each from the lists for all of its neighbors. (In this case, $v$ and/or $v'$ may also exist, as above.) We also color, but do not delete, every vertex that is adjacent to two other vertices that are colored (and possibly deleted). Call the resulting graph $G'$.

We must verify that in $G'$ the set of vertices with lists of size 2 is independent, and none of them is adjacent to a precolored vertex. Further, we must show that the precolored vertices split the graph into parts, where every vertex with a list of size 2 lies on the outer face boundary of each part that contains it, and that we can color each part of $G'$ by induction. The condition on the set of vertices with lists of size 2 holds by Claims 3 and 4 and the fact that every vertex with 2 colored neighbors gets colored itself. (It is not possible that $y$ and $w_{q+1}$ have neighbors that are adjacent, say $x_1$ and $x_2$; if so, then $w_{q+1}w_{q+2}w_{q+3}yx_1x_2$ is a 6-cycle that separates $z$ from $w_q$, which contradicts Claim 6.)

The set of precolored vertices on the boundary of the outer face of $G'$ includes all those that are precolored in G. But what other vertices might be precolored in $G'$? Possibilities might appear to include vertices in $N(w_i) \cap N(w_j)$ where $i, j \in \{1, \ldots, 9\}$. But recall that Claim 4 excludes all of these except for $\{i, j\} = \{4, 7\}$ and $\{i, j\} = \{3, k\}$; in this case the vertex is $v$ or $v'$, as described above. Another possibility is a vertex in $N(w_i) \cap (N(y) \cup N(z))$. These cases are not excluded by Claim 5, but it does imply that $i \notin \{1, q\}$. As a result, no path among the precolored vertices contains more than 6 vertices; see Figure 11.14. Thus, we can split $G'$ into parts and color each part by induction. ♢

Claim 8 completes the proof of Theorem 11.13. □

## 11.5 $(I, F)$-Coloring of Planar Graphs with Girth at least 5

An $(I, F)$-*coloring* of a graph $G$ is a partition of $V(G)$ into sets $I$ and $F$ such that $I$ is an independent set and $G[F]$ is a forest. In this section we prove the following result. $(I, F)$-coloring

**Theorem 11.14.** *If $G$ is a planar graph with girth at least 5, then $G$ has an $(I, F)$-coloring.*

For context, it is helpful to note that Theorem 11.14, like Theorem 11.13, yields a very short proof of Grötzsch's Theorem. Again, by the Folding Lemma, we assume $G$ has girth at least 5, so by Theorem 11.14 has an $(I, F)$-coloring. Now we color $I$ with 1 and we color $F$ with 2 and 3. This proves Grötzsch's Theorem.

It is easy to prove Theorem 11.14 for planar graphs with girth at least 6. By Corollary 1.7, such a graph $G$ contains a vertex $v$ of degree at most 2. By induction $G-v$ has an $(I, F)$-coloring. If all neighbors of $v$ are colored with $F$, then we color $v$ with $I$. Otherwise, we color $v$ with $F$; this cannot create cycles since $d(v) \leqslant 2$, so $v$ has at most one neighbor colored with $F$. Thus, the most interesting case is when $G$ has girth 5.

Rather than proving Theorem 11.14 directly, we actually prove the following more technical result, which is designed to facilitate proof by induction.

**Theorem 11.15.** *Let $G$ be a plane graph of girth at least 5. Let $I_0$ and $S$ be disjoint, possibly empty, vertex sets on the boundary $C$ of the outer face. (Here all vertices of $I_0$ will be colored with $I$, and $S$ is a set of at most two "special" vertices.) Now $G$ has an $(I, F)$-coloring that extends the precoloring of $I_0$ and $S$, whenever $I_0$ and $S$ satisfy the following 4 conditions:*

1. $|S| \leqslant 2$ *and either*
   - *(a)* $|S| = 0$
   - *(b)* $|S| = 1$ *and its vertex is colored with $F$; or*
   - *(c)* $|S| = 2$ *and $S$ contains adjacent vertices, both colored with $F$; or*
   - *(d)* $|S| = 1$ *and its vertex is colored with $I$.*
2. *All vertices of $I_0$ are colored with $I$.*
3. *All vertices colored with $I$ form an independent set.*
4. *$G$ has no path with three vertices, each not in $S$, such that each has a neighbor in $I_0$. Such a path is called a* bad 3-path. *bad 3-path*

(♣) *Furthermore, if 1(d) applies and $S = \{s\}$, then the following result holds, too. For each $t \in N(S) \cap N(I_0)$ on $C$, we can extend the precoloring to an $(I, F)$-coloring such that there is no path colored $F$ from $t$ to any other neighbor of $s$.*

Perhaps the hypothesis above that is easiest to explain is 4. The reason we must forbid bad 3-paths is that if we do allow them, then the theorem becomes false. Specifically, consider a 5-cycle such that 3 consecutive vertices each have an off-cycle neighbor colored I and the remaining two vertices are colored F, as in (1c). Such a (precolored) graph has no extension to an $(I, F)$-coloring.

Before we begin the proof, we provide intuition about why certain parts of the theorem are helpful as we proceed by induction. We think of S as allowing us to require that up to 2 vertices are colored with F. However, on occasion we forgo this freedom and instead require one additional vertex $v$ (beyond those in $I_0$) to be colored I. The point is that now $v$ is not subject to the constraints imposed on vertices in $I_0$ by condition 4 and by (♣).

As usual, we can use our precolored vertices to handle the case when C has a chord or, more generally, a short path the vertices of which are a vertex cut. We also want to handle the case that G has a separating 5-cycle. This is where (♣) is most useful. Suppose we have a separating 5-cycle $\widehat{C}$ and let $G_{in}$ and $G_{out}$ denote the subgraphs of G induced by the vertices of $\widehat{C}$ and the vertices inside and outside, respectively.

We would like to find an $(I, F)$-coloring $\varphi_{out}$ of $G_{out}$ using the given precoloring $\varphi_0$, and then find a precoloring $\varphi_{in}$ of $G_{in}$ that agrees on $\widehat{C}$ with $\varphi_{out}$. Ideally, $\varphi_{out} \cup \varphi_{in}$ will be the desired $(I, F)$-coloring of G. But a possible difficulty is if vertices $x, y \in V(\widehat{C})$ are both colored F in $\varphi_{out}$, and $\varphi_{out}$ has an $x, y$-path $P_1$ colored F that does not lie on $\widehat{C}$. It is possible that $\varphi_{in}$ will also contain an $x, y$-path $P_2$ colored F. But now $P_1 \cup P_2$ is a cycle colored F, contrary to what we aimed for. So the point of (♣) is to forbid the appearance of $P_2$ in $\varphi_{in}$.

In our main induction step, we pick a vertex $w$ on C. We color $w$ with F, add all but one neighbor of $w$ into $I_0$, delete $w$, and aim to finish by induction. We can assume that this approach fails, so we must have created a bad 3-path. But where does this bad 3-path appear? Ultimately, we show that our bad 3-path can be extended to a short path P joining two vertices in $I_0$. But now P is exactly the type of short path forbidden in the first sentence of the previous paragraph. This contradiction finishes the proof.

*Proof.* Suppose the theorem is false and choose G to minimize its number of edges, $\|G\|$. Subject to this value of $\|G\|$, choose G to maximize $|I_0|$. More formally, among all counterexamples we choose G to lexicographically minimize the ordered pair $(\|G\|, -|I_0|)$. The case $\|G\| \leqslant 1$ is trivial. For convenience, when some graph $G'$ is smaller than G in this ordering of graphs, we typically say that we color $G'$ "by induction".

We begin with the following easy observation.

**Claim 1.** *Every vertex of* $I_0$ *has degree* 1.

Proof. Suppose, to the contrary, that $x \in I_0$ and $d(x) = k$ with $k \geqslant 2$. Now we delete $x$ and add vertices $x_1, \ldots, x_k$, with each $x_i \in I_0$ and each $x_i$ inheriting a distinct neighbor of $x$ in G. This contradicts the maximality of $|I_0|$. ♢

**Claim 2.** G *is connected, and every vertex not in* $I_0$ *has degree at least* 2.

Proof. If $G$ has an isolated vertex $v$, then we color $G - v$ by induction and it is easy to extend the coloring to $v$. So we assume $\delta(G) \geqslant 1$. If $G$ is disconnected, then we color each component separately by induction. Suppose $G$ has a vertex $v$ with $d(v) = 1$. If $v$ is either precolored with $F$ or not precolored, then we color $G - v$ by induction and finish, if needed, by coloring $v$ with $F$. So suppose instead that $v$ is precolored with $I$. If $v \in I_0$, then we are done. So assume $v \in S$ in 1(d). Now we let $G' := G - v$, with the neighbor $w$ of $v$ (in $G$) precolored with $F$ in 1(b). We color $G'$ by induction; together with $F$ on $v$, this gives the desired coloring of $G$. $\diamondsuit$

**Claim 3.** $G$ *has a block* $B$ *which has a cycle.*

Proof. If $G$ is a tree, then we color with $F$ every vertex not precolored with $I$. $\diamondsuit$

We will often split the graph $G$ into two subgraphs $G_1$ and $G_2$. We first color $G_1$ by induction, typically taking as $I_0$ and $S$ the restrictions of those sets (in $G$) to $V(G_1)$. We next color $G_2$ by induction, typically taking as $I_0$ and $S$ a combination of the restrictions of those sets in $G$ to $V(G_2)$ together with the coloring of $V(G_1) \cap V(G_2)$ determined by the solution for $G_1$. We will provide more details below when needed, to clarify any possible points of ambiguity.

**Claim 4.** *If* $v$ *is a cut-vertex of* $G$*, then* $G - v$ *has exactly two components, and one of these is a single vertex of* $I_0$.

Proof. Suppose the contrary. So $G = G_1 \cup G_2$ where $G_1, G_2$ are subgraphs with $|G_1| \geqslant 3$ and $|G_2| \geqslant 3$ and $V(G_1) \cap V(G_2) = \{v\}$. If $v$ is precolored, with either $v \in I_0$ or $v \in S$, then we inductively color $G_1$ and $G_2$; in each case, we take as $I_0$ and $S$ the restrictions to $V(G_i)$ of those sets in $G$. Assume instead that $v$ is not precolored. By symmetry, we assume that $G_2$ has no special vertex. Inductively, we first get a coloring $\varphi_1$ of $G_1$ with a new vertex $w$ adjacent to $v$ and $w \in I_0$ if $G_2$ contains such a neighbor $w$ of $v$. Next, we inductively get a coloring $\varphi_2$ of $G_2$ with $v \in S$, where $v$ is precolored to agree with its color in $\varphi_1$. Note that (♣) is satisfied in both $G_1$ and $G_2$. Thus, $\varphi_1 \cup \varphi_2$ is the desired coloring of $G$. $\diamondsuit$

By Claim 3, $G$ has a block $B$ with an outer cycle $C$; we denote the vertices of $C$ by $w_1, w_2, \cdots, w_k$. By Claim 4, each vertex of $G$ that is not in $B$ is a degree 1 vertex in $I_0$ with a neighbor on $C$.

**Claim 5.** $C$ *has no chord.*

The proof is like that of Claim 4, but requires more details to show (♣) holds for $G$.

Proof. Assume instead that $C$ has a chord $w_i w_j$. By symmetry, we assume $i < j$, and that $G_2$ contains no vertices of $S$, except for possibly in $\{w_i, w_j\}$. Now $w_i w_j$ separates $G$ into connected subgraphs $G_1, G_2$ with $V(G_1) \cap V(G_2) = \{w_i, w_j\}$. By induction, we first get a coloring $\varphi_1$ of $G_1$ (plus one or two new vertices in $I_0$ adjacent to $w_i$ and/or $w_j$ if $G_2$ contains such vertices); as usual, $I_0$ and $S$ are now the restrictions to $V(G_1)$ of those sets in $G$. By induction, we next get a coloring $\varphi_2$ of $G_2$, with $w_i$ and/or $w_j$ being special vertices to match their colors in $\varphi_1$.

(We call one of these vertices special if it is colored I in $\varphi_1$; otherwise, we call both special since they are colored F.) Note that adding special vertices to $G_2$ cannot create a bad 3-path.

We must also check that (♣) holds in G. So suppose that 1(d) holds; that is, $S = \{s\}$ and $v$ is colored with I. If $\{s,t\} = \{w_i, w_j\}$, then (♣) holds in both $G_1$ and $G_2$, so it also holds in G. So assume not. Now (♣) holds for $G_1$. Either one or both of $w_i, w_j$ are colored F. But since $w_iw_j \in E(G)$, every two vertices in $G_1$ are connected in G by a path colored F if and only if they are connected by such a path in $G_1$. (A hypothetical such path in G that includes vertices in $V(G_2) \setminus \{w_i, w_j\}$ could be "shortcut" using edge $w_iw_j$ to give such a path in $G_1$.) Thus, (♣) also holds for G, as desired. ♢

**Claim 6.** *Every vertex $w$ in $V(G) \setminus (I_0 \cup S)$ has $d_G(w) \geqslant 3$ unless S consists of a single vertex s that is colored I, vertex t exists, and $w$ is a neighbor of s.*

Proof. Fix a vertex $w \in V(G) \setminus (I_0 \cup S)$. By Claim 2, we have $d_G(w) \geqslant 2$. If $d_G(w) = 2$, then we inductively get a coloring $\varphi'$ of $G - w$. To extend the coloring to G, we color $w$ with I if both its neighbors are colored F in $\varphi'$; otherwise, we color $w$ with F. This cannot create a cycle colored F. So the only possible problem is if s and t exist, and we create a path colored F from t to $w$, which is a neighbor of s. ♢

**Claim 7.** $I_0$ *is nonempty.*

Proof. Suppose instead that $I_0 = \emptyset$. Recall that $|S| \leqslant 2$ and if $|S| = 2$, then its two vertices are adjacent. Also $|C| \geqslant 5$, and C has no chords. Thus, there exists $x \in C$ that is not in $N[S]$. By adding $x$ to $I_0$, we contradict the minimality of our choice of G. ♢

**Claim 8.** C *has length at least* 6*, and* G *has no separating* 5*-cycle.*

Proof. Suppose that $|C| = 5$. By Claim 7, we know $I_0 \neq \emptyset$. And by the definition of $I_0$, all its vertices lie on the outer face. So if G has some vertices inside C, then we are in the case of a separating 5-cycle, which we handle below. Assume instead that G does not have vertices inside C. Now G consists of a 5-cycle, in which some vertices are adjacent to 1-vertices outside of C, and each such 1-vertex lies in $I_0$. Also possibly 1 or 2 adjacent vertices on C lie in S and are colored F, or one vertex lies in S and is colored I.

It is straightforward to verify that in each case, the precoloring can be extended, precisely because G has no bad 3-path. (For example, color the vertices of C in order around the cycle, coloring each vertex $w_i$ with color I when we reach $w_i$ if that yields a valid partial coloring.) When we are not in case 1(d), it suffices to find some vertex on C that can be colored with I. Case 1(d) requires a bit more care, but is not difficult.

Now suppose instead that G has a separating 5-cycle $C'$ with vertices $x_1, x_2, x_3, x_4, x_5$. Let $G_{out}$ and $G_{in}$ denote the subgraphs of G induced by the vertices of $C'$ and, respectively, those outside of or those inside of $C'$. We will first color $G_{out}$, by induction, and thereafter color $G_{in}$, also by induction, so that the colorings agree on $C'$. But the details require care to ensure that in G the union of these colorings satisfies (♣). The key idea is to prove that (♠) no two vertices in $C'$ are joined by a path in $G_{in}$ colored F unless those vertices are already joined by such a path in $C'$. This will ensure that no two vertices in $G_{out}$ are joined in G by a path colored F that crosses (at least twice) cycle $C'$.

We first color $G_{out}$ by induction, with S, $I_0$, and (possibly) t as given in (♣). Recall that $I_0 \neq \emptyset$, so $G_{in}$ is smaller than G; also G has vertices inside C, so $G_{out}$ is smaller than G. Up to rotation (relabeling $x_i$'s), the coloring $\varphi_{out}$ of $G_{out}$ restricted to $C'$ is either (a) $\varphi_{out}(x_1) = I$ and $\varphi_{out}(x_i) = F$ for all $i \in \{2, 3, 4, 5\}$ or else (b) $\varphi_{out}(x_i) = I$ for all $i \in \{1, 3\}$ and $\varphi_{out}(x_j) = F$ for all $j \in \{2, 4, 5\}$.

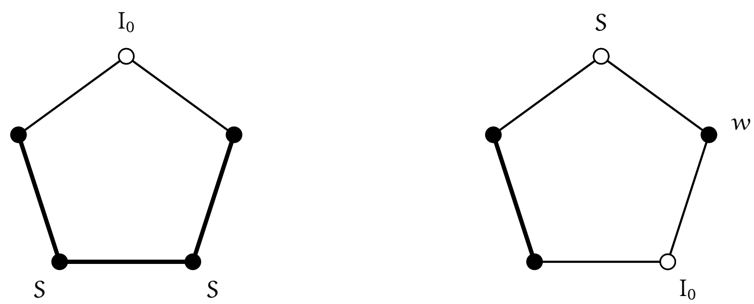



Figure 11.15: The vertices of a separating 5-cycle $C'$ are, clockwise from top, $x_1, \cdots, x_5$. Up to rotation, $C'$ has two possible colorings that it inherits from the coloring $\varphi_{out}$ of $G_{out}$. (a) $\varphi_{out}(x_1) = I$ and $\varphi_{out}(x_i) = F$ for all $i \in \{2, 3, 4, 5\}$ and (b) $\varphi_{out}(x_1) = \varphi_{out}(x_3) = I$ and $\varphi_{out}(x_i) = F$ for all $i \in \{2, 4, 5\}$.

In case (a), when coloring $G_{in}$, we let $I_0 = \{x_1\}$ and $S = \{x_3, x_4\}$, with both vertices in S colored F; see the left of Figure 11.15. Note that this precoloring will enforce that $\varphi_{in}$ agrees on $C'$ with $\varphi_{out}$. Observe that if we are in Case 1(d) for G, then we must have $w \notin C'$, since $x_2$ and $x_5$ are joined by a path colored F. But now $\varphi_{out}$ has no path colored F from $w$ to $V(C') \setminus \{x_1\}$, since $C'$ has a path colored F from $x_2$ to $x_5$. Thus, we do not require that $\varphi_{out}$ satisfies (♠). Nonetheless, $\varphi_{out} \cup \varphi_{in}$ is the desired $(I, F)$-coloring of G.

Now instead we consider case (b) above; see the right of Figure 11.15. When we color $G_{in}$ by induction, we let $S := \{x_1\}$, let $I_0 := \{x_3\}$, and let $w = x_2$. First we check that any extension $\varphi_{in}$ of this precoloring must agree on $C'$ with $\varphi_{out}$. Second, because (♣) holds for $G_{in}$, we know that $\varphi_{in}$ does not contain an $x_2, x_5$-path colored F. So also, $\varphi_{in}$ does not contain such an $x_2, x_4$-path, since it could be extended via edge $x_4x_5$. Thus, $\varphi_{in}$ satisfies (♠). As a result, $\varphi_{out} \cup \varphi_{in}$ is the desired coloring of G. ♢

**Claim 9.** $S \neq \emptyset$ *and* $d(x) \geqslant 2$ *for all* $x \in S$.

Proof. Suppose not; by Claim 2 we must have $S = \emptyset$. By Claim 7, $I_0$ is nonempty, and Claim 1 gives $d(x) = 1$ for all $x \in I_0$. We thus choose some $y \in I_0$, delete $y$, and add its neighbor $y'$ to S, with $y'$ precolored F. We color the resulting graph by induction, and extend the coloring to G by using I on $x$. ♢

I/F-special
peripheral

If a vertex $v$ in $S$ is precolored $I$ (resp. $F$), then $v$ is $I$-*special* (resp. $F$-*special*). A path $P$ in $G$ is *peripheral* if it has no edge in the interior of $C$; otherwise $P$ is *non-peripheral*.

**Claim 10.** $G$ *has no non-peripheral path with at most* 6 *vertices that starts and ends in* $I_0$.

Proof. Suppose to the contrary that such a path $P$ exists, with $P = x_1w_ix_2x_3w_jx_4$ with $x_1, x_4 \in I_0$. (If $P$ has 5 vertices, the proof is similar. Since $C$ is chordless, by Claim 5, $P$ cannot have 4 vertices.) Now $P$ divides $G$ into subgraphs $G_1, G_2$ with $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = V(P)$; see Figure 11.16. By symmetry, assume that $S \subseteq V(G_1)$. Now we first color $G_1$ by induction (keeping $S$ unchanged and restricting $I_0$ to $V(G_1)$); call this coloring $\varphi_1$. Afterward, we color $G_2$ by induction, with the precoloring as follows.

Note that $\varphi_1(w_i) = \varphi_1(w_j) = F$ and $\varphi_1$ colors at most one of $x_2, x_3$ with $I$. If $\varphi_1(x_2) = \varphi_1(x_3) = F$, then when coloring $G_2$, we let $S = \{x_2, x_3\}$, with both vertices precolored $F$. If we are in Case 1(d) for $G$, then (♣) must hold for $G$ because it holds for $G_1$. So assume instead that $x_2$ or $x_3$ is colored $I$ by $\varphi_1$. We handle the case where $\varphi_1(x_2) = I$ and $\varphi_1(x_3) = F$, but the other case is nearly identical. For $G_2$, we let $S = \{x_2\}$, with $x_2$ I-special, and we let $t = w_i$. Again, (♣) holds for $G$ because no path colored $F$ is contained in $G_2$ with both endpoints in $V(P)$, except for edge $x_3w_j$, which is also contained in $G_1$. ♢

Since $S \neq \emptyset$ by Claim 9, we choose the notation so that $w_1 \in S$ and $w_2 \notin S$. So if $w_1$ is F-special, then possibly also $w_k$ is F-special. If $w_1$ is I-special and $t$ exists (as in (♣)), then we must have $t \in \{w_2, w_k\}$.

**Claim 11.** *Let* $w_i$ *be a vertex on* $C$ *having a neighbor in* $I_0$ *such that* $w_{i-1}$ *has no neighbor in* $I_0$. *Now one of the following* 3 *cases holds:*

- *(c₁)* $i = 2$ *and either* $w_1$ *is* $F$-*special or else* $(w_1, w_2) = (s, t)$; *or*
- *(c₂)* $i = 3$ *and* $w_1$ *is* $I$-*special; or*
- *(c₃)* $i \geqslant 6$ *and each of* $w_{i-3}, w_{i-4}$ *has a neighbor in* $I_0$ *and none of* $w_{i-1}, w_{i-2}$ *has a neighbor in* $Q$.

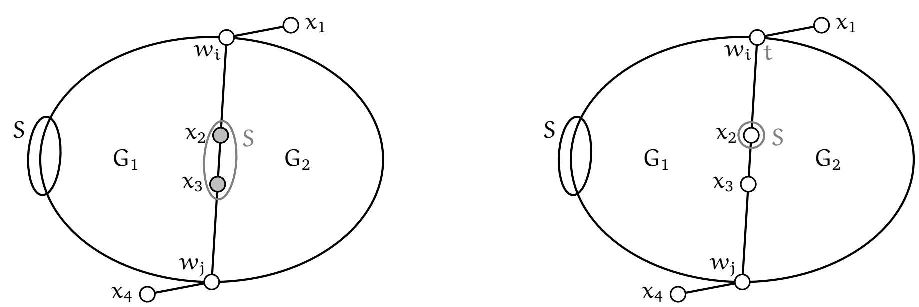



Figure 11.16: The proof of Claim 10. Left: When $\varphi_1(x_2) = \varphi_1(x_3) = F$, we take $x_2$ and $x_3$ to be F-special when coloring $G_2$. Right: When $\varphi_1(x_2) = I$, we take $x_2$ to be I-special, with $t := w_i$, when coloring $G_2$.

<u>Proof.</u> If possible, we give $w_{i-1}$ color I and add it to $I_0$, which contradicts the maximality of $I_0$. So we assume this attempt fails. First suppose $i \geqslant 4$. (For example, perhaps $i = 6$, as in the top right of Figure 11.17.) By hypothesis, $w_{i-1}$ has no neighbor in $I_0$. And by our choice of notation, $w_{i-2} \notin S$. So the reason we fail when trying to add $w_{i-1}$ to $I_0$ is that doing so creates a bad 3-path P. By Claim 10, this path P must be peripheral (recall that $w_i$ has a neighbor in $I_0$). And P cannot start with $w_i$, since $w_i$ already had a neighbor in $I_0$. So P must start with $w_{i-2}$. Thus $w_{i-3}$ and $w_{i-4}$ each have a neighbor in $I_0$. By hypothesis, $w_{i-1}$ has no such neighbor. And since P just became bad when adding $w_{i-1}$ to $I_0$, neither does $w_{i-2}$ have such a neighbor. That $i \geqslant 6$ holds because $w_1 \in S$, so $w_1 \notin V(P)$.

The cases when $i \leqslant 3$ are similar, but now we must consider the role of S; recall that $w_1 \in S$ and $S \subset \{w_1, w_k\}$. If $i = 3$, then $w_2 w_1 w_k$ cannot become a bad 3-path, by definition, because $w_1 \in S$. So we must be unable to add $w_2$ to $I_0$ for some other reason. By hypothesis, $w_2$ has no neighbor in $I_0$. Thus, $w_2$ must have a neighbor that is I-special, and this must be $w_1$. Suppose $i = 2$ and $w_1$ is not F-special. Since $w_1 \in S$, we know $w_1$ is I-special. (Consider the top right of Figure 11.17.)

Now we try to delete the neighbor $y$ of $w_2$ in $I_0$, and finish by induction. This works unless $w_1 = s$ and $w_2 = t$; now after deleting $y$ we may no longer have $w_2 \in N(S) \cap N(I_0)$. Suppose $i = 1$. By assumption $w_1$ has a neighbor $y$ in $I_0$. So $w_1$ must be F-special. But now we can simply delete $y$, and finish by induction. Thus, we conclude that $i \neq 1$. ♢

The previous claim yields a precise description of the sets $I_0$ and S.

**Claim 12.** *The following 3 statements hold (see Figure 11.17):*

1. *If $w_1$ is I-special, then the vertices on C with a neighbor in $I_0$ are precisely $w_3, w_4, w_7, w_8, \ldots, w_{k-2}, w_{k-1}$ (if no t exists) or $w_2, w_3, w_6, w_7, \ldots, w_{k-2}, w_{k-1}$ (if $t = w_2$).*

2. *If $w_1, w_k$ are F-special, then the vertices on C with a neighbor in $I_0$ are precisely $w_2, w_3, w_6, w_7, \ldots, w_{k-2}, w_{k-1}$.*

3. *If $w_1$ is F-special but $w_k$ is not F-special, then the vertices on C with a neighbor in $I_0$ are precisely $w_2, w_3, w_6, w_7, \ldots, w_{k-1}, w_k$.*

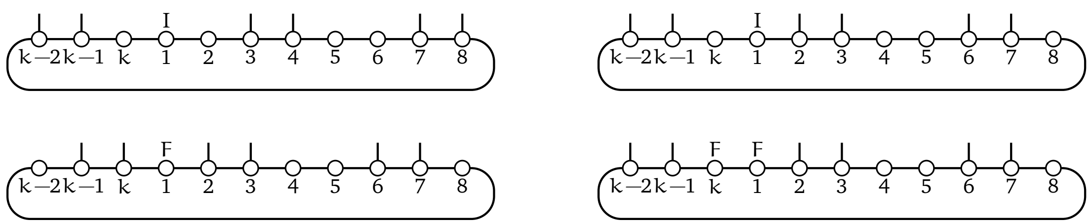


Figure 11.17: 4 possibilities for which vertices on C near $w_1$ have neighbors in $I_0$; every pendent edge leads to a vertex in $I_0$, although these vertices are not drawn explicitly in the figure. Clockwise from top left: (a) $w_1$ is I-special and $w_1 = s$, but no neighbor t of $w_1$ is specified; (b) $w_1$ is I-special and $w_2 = t$; (c) $w_1$ and $w_k$ are both F-special; and (d) $w_1$ (but not $w_k$) is F-special.

We will not need quite all of this detailed structure near $S$ that is provided by Claim 12, but mainly we will use the fact that far away from $S$ the vertices with and without neighbors in $I_0$ alternate (2 "on", then 2 "off") around the cycle.

Proof. We prove the statement for $w_i$ by induction on $i$; the base case is either $(c_1)$ or $(c_2)$ in Claim 11, and the induction step is $(c_3)$. The full details of the proof are a bit tedious, so we just sketch the ideas. Note that by $(c_3)$, if there exists $i$ such that $w_i, w_{i-1}, w_{i-2}$ each have no neighbors in $I_0$, then this is true for all $j \geqslant i$. Similarly, this is true if $w_i, w_{i-2}, w_{i-3}$ each have no neighbors in $I_0$ (or $w_i, w_{i-1}, w_{i-3}$).

Note also that by symmetry, we could index the vertices the other way around the cycle, except for a wrinkle when $(w_1, w_2) = (s, t)$. So the fact that $w_i$ has a neighbor in $I_0$ for some $i \in \{2, 3\}$ implies that vertices with and without neighbors in $I_0$ alternate (2 "on", then 2 "off") around the cycle, in each direction. In the case when $(w_1, w_2) = (s, t)$, going around the cycle in the other direction is akin to the case when $w_1$ is I-special. $\diamondsuit$

We can now complete the proof of the theorem. We first outline the argument and then provide more details. We choose a vertex $w_i$ on $C$ such that $w_i, w_{i+1}$ each have no neighbors in $I_0$, but $w_{i-1}, w_{i+2}$ each have neighbors in $I_0$, and also $w_i, w_{i+1}$ have no neighbors in $S$. (For example, let $i := 5$ in the top left of Figure 11.17 or let $i := 4$ in any other part of that figure.) We will color $w_i$ with $F$, then delete $w_i$ and add all neighbors of $w_i$ to $I_0$, except for $w_{i-1}$. For this, we let $J := N_G(w_i) \setminus \{w_{i-1}\}$. If we can color the resulting smaller graph $G'$ by induction, then we are done since at most one neighbor of $w_i$ can be colored $F$, namely $w_{i-1}$, so no cycle colored $F$ can include $w_i$. So we assume that $G'$ does not satisfy the induction hypothesis. More precisely, $G'$ must contain some bad 3-path $P$. We consider which of the vertices in $P$ have neighbors in $J$ that were added to $I_0$.

$J$, $G'$

It is not possible that none of the vertices in $P$ have neighbors in $J$, since then $P$ would have been a bad 3-path in $G$, a contradiction. It is also not possible that all 3 vertices of $P$ have neighbors in $J$, as follows. In this case, each two successive vertices of $P$, together with their neighbors in $J$, along with $w_i$, form a 5-cycle. By Claim 8, each such 5-cycle has no vertices in its interior. But then the vertex $x$ in $J$ adjacent to the middle vertex of $P$ lies on two 5-faces, so $d_G(x) = 2$, which contradicts Claim 6. Thus, precisely 1 or 2 vertices of $P$ have neighbors in $J$. We consider these cases below. In each case we find a non-peripheral path on at most 7 vertices, and conclude similarly to our proof of Claim 10.

We denote the bad 3-path $P$ by $xyz$, but if some vertex in $\{x, y, z\}$ is on $C$, then we will call it $w_\ell$ for some subscript $\ell$. Our plan is to find a non-peripheral path $P'$ with at most 7 vertices, which we call the separating path derived from $P$; we then split $G$ into two parts by $P'$, say $G_1$ and $G_2$, and color each part $G_j$ by induction, similar to the proof of Claim 10. We assume that $G_1$ contains $S$ and call $G_1$ the *major part*, and we call $G_2$ the *minor part*. The key point is that, in order to color each part $G_j$ by induction, that part must not contain a bad path. So, if we have any choice about our derived path $P'$, we choose it to ensure that neither the major nor minor part contains a bad 3-path.

major part
minor part

So what are the possibilities for which vertices of $P$ have neighbors in $J$? We might have $P$

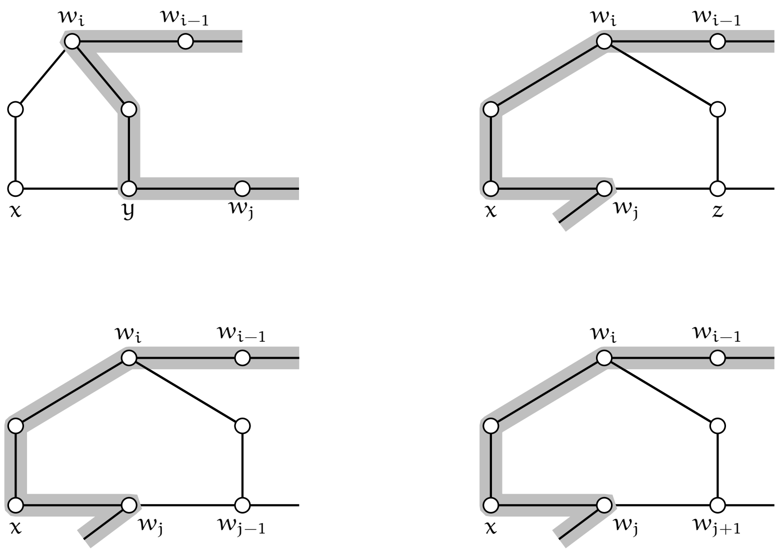



Figure 11.18: 4 possibilities for a bad 3-path $P$ and its derived separating 7-path, clockwise from top left: (a) $xyw_j$, (b) $xw_jz$, (c) $xw_jw_{j+1}$, and (d) $xw_jw_{j-1}$.

as (a) $xyw_j$, (b) $xw_jz$, (c) $xw_jw_{j+1}$, or (d) $xw_jw_{j-1}$; in each case, we assume that the $w_j$'s already have neighbors in $I_0$ and that the remaining vertices of the 3-path (called $x$, $y$, or $z$) have neighbors in $J$ that we plan to add to $I_0$. See Figure 11.18. We call the bad 3-path $P$ *very bad* if the major part must contain all vertices of $P$. very bad

We first assume that $G$ has no 3-path that is very bad, and show how to finish. Afterward, we will explain why we can assume we are in this case. If the minor part contains all vertices of the bad 3-path (in $G$), this will actually not be a problem when handling $G_2$ by induction, since we can use $S$ in $G_2$ to force the desired precoloring of $P$ without $P$ being a bad 3-path. We will provide more details below when we handle this case.

Suppose that $G$ has no very bad 3-path. Let $P$ be a bad 3-path such that the resulting minor part is relatively maximal; this ensures that the major part has no bad 3-path (if it did, then $P$ would be a very bad 3-path, contradicting our assumption). For example, possibly the derived separating 7-path $P'$ is $v_1w_{i-1}w_iuxw_jv_2$, where $j > i$ and $v_1, v_2 \in I_0$. Now $S$ lies on one side of $P'$ (since $w_1 \in S$) and $z$ lies on the other. Thus, the major part $G_1$ does not contain $z$, so $G_1$ can be colored by induction since it has no bad 3-path. It is also possible that $G$ has another bad 3-path whose minor part is relatively maximal; in this case, the derived separating 7-path $P''$ is $v_1w_{i-1}w_iu'x'w_mv_3$, where $m < i$ and $v_3 \in I_0$. If $P''$ exists, then when forming $G_1$ we also delete the minor part for $P''$ (excluding the vertices of $P''$ itself). When we color $G_1$ by induction, we use in place of $I_0$ the set $(I_0 \cup J) \cap V(G_1)$, keeping $S, t$ as defined originally.

Now we color the minor part (or two minor parts) by induction. We just consider the case of one minor part, since the case of two is nearly identical. When $G_1$ is colored by induction, call the coloring $\varphi$; now the derived 7-path $P'$, say $v_1 w_{i-1} w_i u x w_j v_2$, is also colored. In particular, $\varphi$ colors $w_{i-1}, w_i, x, w_j$ with color $F$ (and $u$ with color $I$). When we color $G_2$ by induction, we uncolor the vertices of $P'$ and we let $S := \{u\}$ with $u$ being I-special. This will force the desired coloring in $G_2$ on the path $P'$, but $G_2$ will not have any bad 3-path. We restrict $I_0$ to $V(G_2)$, let $t := w_i$, and add a new neighbor of $w_i$ to the set $I_0$. The coloring $\varphi_2$ of $G_2$ colors $x$ and $w_j$ with $F$, but because (♣) holds in both $G_1$ and $G_2$, the vertices colored $F$ must induce a forest in $G$ and further (♣) holds in $G$.

Finally, we consider the case that $G$ has a very bad path $P$. We may assume that $P$ has the form $xyw_j$ or $xw_jw_{j+1}$ with $j > i$ (or $xyw_m$ or $xw_mw_{m-1}$ with $m < i$) when we add all of $N_G(w_i) \setminus \{w_{i-1}\}$ to $I_0$. Now we repeat the argument above with $w_{j-2}$ rather than $w_i$. By Claim 12, we know that $w_{j-2}$ has no neighbor in $I_0$, but $w_{j-3}$ does have a neighbor in $I_0$. We assume that adding $N_G(w_{j-2}) \setminus \{w_{j-3}\}$ to $I_0$ again creates a bad 3-path in the resulting major part $G_1$. But now $w_{j-3}$ and its neighbor in $I_0$ combine with vertices in this new bad 3-path to give a new derived separating 7-path.

The key observation is that the resulting major part is bigger than before (we can choose this new derived separating 7-path so that it does not cross the old one). By repeatedly applying this argument, we eventually reach a contradiction (since $G$ is finite). Thus, we can assume $G$ has no very bad 3-path, so the argument above finishes the proof. □

## Notes

We have already discussed list-coloring at length in the Notes of Chapter 1, so here we just summarize. List-coloring was introduced in the late 1970s by Vizing [402] and by Erdős, Rubin, and Taylor [152]. The latter paper conjectured that (a) every planar graph is 5-choosable[5] and (b) there exist planar graphs that are not 4-choosable. Voigt [403] proved (b) in 1993. And Thomassen [376] proved (a) in 1994. It is the result of [376] that we presented in Lemma 11.2. The proof is striking for its simplicity and elegance and has been often called [5, Chapter 30] a "Proof from The Book".[6]

The paper [376] has been highly influential. Most importantly, for our purposes here, it inspired all of the other proofs in this chapter. The result itself is also surprisingly robust. For example, we can allow various "alterations" to a planar graph and still conclude that the resulting graph is 5-choosable. This is true for graphs with 2 crossings [136], graphs with arbitrarily many crossings that are far apart [134], and locally planar[7] graphs [112]. We can also prove strengthened 5-choosability results for planar graphs with some precolored vertices

[5]This was also conjectured, privately, by Vizing in 1975; see [228, p. 19].

[6]The authors of [5] write: "Paul Erdős liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems, following the dictum of G.H. Hardy that there is no permanent place for ugly mathematics. Erdős also said that you need not believe in God but, as a mathematician, you should believe in The Book."

[7]Formally, for every surface $S$ there exists a constant $g_S$ such that if every noncontractible cycle in a graph $G$ embedded in $S$ has length at least $g_S$, then $G$ is *locally planar*. Intuitively, by forbidding short non-contractible cycles, we can recover many properties of planar graphs.

that are sufficiently far apart [135]. Thomassen [381] later extended his ideas from [376] to show that planar graphs have exponentially many 5-colorings (a result that we recovered via different methods in Theorem 8.27). That is, given a planar graph G and a 5-assignment L, graph G has at least $2^{|G|/9}$ distinct L-colorings.

Thomassen [379] and Postle and Thomas [336, 337] also proved results for 5-list-coloring graphs on arbitrary surfaces, without restrictions on the length of non-contractible cycles. Now it is no longer true that all such graphs are 5-choosable. So the focus is on classifying obstructions and on designing efficient algorithms to decide if a given graph is 5-choosable or is L-choosable from a given 5-assignment L.

Lemma 11.3 is useful in our proof of Theorem 3.28: For each planar graph G, all 5-colorings of G are Kempe equivalent. The lemma is not necessary for the proof; indeed, the result predates the lemma by about 15 years. But the lemma enables a proof that is simpler and more satisfying. (We want to transform a 5-coloring $\varphi_1$ of a planar graph G into another another 5-coloring $\varphi_2$ of G by a sequence of Kempe swaps. This lemma proves the existence of an "intermediate" 5-coloring that is Kempe equivalent to both $\varphi_1$ and $\varphi_2$. So we are done.)

The term $(k, \ell)$-decomposable was first introduced formally in [82]. However, the idea of decomposing a graph into simpler subgraphs is very natural and has been studied for many years. Edge-coloring is simply decomposing into matchings. And the Tree-Packing Theorem (Theorem A.11) characterizes when a graph decomposes into k forests. In fact, we have already essentially seen the notion of $(k, \ell)$-decomposable in Chapter 9, just using different terminology. In Lemma 9.30 (resp. 9.27) we proved that every planar (resp. outerplanar) graph is $(3, 8)$-decomposable (resp. $(1, 3)$-decomposable). One reason for defining $(k, \ell)$-decomposability is its application to game coloring. It was proved in [193] that if G is $(1, \ell)$-decomposable, then G has game chromatic number $\chi_g(G)$ at most $4 + \ell$.

In [82], Cho, Choi, Kim, Park, Shan, and Zhu studied the minimum values $\ell_k$ such that every planar graph is $(k, \ell_k)$-decomposable. Since all planar graphs are 5-choosable, this problem is of interest only when $k \in [4]$; the graph $K_{2,n}$ witnesses that $\ell_1 = \infty$. The authors of [82] showed that $\ell_4 = 1$, $\ell_3 = 2$, and $\ell_2 \in \{4, 5, 6\}$. The result that $\ell_3 = 2$ is what we presented as Theorem 11.6, and our proof here closely follows that in [82]. The proof that $\ell_2 \leqslant 6$ uses similar ideas, but requires significantly more details.

All contents of Section 11.3 are due to Zhu [438]. The main result, that planar graphs are $(4, 2)$-choosable, answers affirmatively a question of Ktratochvíl, Tuza, and Voigt [277].

Theorem 11.13 was proved by Thomassen [380]. The main corollary, that girth 5 planar graphs are 3-choosable, was proved earlier, also by Thomassen [378] (using a different strengthening that is also amenable to proof by induction). But the proof we presented, which follows [380], is shorter and easier.

Finally, Theorems 11.14 and 11.15 are due to Kawarabayashi and Thomassen [241]. They use Theorem 11.15 to prove two strengthenings of Grötzsch's Theorem that allow (a) specifying some vertices on the outer face to be in one color class or (b) the planar graph to have many triangles, each intersecting the outer face, that are pairwise at distance at least 5.

## Exercises

11.1. Modify Thomassen's proof that every planar graph is 5-choosable (Theorem 11.1) to show that every planar graph is 5-paintable. [354]

11.2. In case 2 of the proof of Lemma 11.2, we delete a vertex $w_k$, but do not immediately color $w_k$, instead waiting until after we have colored the rest of the graph. Modify the proof to immediately color the vertices of some path along $C$, delete them, and proceed by induction. [127, Section 2.2]

11.3. Use the proof of Theorem 11.6 to recursively construct a $(3, 2)$-decomposition of the icosahedron.

11.4. We mentioned near the start of Section 11.3 that that the hypothesis $|L(v) \cap L(w)| \leqslant 2$ is best possible (to guarantee that a planar graph is colorable from a 4-assignment $L$). In Exercise 11, we presented a construction, due to Mirzakhani [300], of a 63-vertex planar graph $G$ and a 4-assignment $L$ witnessing that $G$ is not 4-choosable. Most of the edges $vw \in E(G)$ satisfy $|L(v) \cap L(w)| \leqslant 3$. However, there exist a few edges such that $L(v) = L(w)$, with $|L(v)| = |L(w)| = 4$. Thus, this example does not prove our claim. Modify the example slightly to prove the claim.

# Chapter 12

# The Potential Method

potential: latent qualities or abilities that may be developed and lead to future success or usefulness

—New Oxford American Dictionary

Most vertex coloring problems can be viewed as partitioning the vertex set $V$ into subsets $V_1, V_2, \ldots$ such that each $V_i$ satisfies certain constraints. When each $V_i$ must induce an independent set, we get standard vertex coloring. A weaker requirement is that each $V_i$ induce a subgraph of bounded degree, say $d_i$. Other possibilities include that each $V_i$ induces a forest, that each $V_i$ induce components of bounded order, that each $V_i$ is a 2-independent set (each pair of vertices in $V_i$ has distance in $G$ greater than 2), or that each $V_i$ is an independent set and each pair $V_i, V_j$ satisfies some constraint (such as inducing a forest or a forest of stars).

If we put almost no restriction on the structure of $G$, then a natural hypothesis for theorems of this type is to bound the number of edges, $\|G\|$, in terms of the order, $|G|$. Since each induced subgraph $H$ of $G$ inherits its own partition from the partition of $G$, we should also require that $H$ satisfies the hypothesis. This framework is quite general. For each set of conditions on $V_i$, we ask for the weakest hypotheses such that if $G$ and each of its induced subgraphs satisfy these hypotheses, then we can partition $V(G)$ as desired.

In this chapter, we study the Potential Method, which works especially well when each $V_i$ has the same requirement. The best example of this is a lower bound of Kostochka and Yancey on the number of edges in a $k$-critical graph. (A graph $G$ is $k$-*critical* if $\chi(G) = k$ and $\chi(G - e) < k$ for each edge $e$ in $G$.)

## 12.1 Generalizing Grötzsch's Theorem

In this section we show that if neither $G$ nor any of its subgraphs is too dense, then $G$ is 3-colorable. As a corollary, we will get a short proof of Grötzsch's Theorem, which we proved in Section 4.1.1. Our previous proof relied heavily on planarity. But here we will use planarity only to get a bound on the number of edges in G, and each of its subgraphs. To measure the denseness of G, for each $R \subseteq V(G)$, we define its *potential*, $\rho_G(R)$, where

potential

$$\rho_G(R) := 5\,|R| - 3\|G[R]\|$$

$\rho(R)$ and $\rho(G) := \min_{\emptyset \neq R \subseteq V(G)} \rho_G(R)$; we also call $\rho(G)$ the potential of G. When the context is clear, we write $\rho$ for $\rho_G$.

Intuitively, the potential $\rho(R)$ measures how many edges we can add to $G[R]$ before the resulting graph has average degree $10/3$. If the potential is large, then we can add many edges. If it is small, then not many edges. The reader will be served well by internalizing the following simple observation: $\rho(R) > 0$ if and only if $G[R]$ has average degree less than $10/3$; more generally, $\rho(G) > 0$ if and only if $\mathrm{mad}(G) < 10/3$.

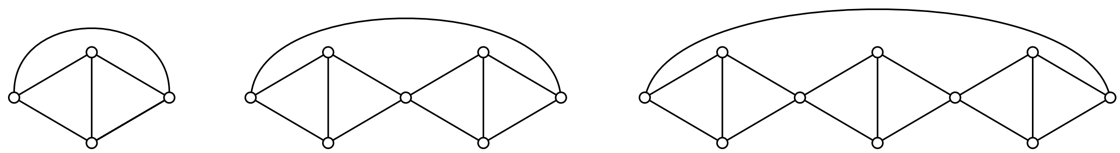


Figure 12.1: The three smallest necklaces; each is 4-critical and has $\rho(G) = 2$.

**Example 12.1.** It is easy to construct 4-chromatic graphs G with $\rho(G) = 2$. Note that $\rho(K_4) = 5(4) - 3(6) = 2$. Generalizing $K_4$, consider the "necklace" G formed from an odd cycle by expanding each vertex of some maximum independent set into an edge, with both endpoints of each new edge inheriting the two neighbors of the original vertex; see Figure 12.1. It is easy to check that $\chi(G) = 4$ and $\rho(G) = 2$. If the original cycle has $2s+1$ vertices, then $|G| = 3s+1$ and $\|G\| = 5s+1$, so $\rho_G(V(G)) = 5(3s+1) - 3(5s+1) = 2$. As a result, our next theorem is best possible. ♢

**Theorem 12.2.** *If G is a graph with $\rho(G) \geqslant 3$, then G is 3-colorable.*

Before we prove Theorem 12.2, we give a definition and a few lemmas.

**Definition 12.3.** Given a graph G, a set $R \subsetneq V(G)$, and a 3-coloring $\varphi$ of $G[R]$, form the graph $H(G,R,\varphi)$ from G by contracting each color class $i$ of $\varphi$ to a single vertex $x_i$ (adding an isolated $x_i$ if color class $i$ is empty), making vertices $x_1, x_2, x_3$ pairwise adjacent, and deleting any multiple edges formed in the process. Let $X := \{x_1, x_2, x_3\}$. See Figure 12.2.

$H(G,R,\varphi)$

$X$

**Lemma 12.4.** *If G has no 3-coloring, then for each set $R \subsetneq V(G)$ and each 3-coloring $\varphi$ of $G[R]$, the graph $H(G,R,\varphi)$ also has no 3-coloring.*

*Proof.* Given a 3-coloring $\varphi'$ of $H(G,R,\varphi)$, we get a 3-coloring of G by "uncontracting" $H(G,R,\varphi)$. That is, for each color class $i$ of $\varphi$, we give to its vertices, $\varphi^{-1}(i)$, the color $\varphi'(x_i)$ given to their image in $\varphi'$, and each vertex outside R keeps its color from $\varphi'$. □

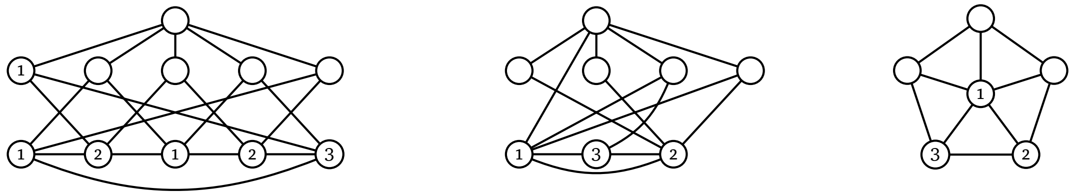



Figure 12.2: Left: A 4-critical graph $G$ and a 3-coloring $\varphi$ of the subgraph induced by a subset $R$ of size 6. Center: The graph $H(G, R, \varphi)$. Right: A more standard drawing of the 4-critical subgraph of $H(G, R, \varphi)$, formed from $H(G, R, \varphi)$ by deleting its two 2-vertices.

The next two lemmas can both be seen easily from the definitions, via direct computation. We record the first just for easy reference. Deleting edges from a graph can only increase its potential. Thus, this first lemma shows that every graph of order at most 3 has potential at least 5. The second is a fundamental property of potential, and it plays a key role in the proofs of Lemmas 12.7 and 12.11, which are central to the proof of Theorem 12.2.

**Lemma 12.5.** *We have* $\rho_{K_1}(V(K_1)) = 5$, $\rho_{K_2}(V(K_2)) = 7$, *and* $\rho_{K_3}(V(K_3)) = 6$.

**Lemma 12.6.** *If* $G$ *is a graph with* $A, B \subseteq V(G)$ *and* $A \cap B = \emptyset$, *then* $\rho_G(A \cup B) = \rho_G(A) + \rho_G(B) - 3|E_G(A, B)|$.

*Proof.* This is true because the contribution to $\rho_G(A \cup B)$ by each vertex and edge of $G[A \cup B]$ is counted by exactly one of $\rho_G(A)$ and $\rho_G(B)$, except for the edges of $E_G(A, B)$, the contribution of which are counted by neither. $\square$

### 12.1.1 Proving Theorem 12.2

Suppose Theorem 12.2 is false, and let $G$ be a counterexample minimizing $|G| + \|G\|$. Since $G$ is minimal, every proper subgraph is 3-colorable. The central idea of the proof is the following, which we formalize in Lemmas 12.7 and 12.8.

We will show, for each $R \subsetneq V(G)$ with $|R| \geqslant 2$, that $\rho_G(R) \geqslant 6$. Suppose this is true. Now for some edge $e$ not in $G[R]$, let $G^+ := G[R] + e$. Since $\rho(G^+) \geqslant \rho(G) - 3(1) \geqslant 6 - 3 = 3$, the minimality of $G$ allows us to 3-color $G^+$. This gives us more control over the 3-coloring of $G[R]$, allowing us to extend it a 3-coloring of $G$. So we must show that $\rho_G(R) \geqslant 6$.

**Lemma 12.7 (Gap Lemma).** *If* $R \subsetneq V(G)$ *with* $|R| \geqslant 2$, *then* $\rho_G(R) \geqslant 6$.

*Proof.* Choose $R \subsetneq V(G)$, with $|R| \geqslant 2$, to minimize $\rho(R)$. When $|R| \leqslant 3$ the bound holds, $R$
by Lemma 12.5 (since deleting edges only increases potential). So assume $|R| \geqslant 4$. Let $\varphi$
be a 3-coloring of $G[R]$ and let $G' := H(G, R, \varphi)$. By Lemma 12.4, we know that $G'$ has no $G'$
3-coloring. By the minimality of $G$, there exists $R' \subseteq V(G')$ such that $\rho_{G'}(R') \leqslant 2$. We will $R'$
use $R'$ to show that $\rho(G) \leqslant 2$, so $G$ is not a counterexample. Specifically, let $S := (R' \setminus X) \cup R$, $S$

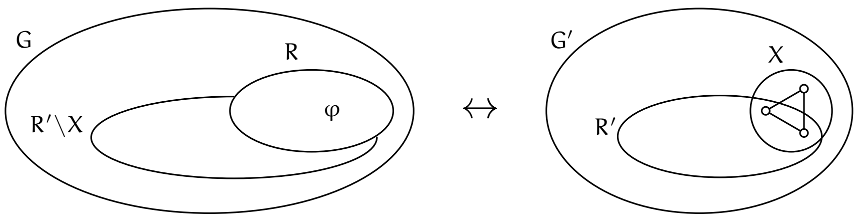



Figure 12.3: The proof of the Gap Lemma (Lemma 12.7).

with $X$ as in Definition 12.3; see Figure 12.3. Intuitively, $S$ is the preimage of $R' \cup X$ under the contraction used to form $G'$ from $G$. We get that[1]

$$\rho_G(S) \leqslant \rho_{G'}(R') - \rho_{G'}(R' \cap X) + \rho_G(R), \tag{12.1}$$

as we now show. Note that $|E_{G'}(R' \setminus X, R' \cap X)| \leqslant |E_G(R' \setminus X, R)|$. By Lemma 12.6, we have $\rho_G(R' \setminus X) = \rho_{G'}(R' \setminus X) = \rho_{G'}(R') - \rho_{G'}(R' \cap X) + 3|E_{G'}(R' \setminus X, R' \cap X)| \leqslant \rho_{G'}(R') - \rho_{G'}(R' \cap X) + 3|E_G(R' \setminus X, R' \cap X)|$. By combining this inequality with another application of Lemma 12.6, we get $\rho_G(S) = \rho_G((R' \setminus X) \cup R) = \rho_G(R' \setminus X) - 3|E_G(R' \setminus X, R)| + \rho_G(R) \leqslant \rho_{G'}(R') - \rho_{G'}(R' \cap X) + \rho_G(R)$, as claimed.

We also provide an intuitive explanation. This inequality follows directly from Lemma 12.6, since in moving from $R'$ (in $G'$) to $S$ (in $G$), we replace all vertices and edges in $G'[R' \cap X]$ by vertices and edges in $G[R]$. The vertices and edges in $G[R' \setminus X]$ stay unchanged, and each edge in $E_{G'}(R' \setminus X, R' \cap X)$ has at least one preimage edge in $E_G(S \setminus R, R)$. (This inequality may be strict since some vertex in $S \setminus R$ may have multiple neighbors in $R$ with the same color $i$ under $\varphi$ but only one neighbor in $X$ with color $i$.)

If $R' \subseteq V(G)$, then $3 \leqslant \rho(G) \leqslant \rho_G(R') \leqslant \rho_{G'}(R') \leqslant 2$, which contradicts the hypothesis. Thus, $R' \not\subseteq V(G)$. That is, $R' \cap X$ is nonempty. Because $G'[X] = K_3$, Lemma 12.5 shows that $\rho_{G'}(R' \cap X) \geqslant 5$. Now (12.1) gives $\rho_G(S) \leqslant 2 - 5 + \rho_G(R)$. In particular, $\rho_G(S) < \rho_G(R)$. Since $|R| \geqslant 4$ and $R \subseteq S$, also $|S| \geqslant 4$. Since we chose $R$ to minimize $\rho_G(R)$, and $\rho_G(S) < \rho_G(R)$, we conclude that $S = V(G)$. Now if $\rho_G(R) \leqslant 5$, then $\rho(G) = \rho_G(S) \leqslant 2 - 5 + 5 = 2$, so $G$ is not a counterexample. Thus, we must have $\rho_G(R) \geqslant 6$. $\square$

The Gap Lemma, specifically (12.1), is the key idea in the proof of Theorem 12.2. Our next three lemmas are immediate corollaries of the Gap Lemma.

**Lemma 12.8.** *If* $R \subsetneq V(G)$ *and* $|R| \geqslant 2$ *and* $e \notin E(G[R])$, *then* $G[R] + e$ *is* 3*-colorable.*

*Proof.* Let $G^+ := G[R] + e$. By the minimality of $G$, it suffices to show that $\rho_{G^+}(S) \geqslant 3$ for each nonempty $S \subseteq V(G^+)$. If $|S| = 1$, then $\rho_{G^+}(S) = 5$. Otherwise $|S| \geqslant 2$, so Lemma 12.7 gives $\rho_{G^+}(S) \geqslant \rho_G(S) - 3 \geqslant 6 - 3 = 3$. $\square$

**Lemma 12.9.** $G$ *has no copy of* $K_4 - e$.

[1]Inequality (12.1), like its analogues in later sections, is sometimes called a Potential–Extension Lemma.

*Proof.* This follows directly from Lemma 12.7, since $\rho(K_4 - e) = 5(4) - 3(5) = 5$. □

**Lemma 12.10.** *No triangle contains two (or three) 3-vertices.*

*Proof.* Assume the contrary. Let $S := \{v_1, v_2, v_3\}$, where $S$ induces a triangle and $d(v_1) = d(v_2) = 3$. Let $w_1$ and $w_2$ be the neighbors of $v_1$ and $v_2$ outside of $S$. Let $G^+ := G \setminus \{v_1, v_2\} + w_1w_2$; see Figure 12.4. Lemma 12.8 shows that $G^+$ has a 3-coloring $\varphi$. Now $\varphi(w_1) \neq \varphi(w_2)$, so we can extend $\varphi$ to $G$, since the uncolored vertices, $v_1$ and $v_2$, each have at least one available color, and if both have exactly one, then these colors are distinct. □

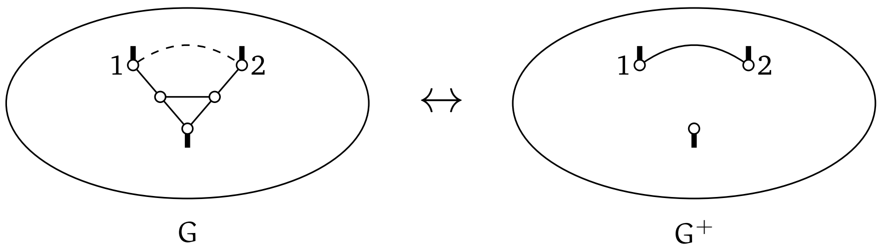



Figure 12.4: The proof of Lemma 12.10. The bold half-edges incident to three vertices indicate that the degrees of these vertices are not prescribed, in contrast to the degrees of the other two 3-vertices.

In fact, we can push the ideas in the proof of Lemma 12.7 a little farther. This improved bound is not enough to allow us to 3-color subgraphs after adding *two* edges, but it does allow us to 3-color subgraphs after alterations more complex than adding a single edge.

**Lemma 12.11 (Strong Gap Lemma).** *If* $R \subsetneq V(G)$ *and* $|R| \geqslant 2$ *and* $G[R] \neq K_3$*, then* $\rho_G(R) \geqslant 7$.

*Proof.* Suppose the lemma is false and choose $R$, satisfying the hypotheses, that minimizes $\rho(R)$. Lemma 12.7 implies $\rho(R) = 6$. Note that $G$ must be 2-connected, since otherwise we can 3-color two subgraphs of $G$, by the minimality of $G$, and permute colors to agree on a cut-vertex. So there exist $v_1, v_2 \in R$ with neighbors in $V(G) \setminus R$. Let $G^+ := G[R] + v_1v_2$ (if $v_1v_2 \in E(G)$, then $G^+ := G[R]$). Lemma 12.8 shows that $G^+$ has a 3-coloring $\varphi$.

As in the proof of Lemma 12.7, let $G' := H(G, R, \varphi)$. Now repeating that proof[2] we again get (12.1), which gives $\rho_G(S) \leqslant \rho_{G'}(R') - \rho_{G'}(R \cap X') + \rho_G(R) \leqslant 2 - 5 + 6 = 3$. (We omit a figure, since it would be nearly identical to Figure 12.3.) So again $S = V(G)$. (Recall that $S = (R' \setminus X) \cup R$.) Further, when $|R' \cap X| \geqslant 2$, we have $\rho_{G'}(R' \cap X) \geqslant 6$, so $\rho_G(V(G)) \leqslant 2$, which contradicts the hypothesis. Thus, we assume that $|R' \cap X| = 1$. By symmetry, say $v_2 \notin R' \cap X$. But now the edge from $v_2$ to its neighbor outside $R$ is counted on the left side of (12.1) but not on the right side; hence, the inequality is strict. This implies that $\rho_G(V(G)) \leqslant 2$, which again contradicts the hypothesis $\rho_G(V(G)) \geqslant 3$. □

[2]When proving that $\rho_G(R) \geqslant 7$, our reduction to the case $|R| \geqslant 4$ uses the hypothesis $G[R] \neq K_3$, since $\rho(K_3) = 6$.

**Lemma 12.12.** *If $v_1$ and $v_2$ are adjacent 3-vertices, then each is in a triangle.*

$v_1, w, x$ $G'$ $w * x$

*Proof.* Suppose, to the contrary, that $v_1$ is not in a triangle. Let $w$ and $x$ be the other neighbors of $v_1$. We form a new graph $G'$ from $G \setminus \{v_1, v_2\}$ by identifying $w$ and $x$, and we let $w{*}x$ denote this new vertex.

If $G'$ has a 3-coloring $\varphi'$, then $\varphi'$ induces a 3-coloring $\varphi$ of $G \setminus \{v_1, v_2\}$ with $\varphi(w) = \varphi(x)$, which we can extend to a 3-coloring of G by greedily coloring $v_2$, followed by $v_1$. So assume that $G'$ has no 3-coloring. By the minimality of G, there exists $R \subseteq V(G')$ such that $\rho_{G'}(R) \leqslant 2$. Since $\rho(G) \geqslant 3$, we know that $R \not\subseteq V(G)$, which implies that $w * x \in R$. Let $R' := (R \setminus \{w * x\}) \cup \{v_1, w, x\}$. Since $G[R']$ has two more vertices than $G'[R]$ and at least two more edges, $\rho_G(R') \leqslant \rho_{G'}(R) + 5(2) - 3(2) \leqslant 2 + 4 = 6$. But this contradicts Lemma 12.11, since $G[R']$ is not a copy of $K_3$. (Our reason for deleting $v_2$ when forming $G'$ is to ensure that $R' \subsetneq V(G)$, which allows us to apply the Strong Gap Lemma.) □

**Lemma 12.13.** *Each 3-vertex has at most one 3-neighbor.*

*Proof.* If a 3-vertex $v$ has a 3-neighbor, then $v$ is in a triangle, by Lemma 12.12. So its two other neighbors are not 3-vertices, by Lemma 12.10. □

*Proof of Theorem 12.2.* Note that $2\|G\| = \sum_{v \in V(G)} d(v) \geqslant \frac{10}{3}|G|$ is equivalent to $\rho(G) = 5|G| - 3\|G\| \leqslant 0$. So, to contradict our hypothesis that $\rho(G) \geqslant 3$, it suffices to show that $\sum_{v \in V(G)} d(v) \geqslant \frac{10}{3}|G|$. We use discharging, with initial charge $d(v)$ for each vertex $v$. We use only a single discharging rule: Each 3-vertex takes $\frac{1}{6}$ from each $4^+$-neighbor. Now we show that all vertices are happy. Since G is 4-critical, clearly $\delta(G) \geqslant 3$. By Lemma 12.13, each 3-vertex ends with at least $3 + 2(\frac{1}{6}) = \frac{10}{3}$. Each $4^+$-vertex loses at most $\frac{1}{6}$ to each neighbor, so ends with at least $d(v) - \frac{1}{6}d(v) = \frac{5}{6}d(v) \geqslant \frac{20}{6} = \frac{10}{3}$, as desired. □

Theorem 12.2 is equivalent to Corollary 12.14. (Deducing the theorem from the corollary is easy.) Recall that a graph G is $k$-*critical* if $\chi(G) = k$ and $\chi(G - e) < k$ for each edge $e$ in G.

k-critical

**Corollary 12.14.** *If* G *is* 4*-critical, then* $5|G| - 3\|G\| \leqslant 2$.

*Proof.* Let G be 4-critical. By Theorem 12.2, there exists $\emptyset \neq R \subseteq V(G)$ such that $\rho_G(R) \leqslant 2$; choose R to minimize $\rho_G(R)$. If $\rho_G(V(G)) = \rho_G(R)$, then we are done, so assume this is false. Now we exactly repeat the proof of Lemma 12.7 (using Theorem 12.2 in place of the minimality of G). This implies that $\rho_G(R) \geqslant 6$, which yields a contradiction. □

Corollary 12.14 gives a short proof of Grötzsch's Theorem. Let G be a triangle-free planar graph. If G has a 4-face f, then we can identify one of the pairs of non-adjacent vertices on f to get a smaller triangle-free planar graph, and proceed by induction. But if G has no 4-face, then a simple counting argument using Euler's formula shows that $\|G\| \leqslant \frac{5|G|-10}{3}$. This contradicts

Theorem 12.2, and thus finishes the proof. We formalize the idea of contracting 4-faces in the next lemma[3], and thereafter we prove Grötzsch's Theorem.

**Lemma 12.15.** *Let $G$ be a plane graph with a 4-face $f$. Let $G_1$ and $G_2$ be the plane graphs formed from $G$ by identifying the two pairs of non-adjacent vertices on $f$. If no edge of $f$ lies in a 3-cycle, then either $G_1$ or $G_2$ has no more 3-cycles than $G$.*

*Proof.* Let $v_1, v_2, v_3, v_4$ be the vertices of $f$, in order. See Figure 12.5. Form $G_1$ from $G$ by identifying $v_1$ and $v_3$, and form $G_2$ from $G$ by identifying $v_2$ and $v_4$. Suppose that both $G_1$ and $G_2$ have more 3-cycles than $G$. Since $G_1$ does, there exists in $G$ a path $v_1w_1w_3v_3$. Similarly, since $G_2$ does, there exists in $G$ a path $v_2w_2w_4v_4$. By planarity, some vertex of $\{w_1, w_3\}$ is also in $\{w_2, w_4\}$. By symmetry, say $w_2 = w_3$. Now $v_2w_2v_3$ is a 3-cycle of $G$. □

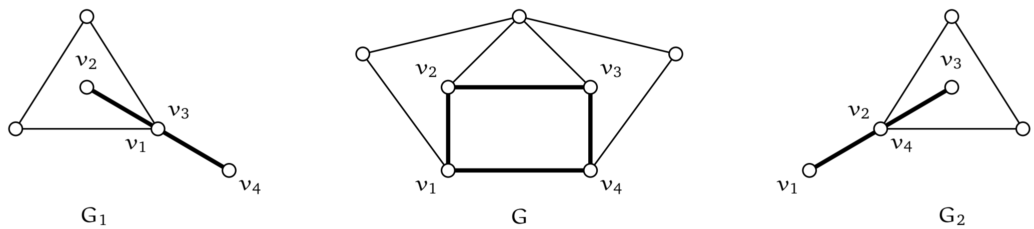



Figure 12.5: The proof of Lemma 12.15. We assume that all bold edges are present, and we succeed unless all other edges shown are also present. We try identifying $v_1$ with $v_3$, and also try identifying $v_2$ with $v_4$. If each possibility creates a 3-cycle, then some edge of $v_1v_2v_3v_4$ already lies on a 3-cycle.

**Theorem 12.16 (Grötzsch's Theorem).** *If $G$ is planar and has no 3-cycle, then $G$ is 3-colorable.*

*Proof.* We use induction on $|G| + \|G\|$, with the trivial base case $|G| + \|G\| \leqslant 3$. If $G$ has a 4-face $f$, then by Lemma 12.15, we can identify some pair of non-adjacent vertices on $f$, say $v_1$ and $v_3$, to get a smaller triangle-free plane graph $G'$, with new vertex $v_1 * v_3$. By induction $G'$ has a 3-coloring $\varphi'$. But $\varphi'$ naturally yields a 3-coloring of $G$, by giving both $v_1$ and $v_3$ color $\varphi'(v_1 * v_3)$, and giving each other vertex its color under $\varphi'$.

So suppose $G$ has no 4-face. Assume, contrary to the theorem, that $G$ has no 3-coloring. By induction every proper subgraph of $G$ is 3-colorable. So $G$ is 4-critical, and Corollary 12.14 implies that $\|G\| \geqslant \frac{5|G|-2}{3}$. Since $G$ has no 4-face, counting edges by faces gives $5\,|F| \leqslant 2\|G\|$, where $F$ is the set of all faces. Now Euler's formula, $|F| - \|G\| + |G| = 2$, gives $10 - 5\,|G| + 5\|G\| \leqslant 2\|G\|$. This inequality simplifies to $\|G\| \leqslant \frac{5|G|-10}{3}$, which contradicts the inequality above coming from Corollary 12.14. □

[3]This is essentially a very special case of the Folding Lemma, from Section 4.3. To keep our proof self-contained, we reprove what we use here.

### 12.1.2 An Outline of the Potential Method

We end this section with an outline of how to prove theorems using the potential method.

1. Find Extremal Examples
2. Define Potential
3. State the Theorem using Potential
4. Find the Gadgets (if needed, Generalize the theorem to allow Precoloring)
5. Prove a Gap Lemma (if needed, leverage this first "weak" Gap Lemma to prove a "strong" Gap Lemma)
6. Use the Gap Lemma to discover new Reducible Configurations
7. Finish with Discharging

To illustrate this approach, we highlight where each step takes place in the previous proof.

1. The extremal examples are the necklaces, described in Example 12.1. In fact, there are many other extremal examples, too. But the important thing is to find an infinite family, generally with the difference in orders of successive examples bounded by a constant.

2. The necklaces have average degree tending to $10/3$. This motivates the potential function $\rho(R) := 5|R| - 3\|G[R]\|$. In general, if the average degree of the extremal examples tends to $2a/b$, then a natural choice for the potential function is $\rho(R) := a|R| - b\|G[R]\|$.

3. Theorem 12.2 states the theorem using potential. We have two natural ways to phrase our theorems: (i) If $G$ is a graph with $\rho(R) > C$ for each nonempty $R \subseteq V(G)$ then $G$ is "colorable" and (ii) If $G$ is "critical" ($G$ is not colorable, but each of its proper subgraphs is), then $\rho(V(G)) \leqslant C$. Here "colorable" refers to whatever type of coloring we seek, and $C$ is a constant (often close to 0). It is easy to deduce version (i) from version (ii), but less clear how to prove (ii) from (i).

   The language of version (i) is more algorithmic, which can make the argument a bit easier to follow. However, version (ii) appears to give us more power when we color by the induction hypothesis, or criticality. In practice, if we can prove version (i) of a theorem, then we can usually also prove version (ii). To make the proof more accessible, we presented the argument above using version (i). But in the rest of this chapter, we typically favor version (ii). We say more about algorithms in the Notes, but most theorems proved using the potential method translate naturally into polynomial-time algorithms to construct the desired colorings.

4. The most important role of a gadget is to help prove the Gap Lemma. In the previous example, the sole gadget is the triangle induced by vertex set $X$. We need that every valid coloring of $G'[X]$ uses each color on a vertex of $X$. When our coloring is simply a map to

some target graph H, then we generally choose H as the gadget. Our job is easier when $|H|$ is small and $\rho_H(R)$ is "big", for each nonempty $R \subseteq V(H)$. It also helps when H is vertex-transitive or, more strongly, when H is a clique, so that we can freely permute the "colors" of H to make two colorings agree on some clique cutset.

When our coloring is not simply a map to some vertex-transitive H, our colors are no longer "interchangeable", i.e., permuting the color classes of a valid coloring may not yield another valid coloring. In this case, it is often helpful to generalize the theorem to allow "precoloring". A precolored graph can specify the color of certain vertices. We generalize our theorem to a statement for precolored graphs, where each precolored vertex contributes less to the potential of each vertex subset containing it. We can actually simulate these precolored graphs with normal ("un-precolored") graphs, by adding pendent copies of certain "gadgets" at the vertices that are to be precolored. The potential of each gadget tells us the appropriate weight for that type of vertex in the generalized potential function. We first see precoloring in Section 12.4.1.

5. The heart of a proof via the potential method is always its Gap Lemma. Every Gap Lemma has the form: If $R \subsetneq V(G)$ with $a < |R| < |G| - b$, then $\rho(R) > C'$. Here $a, b, C'$ are positive integers and $C' > C$. Recall that our theorem states: If G is critical, then $\rho(V(G)) \leqslant C$. Since we assume G is a counterexample, we have $\rho(V(G)) > C$. The term *gap* refers to the difference $C' - C$. A larger gap gives us more power for finding reducible configurations. The proof of every Gap Lemma is nearly identical to that of Lemma 12.7. It all hinges on inequality (12.1), which in turn builds on Lemma 12.6. The size of our gap is determined primarily by $\min_{\emptyset \neq R \subseteq V(H)} \rho(R)$, where H is our gadget; a larger minimum gives a larger gap.

   Often, we can prove a second Gap Lemma, which guarantees a still larger gap, e.g., Lemma 12.10. We call this a Strong Gap Lemma (and the earlier result a Weak Gap Lemma). The Strong Gap Lemma may need to exclude a few other subgraphs with small potential. The reason we first prove the Weak Gap Lemma (rather than beginning with the Strong Gap Lemma) is that we use it to prove the Strong.

6. Lemmas 12.8, 12.9, 12.10, and 12.12 are all good examples of how we use our Gap Lemmas to discover new reducible configurations.

7. Often, the discharging step is among the easier parts of the proof. We design our initial charges so that their sum is exactly $\rho(V(G))$. Since G is a counterexample, we assume $\rho(V(G)) > C$. To reach a contradiction, we must show that the sum of final charges is at most C. When C is positive, we typically show that each vertex finishes with charge at most 0, which suffices. When C is slightly negative, we must work harder to reach a contradiction. Typically, we show that each vertex finishes with at most 0, and we find a few vertices with final charges summing to less than C.

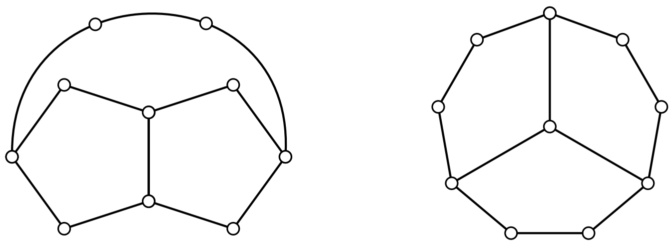


Figure 12.6: Two $C_5$-critical graphs with $\rho = 6(10) - 5(12) = 0$.

## 12.2 $C_5$-Critical Graphs

In the previous section, we considered 3-colorings. If we view a 3-coloring of G as a homomorphism (map, for short) from G to $K_3$, then it is natural to ask about maps from G into other target graphs H. If H is an even cycle (or any forest with at least one edge), then maps into H are simply 2-colorings of G, which are well understood. A more interesting problem comes from letting H be an odd cycle. Here we let $H := C_5$.

map
$C_5$-coloring
$C_5$-colorable
$C_5$-critical

**Definition 12.17.** We typically write *map* to denote homomorphism. A $C_5$-*coloring* of a graph G (or simply *coloring of* G, within this section) is a map from G to $C_5$. A graph G is $C_5$-*colorable* (or simply *colorable*) if G admits such a map. And G is $C_5$-*critical* (or simply *critical*) if G is not $C_5$-colorable, but $G - e$ is $C_5$-colorable, for every $e \in E(G)$. If G is critical, then $\delta(G) \geqslant 2$.

$\rho_G(R)$

In each section of this chapter, we will define a new potential function, analogous to that in Section 12.1. Only the coefficients on $|R|$ and $\|G[R]\|$ will change, to account for the bound we are trying to prove on edge density. In this section, for each $R \subseteq V(G)$, let $\rho_G(R) := 6|R| - 5\|G[R]\|$.

**Theorem 12.18.** *If* G *is* $C_5$-*critical and* $G \neq K_3$, *then* $|E(G)| \geqslant \frac{6}{5}|V(G)|$. *Equivalently, if* G *is* $C_5$-*critical and* $G \neq K_3$, *then* $\rho_G(V(G)) \leqslant 0$.

Theorem 12.18 is sharp, as shown by the two graphs in Figure 12.6. Throughout this section, we will assume the theorem is false and let G be a counterexample with as few vertices as possible. In particular, $\rho_G(V(G)) > 0$.

**Lemma 12.19.** *If* $R \subseteq V(C_5)$ *(and* $R \neq \emptyset$*), then* $\rho_{C_5}(R) \geqslant 5$.

*Proof.* If $|R| \leqslant 4$, then $G[R]$ is a forest, so $\rho(R) \geqslant 6|R| - 5(|R| - 1) = |R| + 5 > 5$. And if $|R| = 5$, then $\rho(R) = 6(5) - 5(5) = 5$. □

To facilitate a proof by induction, we first prove a more technical statement, our "Gap Helper" below, and then show that this immediately implies our desired Gap Lemma.

**Lemma 12.20 (Gap Helper).** *If* $R \subsetneq V(G)$ *with* $G[R] \not\subseteq C_5$, *then* $\rho(R) \geqslant \rho(V(G)) + 2$. *Further, if* G *is not formed from* $G[R]$ *by adding a path on two vertices (and 3 edges), then* $\rho(R) \geqslant \rho(V(G)) + 4$.

*Proof.* We use induction on $|G| - |R|$. The base case, $|G| - |R| = 1$, is easy: $\delta(G) \geqslant 2$ implies $\rho(R) \geqslant \rho(V(G)) - 6 + 2(5) = \rho(V(G)) + 4$.

Now we consider the induction step. By criticality, $G[R]$ has a coloring; choose a coloring $\varphi$ with as few colors as possible. Form $G'$ from $G$ by contracting $R$ to its image under $\varphi$ in $C_5$; that is, each color class of $\varphi$ is contracted to a single vertex of $C_5$ and $G'$ inherits from $C_5$ all edges induced by the corresponding vertices.[4] Let $X$ be this set of new vertices ($R$'s image). $\varphi, G'$ $X$

If $G'$ has a coloring $\varphi'$, then we can permute its color classes to agree with $\varphi$ on $X$. Hence, we can combine $\varphi$ and $\varphi'$ to get a coloring of $G$, which contradicts that $G$ is critical. Thus, we assume that $G'$ has no coloring. So $G'$ has a critical subgraph $G''$; let $S := V(G'')$. Note that $G'', S$

$$\rho_G((S \setminus X) \cup R) \leqslant \rho_{G'}(S) - \rho_{G'}(S \cap X) + \rho_G(R). \tag{12.2}$$

(The proof of (12.2) is essentially identical to that of (12.1) in the previous section, so we do not repeat it.[5] In a word, it is submodularity.) Since $G[R] \not\subseteq C_5$, and $\varphi$ has as few color classes as possible, we have $|R| > |X|$, so $|G| > |G'| \geqslant |G''|$. Thus, Theorem 12.18 holds for $G''$. That is, either $G'' = K_3$ or else $\rho_{G''}(V(G'')) \leqslant 0$. Since $\rho_{K_3}(V(K_3)) = 3$, Lemma 12.19 and inequality (12.2) give $\rho_G((S \setminus X) \cup R) \leqslant 3 - 5 + \rho_G(R)$. That is, $\rho(R) \geqslant \rho((S \setminus X) \cup R) + 2$. If $(S \setminus X) \cup R \neq V(G)$, then $\rho((S \setminus X) \cup R) \geqslant \rho(V(G)) + 2$ by induction (recall that induction is on $|G| - |R|$), since $|(S \setminus X) \cup R| = |S \setminus X| + |R| > |R|$. Thus, $\rho(R) \geqslant \rho((S \setminus X) \cup R) + 2 \geqslant (\rho(V(G)) + 2) + 2$. So assume instead that $(S \setminus X) \cup R = V(G)$. Now $\rho(R) \geqslant 2 + \rho((S \setminus X) \cup R) = 2 + \rho(V(G))$.

All that remains is to show that if $\rho(R) < \rho(V(G)) + 4$, then $G$ is formed from $G[R]$ by adding a path with 2 internal vertices. If $\rho(R) < \rho(V(G)) + 4$, then $(S \setminus X) \cup R = V(G)$ so (12.2) and Lemma 12.19 imply that $G'' = K_3$. Let $a := |S \cap X|$, and note that $a \in \{1, 2\}$, since $K_3 \not\subseteq G$ and $K_3 \not\subseteq C_5$. If $a = 1$, then $|V(G'') \setminus X| = 2$, so indeed $G$ is formed from $G[R]$ by adding a path with 2 internal vertices. If $a = 2$, then $\rho(R) = \rho(V(G)) + 2(5) - 6 = \rho(V(G)) + 4$, which proves the lemma. □

**Lemma 12.21 (Gap Lemma).** *If* $R \subsetneq V(G)$ *(and* $R \neq \emptyset$*), then* $\rho(R) \geqslant 3$. *Further, if* $G$ *is not formed from* $G[R]$ *by adding a path with two internal vertices (and* 3 *edges), then* $\rho(R) \geqslant 5$.

*Proof.* If $G[R] \subseteq C_5$, then we are done by Lemma 12.19. And if $G[R] \not\subseteq C_5$, then we are done by the Gap Helper, since $\rho(V(G)) \geqslant 1$. □

A $k$-*thread* is a path with $k$ internal vertices, each of which has degree 2 in $G$. A *weak neighbor* of a 2-vertex $v$ is a $3^+$-vertex $w$ that is an endpoint of the longest thread containing $v$; likewise, $v$ is a *weak neighbor* of $w$. k-thread weak neighbor

**Lemma 12.22.** $G$ *does not contain any* $k$*-thread with* $k \geqslant 3$. *Also,* $\delta(G) \geqslant 2$.

*Proof.* If $G$ contains a $1^-$-vertex $v$, then $G - v$ is colorable by criticality, and we can extend the coloring to $v$. So $\delta(G) \geqslant 2$.

[4] This is analogous to $H(G, R, \varphi)$ in the previous section.

[5] A picture here would be identical to Figure 12.3, except there $X$ induces $K_3$ and here it induces $C_5$.

Now assume that $G$ contains a path $v_0v_1v_2v_3v_4$ with $d(v_i) = 2$ for each $i \in [3]$. By criticality $G \setminus \{v_1, v_2, v_3\}$ has a coloring $\varphi$. It is easy to check that for each pair $w_1, w_2 \in V(C_5)$ there is a walk in $C_5$ of length exactly 4 from $w_1$ to $w_2$, even if $w_1 = w_2$ (there are two directions around $C_5$, and for each $w_2$ at least one of them works). Thus, we can extend $\varphi$ to $G$, a contradiction. $\square$

**Lemma 12.23.** $G$ *has no* 3*-vertex with at least four weak neighbors (that are* 2*-vertices).*

*Proof.* Let $v$ be a 3-vertex with $k$ weak neighbors. We will show that $k \leqslant 3$. We first rule out the case $k \geqslant 5$, using a short counting argument.

**Claim 1.** *Fix* $v \in V(C_5)$ *and* $i \in [4]$*. The number of vertices of* $C_5$ *that are reachable from* $v$ *along walks of length exactly* $i$ *is* $i + 1$*. So the number of vertices not reachable is* $4 - i$*.*

Proof. This can be proved by induction on $i$, but it is simpler to just check the cases for the four values of $i$. (Since $C_5$ is vertex transitive, we can choose $v$ arbitrarily.) $\diamondsuit$

Suppose that $k \geqslant 5$. By Lemma 12.22, we assume that $v$ is the endpoint of a 2-thread, a 2-thread, and a $1^+$-thread. Delete $v$ and all its weak neighbors, and color the resulting graph by criticality; call this coloring $\varphi$. We must extend $\varphi$ to $G$. Note that a walk from one endpoint of an $i$-thread to the other has length $i + 1$. By the claim, the color given by $\varphi$ to the other endpoint of each $i$-thread (with $v$ as one endpoint) forbids $4 - (i + 1) = 3 - i$ colors from use on $v$. So the total number of colors forbidden from use on $v$ is at most $1 + 1 + 2$. Thus, some color can be used on $v$ and we can extend $\varphi$ to $G$, as desired. This finishes the case $k \geqslant 5$.

Now we consider the case that $v$ has exactly 4 weak neighbors, i.e., $k = 4$. So $v$ is the endpoint of either (a) a 0-thread, a 2-thread, and a 2-thread or (b) a 1-thread, a 1-thread, and a 2-thread. We handle these two cases below.

**Case (a): $v$ is the endpoint of a 2-thread, a 2-thread, and a 0-thread.** Denote the vertices along the threads incident to $v$ by $w_1$, $w_2$, $w_3$; $x_1$, $x_2$, $x_3$; and $y_1$ (see Figure 12.7). Form $G'$

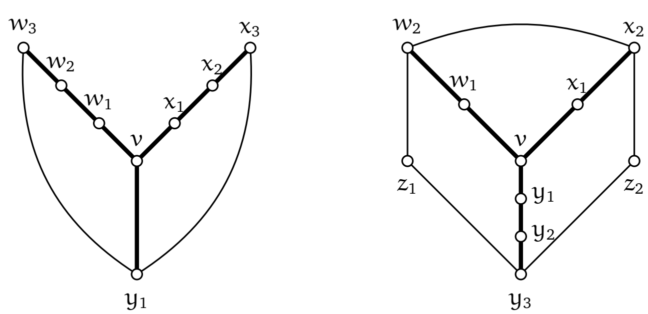



Figure 12.7: Cases (a) and (b) in the proof of Lemma 12.23. We begin assuming that all bold edges are present and we succeed in reducing to a smaller graph unless all other edges shown are also present. Since $\rho > 0$, no further edges are induced by these vertices.

from $G \setminus \{v, x_1, x_2\}$ by identifying $w_1$ with $y_1$; call this new vertex $v'$. If $G'$ has a coloring, then we can easily extend it to a coloring of G, as in the proof of Lemma 12.22. So $G'$ must contain a critical subgraph $G''$; let $S' := V(G'')$. If $G'' \neq K_3$, then by the minimality of G, we have $\rho_{G''}(S') \leqslant 0$. Let $S := (S'-v') \cup \{v, w_1, y_1\}$. Now $\rho_G(S) = \rho_{G''}(S') + 2(6) - 2(5) \leqslant 0 + 2 = 2$. However, this contradicts the Gap Lemma, which says that $\rho_G(S) \geqslant 3$. Thus, we conclude that $\rho_{G''}(S') > 0$. This implies that $G'' = K_3$; hence, $w_3y_1 \in E(G)$. Similarly, $x_3y_1 \in E(G)$.

Let $T := \{v, w_1, w_2, w_3, x_1, x_2, x_3, y_1\}$. Note that $\rho_G(T) \leqslant 8(6) - 9(5) = 3$. This contradicts the Gap Lemma unless either (i) $V(G) = T$ or (ii) G is formed from $G[T]$ by adding a path on 2 vertices and 3 edges. In (i) G is colorable unless it contains at least one more edge. But in that case $\rho_G(T) = 8(6) - 10(5) = -2$, so G is not a counterexample. Instead assume we are in (ii). Now accounting for the additional path on 2 vertices and 3 edges[6], we have $\rho_G(V(G)) \leqslant 10(6) - 12(5) = 0$. Again, G is not a counterexample, which concludes Case (a).

**Case (b): $v$ is the endpoint of a 1-thread, a 1-thread, and a 2-thread.** The analysis is similar to that above for Case (a), but a bit longer. Denote the vertices along the threads incident to $v$ by $w_1$, $w_2$; $x_1$, $x_2$; and $y_1$, $y_2$, $y_3$ (again, see Figure 12.7). Form $G'$ from $G \setminus \{v, y_1, y_2\}$ by identifying $w_1$ with $x_1$; call this new vertex $v'$. If $G'$ has a coloring, then we can easily extend it to a coloring of G, as in the proof of Lemma 12.22. So $G'$ must contain a critical subgraph $G''$; let $S' := V(G'')$. If $G'' \neq K_3$, then by the minimality of G, we have $\rho_{G''}(S') \leqslant 0$. Let $G'', S'$
$S := (S' - v') \cup \{v, w_1, x_1\}$. Now $\rho_G(S) = \rho_{G''}(S') + 2(6) - 2(5) \leqslant 0 + 2 = 2$. But this contradicts the Gap Lemma, which says that $\rho_G(S) \geqslant 3$. So we conclude that $\rho_{G''}(S') > 0$. This implies that $G'' = K_3$; hence, $w_2x_2 \in E(G)$.

Now instead form $G'$ from $G \setminus \{v, w_1, x_1, y_1\}$ by identifying $x_2$ with $y_2$. If $G'$ has a coloring, then we can easily extend it to a coloring of G, as in the proof of Lemma 12.22. So $G'$ must contain a critical subgraph $G''$; let $S' := V(G'')$. If $G'' \neq K_3$, then by the minimality of G, we have $\rho_{G''}(S') \leqslant 0$. Let $S := (S' - v') \cup \{v, x_1, y_1\}$. Now $\rho_G(S) = \rho_{G''}(S') + 2(6) - 2(5) \leqslant 0 + 2 = 2$. But this contradicts the Gap Lemma, which says that $\rho_G(S) \geqslant 3$. Thus, we have $\rho_{G''}(S') > 0$. This implies that $G'' = K_3$; hence, there exists a vertex $z_1$ with $z_1w_2, z_1y_3 \in E(G)$. By the same argument, there exists a vertex $z_2$ with $z_2x_2, z_2y_3 \in E(G)$. Let $T := \{v, w_1, w_2, x_1, x_2, y_1, y_2, y_3, z_1, z_2\}$. Now $\rho_G(T) \leqslant 6(10) - 5(12) = 0$. So T contradicts the Gap Lemma unless $T = V(G)$. In fact, it is easy to check that $G[T]$ is actually the graph on the left in Figure 12.6, which is indeed $C_5$-critical. □

Finally, we use discharging, with Lemmas 12.22 and 12.23, to finish the proof.

*Proof of Theorem 12.18.* We give each vertex $v$ initial charge $6 - \frac{5}{2}d(v)$. Note that the sum of these initial charges is $6|G| - 5\|G\| = \rho(V(G)) \geqslant 1$. To reach a contradiction, we discharge so that each vertex finishes with at most 0. We use a single discharging rule: Each 2-vertex sends $\frac{1}{2}$ to each weak neighbor. Now we show that each vertex finishes nonpositive, which contradicts that $\rho(V(G)) \geqslant 1$.

[6]This case can occur, yielding either of the graphs in Figure 12.6.

Recall that $\delta(G) \geqslant 2$. If $d(v) = 2$, then $v$ finishes with $6-5-2(\frac{1}{2}) = 0$. If $d(v) = 3$, then $v$ has at most 3 weak neighbors by Lemma 12.23. So $v$ finishes with at most $6-\frac{5}{2}d(v)+3(\frac{1}{2}) = 0$. If $d(v) \geqslant 4$, then $v$ receives at most 1 from neighbors in each direction (since G has no $3^+$-threads, by Lemma 12.22), so $v$ finishes with at most $6-\frac{5}{2}d(v)+d(v) = 6-\frac{3}{2}d(v) = \frac{3}{2}(4-d(v)) \leqslant 0$. □

**Corollary 12.24.** *If* $G$ *is planar with girth at least* 12*, then* $G$ *has a map to* $C_5$.

*Proof.* Let G be planar with girth at least 12. By Lemma 1.6, $\text{mad}(G) < 2(12)/(12-2) = 12/5$. Thus, $\rho(R) > 0$ for every $R \subseteq V(G)$. So G has no critical subgraph, and G maps to $C_5$. □

## 12.3 Defective coloring

In this section, we study defective 2-coloring. Now each vertex in one color class is allowed one neighbor of the same color.

k-thread
(1, 0)-coloring
colored with i
(1, 0)-colorable
(1, 0)-critical

**Definition 12.25.** Recall that a k-*thread* is a path with k internal vertices, each of degree 2. A $(1,0)$-*coloring* of a graph G partitions $V(G)$ into sets $V_1$ and $V_0$ such that $\Delta(G[V_1]) \leqslant 1$ and $\Delta(G[V_0]) = 0$; that is, $V_1$ induces a matching and $V_0$ is an independent set. For a $(1,0)$-coloring, $v$ *is colored with* $i$ when $v \in V_i$. A graph G is $(1,0)$-*colorable* (simply, *colorable*, for short) if G has a $(1,0)$-coloring, and is $(1,0)$-*critical* (*critical*, for short) if G has no $(1,0)$-coloring, but every proper subgraph does. For each $R \subseteq V(G)$, let[7]

$$\rho_G(R) := 6|R| - 5\|G[R]\|.$$

potential, $\rho_G(R)$

We call $\rho_G(R)$ the potential of R, and we write $\rho(R)$ when the context is clear. Note that $\rho(V(G)) \geqslant 0$ is equivalent to $\overline{d}(G) \leqslant \frac{12}{5}$.

In this section we prove the following theorem. The bound $\rho(V(G)) \leqslant -3$ is sharp, since there exist infinitely many $(1,0)$-critical graphs G with $\rho_G(V(G)) = -3$. Figure 12.8 shows the first two of these, and Exercise 5 asks the reader to construct the rest.

**Theorem 12.26.** *If* $G$ *is* $(1,0)$*-critical, then* $\rho(V(G)) \leqslant -3$*. Thus* $G$ *is* $(1,0)$*-colorable when* $\textit{mad}(G) \leqslant \frac{12}{5}$.

As in the previous section, our proof assumes a minimal counterexample G and uses discharging to reach a contradiction. Clearly $\delta(G) \geqslant 2$ and G is connected. The main reducible configurations are $3^+$-threads and 2-threads not contained in 3-cycles, Lemma 12.32. Using just these configurations, we give an easy discharging argument to show that $\rho(V(G)) \leqslant 0$. More detailed case analysis[8] yields the stronger bound $\rho(V(G)) \leqslant -3$.

[7]The reader should not give much weight to the fact that the potential functions in Sections 12.2 and 12.3 are identical. The result in the present section is sharp infinitely often; thus, it is essentially best possible. However, the result in the previous section is not (it is sharp only for the two graphs in Figure 12.6). It was chosen to strike a balance between strength of result and difficulty of proof. The Notes mention a stronger result with a harder proof.

[8]This is our first example of the potential method where we show that if G is critical, then $\rho(V(G)) \leqslant C$ with $C < 0$. As $|C|$ grows, the extra work required in the discharging phase grows much faster.

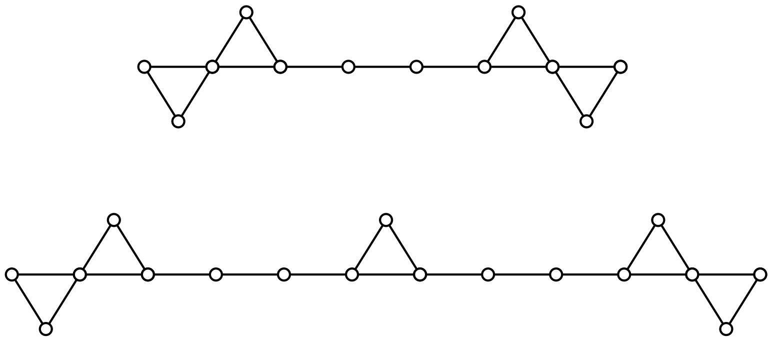


Figure 12.8: The first two $(1,0)$-critical graphs in an infinite family; each has $\rho = -3$.

When proving that threads are reducible, the main intermediate step is our Gap Lemma (Lemma 12.31): $\rho_G(R) \geqslant 1$ whenever $R \subsetneq V(G)$ and $|R| \geqslant 5$ and $R$ does not induce a butterfly (defined below). The idea is similar to the proofs of Lemmas 12.7 and 12.21 in the previous two sections. The main difference is that given a $(1,0)$-coloring of $G[R]$, we collapse $G[R]$ down to a butterfly, rather than a clique. One other difference is that now we refine our partial ordering on graphs, beyond simply ordering by $|G|$ or by $|G| + \|G\|$.

**Definition 12.27.** A *butterfly* $J$ is formed from two vertex-disjoint 3-cycles by identifying one vertex from each. The 5 leftmost vertices on the top of Figure 12.8 induce a butterfly. A butterfly has 5 vertices and 6 edges, so its potential is $6(5) - 5(6) = 0$. Given a graph $G$, a set $R \subsetneq V(G)$ and a $(1,0)$-coloring $\varphi$ of $G[R]$, we form the graph $H(G,R,\varphi)$ from $G$ by identifying all vertices of $R$ colored 0 to form a vertex $x_0$, identifying all vertices of $R$ colored 1 to form a vertex $x_1$, and identifying $x_0$ and $x_1$ with the 4-vertex and some 2-vertex in a butterfly. We also delete any loops and (all but one copy of) each parallel edge formed in this process. Let $B$ denote this butterfly, and let $X := V(B)$.

butterfly

$H(G,R,\varphi)$

$B, X$

A 3-cycle is *special* if it has at least two 2-vertices. A graph $G_2$ is *smaller* than $G_1$ if either (i) $|G_2| < |G_1|$ or (ii) $|G_2| = |G_1|$ and $G_2$ has more special 3-cycles than $G_1$. A smallest counterexample to Theorem 12.26 is a *minimal counterexample*; for short, we say a *minimal* $G$.

special, smaller

minimal counterexample

minimal $G$

Analogous to Lemma 12.6, we have the following.

**Lemma 12.28.** *If* $G$ *is a graph with* $A, B \subseteq V(G)$ *and* $A \cap B = \emptyset$*, then* $\rho_G(A \cup B) = \rho_G(A) + \rho_G(B) - 5|E_G(A,B)|$. *Thus, for all* $A, B \subseteq V(G)$ *(not necessarily disjoint) we have* $\rho_G(A \cap B) + \rho_G(A \cup B) \leqslant \rho_G(A) + \rho_G(B)$.

A short calculation proves the following. We record it for easy reference.

**Lemma 12.29.** *If* $|G| \leqslant 3$ *or* $G$ *is a butterfly, then* $\rho_G(S) \geqslant 0$ *for all* $S \subseteq V(G)$. *If* $|G| = 4$ *and* $G \neq K_4$*, then* $\rho_G(S) \geqslant -1$ *for all* $S \subseteq V(G)$.

The following lemma is key to our approach.

**Lemma 12.30.** *If* $G$ *has no* $(1,0)$*-coloring, then for each set* $R \subsetneq V(G)$ *and each* $(1,0)$*-coloring* $\varphi$ *of* $G[R]$*, graph* $H(G,R,\varphi)$ *also has no* $(1,0)$*-coloring.*

*Proof.* Suppose $\varphi'$ is a $(1,0)$-coloring of $H(G,R,\varphi)$. In every $(1,0)$-coloring of a 3-cycle, exactly one vertex on the 3-cycle is colored 0 and the other two vertices are colored 1. Thus $x_0$ has a neighbor colored 1 in each 3-cycle of the butterfly induced by $X$, so $x_0$ must be colored 0 in $\varphi'$. This implies that $x_1$ is colored 1 in $\varphi'$. To get a $(1,0)$-coloring of $G$, give each vertex of $R$ its color in $\varphi$ and each vertex not in $R$ its color in $\varphi'$. This $(1,0)$-coloring of $G$ contradicts the hypothesis, so instead $H(G,R,\varphi)$ has no $(1,0)$-coloring. □

### 12.3.1 Reducibility

**Lemma 12.31 (Gap Lemma).** *In a minimal* $G$*, if* $R \subsetneq V(G)$ *and* $|R| \geqslant 5$ *and* $R$ *does not induce a butterfly, then* $\rho(R) \geqslant 1$.

$R$ *Proof.* Suppose the lemma is false. Choose $R$ satisfying the hypothesis with $\rho_G(R)$ minimum. So $\rho_G(R) \leqslant 0$. Since $G$ is $(1,0)$-critical and $R$ is a proper subset of $V(G)$, graph $G[R]$ has a $G'$ $(1,0)$-coloring $\varphi$. Let $G' := H(G,R,\varphi)$. Lemma 12.30 implies that $G'$ has no $(1,0)$-coloring. We first show that $G'$ is smaller than $G$, and next show that $\rho_{G'}(S) \geqslant -2$ for all $S \subseteq V(G')$. So $G'$ has a $(1,0)$-coloring by minimality, a contradiction.

If $|R| \geqslant 6$, then $|G'| < |G|$, so $G'$ is smaller than $G$. So assume $|R| = 5$. To prove $G'$ is smaller than $G$, it suffices to show that $G'$ has more special cycles. Since $\rho_G(R) \leqslant 0$, $R$ induces at least 6 edges; we will show that, $G[R]$ contains no special cycles. To see this, we try to construct such a $G[R]$ by starting with a (hypothetical) special cycle $C$ and adding edges until $\|G[R]\| \geqslant 6$. This is impossible, only 1 vertex of $C$ can have more incident edges, but $R$ does not induce a butterfly (by assumption). Clearly $X$ induces two special cycles in $G'$. So to show that $G'$ is smaller than $G$, it suffices to show that every special cycle in $G$ is also special in $G'$.

Consider a special cycle $C$ in $G$. If all vertices of $C$ are outside $R$, then clearly $C$ is also special in $G'$. So assume $C$ has at least one vertex in $R$. Note that $C$ cannot have only a single vertex $v$ outside of $R$. If so, then $\rho_G(R \cup \{v\}) \leqslant \rho_G(R) + 1(6) - 2(5) < \rho_G(R)$. This contradicts the minimality of $R$ unless $R \cup \{v\} = V(G)$; but in that case $\rho_G(V(G)) \leqslant -4$, so $G$ is not a counterexample. Thus, $C$ has at least two vertices outside of $R$. Further, these are both 2-vertices, since each vertex $w$ in $R$ has $d_{G[R]}(w) \geqslant 1$, because $\|G[R]\| \geqslant 6$ and $R$ induces no copy of $K_4$ (since $K_4$ is $(1,0)$-critical). So, $C$ is also special in $G'$, which implies that $G'$ has more special cycles than $G$. Thus, $G'$ is smaller than $G$, as desired.

To show that $G'$ has a $(1,0)$-coloring by minimality, we must show that $\rho_{G'}(R') \geqslant -2$ for all $R' \subseteq V(G')$. Suppose instead that there exists $R' \subseteq V(G')$ with $\rho_{G'}(R') \leqslant -3$. Now[9]

$$\rho_G((R' \setminus X) \cup R) \leqslant \rho_{G'}(R') - \rho_{G'}(R' \cap X) + \rho_G(R). \tag{12.3}$$

Inequality (12.3) follows directly from Lemma 12.28. The proof is nearly the same as that of (12.1), so we omit the details. By assumption, $\rho_G(R) \leqslant 0$. Lemma 12.29 implies that

[9]Again, the picture is nearly identical to Figure 12.3, except that there $X$ induces $K_3$ and here it induces a butterfly.

$\rho_{G'}(R' \cap X) \geqslant 0$. Since $\rho_{G'}(R') \leqslant -3$, inequality (12.3) implies that $\rho_G((R' \setminus X) \cup R) \leqslant -3 - 0 + 0 = -3$, a contradiction. So in fact, $\rho_{G'}(R') \geqslant -2$ for all $R' \subseteq V(G')$. By minimality, $G'$ has a $(1,0)$-coloring, contradicting Lemma 12.30. □

Our next lemma proves the reducibility of our main reducible configuration.

**Lemma 12.32.** *If a minimal* $G$ *contains adjacent* $2$*-vertices* $v$ *and* $w$*, then they are on a special cycle* $uvw$*. In particular,* $G$ *has no* $3^+$*-threads.*

As is standard, we assume that $G$ contains such $v$ and $w$, modify $G$ to get a smaller $G'$, color $G'$ by minimality, and extend the coloring to $G$. Lemma 12.31 helps us show that $\rho_{G'}(R') \geqslant -2$ for all $R' \subseteq V(G')$, which implies that $G'$ is $(1,0)$-colorable.

*Proof.* Suppose to the contrary that $G$ contains a path $uvwx$, where $v$ and $w$ are 2-vertices (and $u \neq x$). Form $G'$ from $G$ by deleting $wx$ and adding $wu$. Note that $G'$ is smaller than $G$, since now $uvw$ is a special cycle. $u, v, w, x$ $G'$

Suppose that $G'$ has a $(1,0)$-coloring $\varphi'$. This gives a partial $(1,0)$-coloring of $G$, with $v$ and $w$ uncolored. We can extend the coloring to $v$ and $w$ as follows. If $\varphi'(u) \neq \varphi'(x)$, then color $v$ with $\varphi'(x)$ and $w$ with $\varphi'(u)$. If $\varphi'(u) = \varphi'(x) = 0$, then color $v$ and $w$ with 1. So assume that $\varphi'(u) = \varphi'(x) = 1$. Since $v$ and $w$ are adjacent in $G'$, either $\varphi'(v) = 1$ or $\varphi'(w) = 1$; thus $u$ has no other neighbor in $G'$ colored 1. So in $G$ we can color $v$ with 1 and $w$ with 0. This gives a $(1,0)$-coloring of $G$, a contradiction.

Thus, we assume that $G'$ has no $(1,0)$-coloring. Now $G'$ has a $(1,0)$-critical subgraph $G''$; let $R' := V(G'')$. We get $\rho_{G'}(R') \leqslant \rho_{G''}(R') \leqslant -3$; the second inequality holds by the minimality of $G$, since $G''$ is smaller than $G$. (Either $|G''| < |G'| = |G|$ or else $|G''| = |G|$; in the second case every special cycle in $G'$ is also special in $G''$, since $G''$ is $(1,0)$-critical, so $\delta(G'') \geqslant 2$.) We must have $\{u, v, w\} \subseteq R'$; otherwise $\rho_G(R') \leqslant \rho_{G'}(R') \leqslant -3$. However, this inequality contradicts either Lemma 12.31 or Lemma 12.29, as we now show. $G''$

Let $S := R' \setminus \{v, w\}$. Now $\rho_G(S) = \rho_{G'}(S) \leqslant \rho_{G'}(R') - 6(2) + 5(3) \leqslant 0$. Note that $x \notin S$, since otherwise $\rho_G(S \cup \{v, w\}) = \rho_{G'}(R') \leqslant -3$, a contradiction. By Lemma 12.31, either $S$ induces a butterfly or $|S| \leqslant 4$. Analogous[10] to $S$, there exists $T \subset V(G) - \{v, w\}$ such that $x \in T$ and $u \notin T$ and $\rho(T) \leqslant 0$. By Lemma 12.28, we have $S$ $T$

$$\rho(S \cup T) + \rho(S \cap T) \leqslant \rho(S) + \rho(T) \leqslant 0. \tag{12.4}$$

Since $u, x \in S \cup T$, we must have $\rho(S \cup T) \geqslant 1$; otherwise $\rho_G(S \cup T \cup \{v, w\}) \leqslant \rho_G(S \cup T) + 2(6) - 3(5) \leqslant -3$, a contradiction. Now (12.4) implies $\rho_G(S \cap T) \leqslant -1$. By Lemmas 12.31 and 12.29, we have $|S \cap T| = 4$ and $\|G[S \cap T]\| = 5$. Since $S \cap T \neq S \cup T$, by symmetry we assume that $|S| \geqslant 5$. But now $S$ cannot induce a butterfly, since $S \cap T$ induces 5 edges among 4 vertices. Thus, by Lemma 12.31, we have $1 \leqslant \rho_G(S) = \rho_{G'}(S)$. This contradicts the fact above that $\rho_{G'}(S) \leqslant 0$. □

[10] We form $G'''$ from $G$ by replacing edge $uv$ with $vx$.

**Lemma 12.33.** *In a minimal* $G$*, no* 3*-vertex lies on a special cycle.*

*Proof.* Suppose to the contrary that $G$ is minimal and $u$ is such a vertex; let $v$ denote its neighbor not on the special cycle $C$. By minimality, $G - V(C)$ has a $(1,0)$-coloring $\varphi'$. To extend $\varphi'$ to $G$, color $u$ to avoid $\varphi'(v)$; now color one 2-vertex on $C$ with 1 and the other with $\varphi'(v)$. This gives a $(1,0)$-coloring of $G$, a contradiction. □

### 12.3.2 Discharging

To show that no minimal $G$ exists, we use discharging with initial charge $5d(v) - 12$. Now $-2 \leqslant \rho(V(G)) = 6|G| - 5\|G\|$, so $\sum_{v \in V(G)} 5d(v) - 12 \leqslant 4$. Since $\delta(G) \geqslant 2$, we mainly just need to get enough charge to 2-vertices. To reach a contradiction, we discharge so that every vertex is happy and a few vertices have extra charge, summing to more than 4.

**Lemma 12.34.** *Consider a minimal* $G$*. If we give each vertex* $v$ *charge* $5d(v) - 12$ *and discharge by (R1) below, then every vertex is happy. Further, every* $3^+$*-vertex* $v$ *ends with at least* 1*, unless* $v$ *is a* 3*-vertex incident to three* 1*-threads.*

(R1) *Every* 2*-vertex on a* 1*-thread takes* 1 *from each neighbor. Every* 2*-vertex on a special cycle takes* 2 *from its* $3^+$*-neighbor.*

*Proof.* Clearly each 2-vertex ends happy. Suppose $d(v) = 3$. By Lemma 12.33, $v$ cannot lie on a special cycle, so $v$ loses 1 to each 2-neighbor, and ends with at least $5(3) - 12 - 3(1) = 0$. Further, this inequality is strict unless $v$ is incident to three 1-threads. Suppose instead that $d(v) \geqslant 4$. Since $v$ loses at most 2 to each neighbor, $v$ ends with at least $5d(v) - 12 - 2d(v) = 3d(v) - 12 = 3(d(v) - 4) \geqslant 0$. Equality holds only if $v$ is a 4-vertex on two special cycles. But then $G$ must be a butterfly, a contradiction. So each $4^+$-vertex ends positive. □

Now we are nearly done. Since the final charges sum to at most 4 (as we saw above), Lemma 12.34 implies that $G$ is very nearly a bipartite graph with 2-vertices in one part and 3-vertices in the other. Such a graph has an obvious $(1,0)$-coloring; color the 3-vertices with 1 and the 2-vertices with 0. Although we still have some cases to check, what remains is really just tying up loose ends; we have already completed the heart of the proof. We now show that $G$ always has a $(1,0)$-coloring that is very close to the one just described.

tough

$s(v)$, $t(v)$

**Definition 12.35.** An edge $uv$ is *tough* if $u$ and $v$ are both $3^+$-vertices. For each vertex $v$, let $s(v)$ denote the number of special cycles containing $v$ and let $t(v)$ denote the number of tough edges incident to $v$ (here $s$ is for special and $t$ is for tough).

**Lemma 12.36.** *No minimal* $G$ *exists. So Theorem 12.26 is true.*

*Proof.* Recall: $\rho_G(V(G)) \leqslant -2$, so $\sum_{v\in V(G)}(5d(v)-12) = 10\|G\|-12|G| = -2\rho_G(V(G)) \leqslant -2(-2) = 4$. Consider a partial coloring $\varphi$ that colors each $3^+$-vertex with 1 and each 2-vertex on a 1-thread with 0. We can extend $\varphi$ to a $(1,0)$-coloring of G unless there exists a vertex $v$ such that

$$s(v) + t(v) \geqslant 2.$$

Suppose there exists such a $v$. We show that if G has no $(1,0)$-coloring, then the charges sum to more than 4, a contradiction.

**Case 1: $d(v) \geqslant 6$.** Now $v$ ends with at least $5d(v) - 12 - 2d(v) = 3(d(v) - 4) \geqslant 6$. Since each other vertex ends happy, the sum of the charge is at least 6, a contradiction.

**Case 2: $d(v) = 5$.** If $s(v) \leqslant 1$, then $v$ ends with at least $5d(v) - 12 - 4 - (d(v) - 2) = 4d(v)-14 = 6$. If $t(v) \geqslant 1$, then $v$ ends with at least $5d(v)-12-2(d(v)-1) = 3d(v)-10 = 5$. Each case is a contradiction. So assume that $s(v) = 2$ and $v$ has a fifth 2-neighbor, not on a special cycle. Since $v$ ends with 4, Lemma 12.34 implies that every other $3^+$-vertex is a 3-vertex incident to three 1-threads. So we color $v$ with 0 and all of its neighbors with 1 (and all 3-vertices with 1 and all other 2-vertices with 0). This gives a $(1,0)$-coloring of G.

**Case 3: $d(v) = 4$.** Since G is connected, $s(v) \leqslant 1$. If $t(v) \geqslant 2$, then $v$ ends with at least 4, and two of its neighbors end positive, a contradiction; so $t(v) \leqslant 1$. Thus $s(v) = t(v) = 1$. Now again color $v$ with 0 and its neighbors with 1 (and the other vertices as in Case 2).

**Case 4: $d(v) = 3$.** Lemma 12.33 implies that $s(v) = 0$. If $t(v) = 3$, then $v$ ends with 3 and each of its neighbors ends positive, a contradiction. So assume $t(v) = 2$. Now $v$ ends with 2 and its two neighbors along tough edges end positive. So no other vertex $w$ has $s(w) + t(w) \geqslant 2$. Now we again color $v$ with 0 and color its 2-neighbor with 1 (and the other vertices as above). □

## 12.4 Two More Gaps: $(I^*, F)$-coloring and $K_3$-free 4-critical graphs

For our next two example, we do not give full proofs. In particular, we omit most details of the reducible configurations and the discharging, which should be familiar by now. Instead, we emphasize steps 1–5 in the list from Section 12.1.2. We specifically focus on the proofs of the Gap Lemmas, which nicely illustrate a few tricks that are common in such proofs.

### 12.4.1 $(I^*, F)$-coloring

In this section, we want to color a graph G with colors I and F such that the vertices colored F induce a forest and those colored I induce a 2-*independent set*; that is, $\text{dist}(x, y) > 2$ for each pair $x, y$ colored I. Such a coloring is an $(I^*, F)$-*coloring* (we use $I^*$ as a reminder that this set is not just independent, but actually 2-independent). We define $(I^*, F)$-*colorable* and $(I^*, F)$-*critical* in the natural way. But for short we often just write *colorable* and *critical*.

2-independent set

$(I^*, F)$-coloring

We begin by looking for the sparsest critical graphs. A *paw* is formed from $K_3$ by adding a pendent edge at one vertex. Consider the graph $G_t$ formed from the cycle $C_t$ by identifying

paw

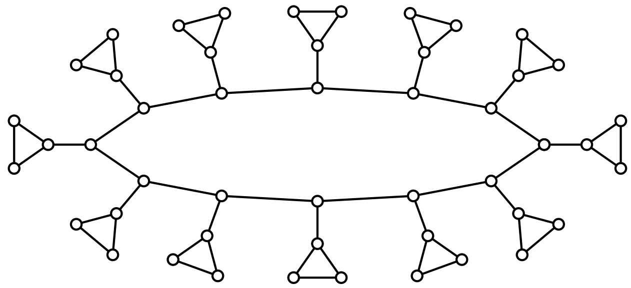


Figure 12.9: An $(I^*, F)$-critical graph with $\rho = 0$. To get other similar examples, we can start with any cycle and add a pendent paw at each vertex of the cycle.

each vertex with the 1-vertex in a paw. If $G_t$ was colorable, then some vertex $v$ on the long cycle would be colored I, and each other vertex in the paw belonging to $v$ would be colored F, a contradiction. So $G_t$ has no coloring. It is easy to check that $G_t$ is indeed critical; $G_t$ has four equivalence classes of edges, and deleting an edge from any one of these allows a coloring. Since $\|G_t\| = 5|G|/4$, we get that each $G_t$ has average degree $5/2$. This motivates our potential function.

For each $R \subseteq V(G)$, let

$$\rho^0(R) := 5|R| - 4\|G[R]\|.$$

So we naturally conjecture the following theorem.

**Theorem 12.37.** *If* $G$ *is* $(I^*, F)$*-critical, then* $\rho^0(V(G)) \leqslant 0$.

Since colors I and F are not equivalent, it will be nice to extend the theorem to allow precoloring, as mentioned in step (4) of the outline in Section 12.1.2. We write $I_0$ and $F_0$ (and $U_0$) to denote the sets of vertices in a precolored graph that are already colored I or F (and $U_0 := V(G) \setminus (I_0 \cup F_0)$). By examining our sharpness examples, we see that the paw serves as an F-gadget. More precisely, every coloring of the paw colors its 1-vertex with F. The paw has potential $5(4) - 4(4) = 4$, which suggests the more general potential function $\rho(R) := 5|R \cap U_0| + 4|R \cap F_0| + a|R \cap I_0| - 4\|G[R]\|$. A natural way to guarantee that I is used on an uncolored vertex $v$ is to include $v$ in a 3-cycle with two vertices colored F. The potential of such a precolored graph is $5(1) + 4(2) - 4(3) = 1$. So we try 1 for the weight (coefficient in $\rho$) of each vertex precolored I.

Before we can state our generalization to precolored graphs, we must extend our definition of critical graphs. Formally, a *precolored graph* $G$ is a graph with vertex subsets $I_0$ and $F_0$, where $I_0 \cap F_0 = \emptyset$. An $(I^*, F)$-coloring $\varphi$ of a precolored $G$ is an $(I^*, F)$-coloring of $G$ such that
critical $\varphi(v) = I$ for all $v \in I_0$ and $\varphi(v) = F$ for all $v \in F_0$. A precolored graph is *critical* if (i) it has no $(I^*, F)$-coloring, (ii) $G - e$ has an $(I^*, F)$-coloring for each $e \in E(G)$, and (iii) $G'$ has an $(I^*, F)$-coloring whenever $G'$ is formed from $G$ by removing the color on any vertex $v$ in $I_0 \cup F_0$. Condition (iii) corresponds to removing some edge from the gadget enforcing the precoloring

on $v$. (It is easy to check that both the F-gadget and the I-gadget are edge-minimal in the following sense: for any vertex $v$ and edge $e$ in the gadget $H$, we have valid colorings $\varphi_1$ and $\varphi_2$ of $H - e$ with $\varphi_1(v) = I$ and $\varphi_2(v) = F$.) Now we can state our theorem.[11]

**Theorem 12.38.** *Let*

$$\rho_G(R) := 5|R \cap U| + 4|R \cap F| + |R \cap I| - 4\|G[R]\|.$$

*If* $G$ *is a precolored* $(I^*, F)$*-critical graph, then* $\rho_G(V(G)) \leqslant 0$.

We assume the theorem is false and choose $G$ to be a counterexample minimizing $|G| + \|G\|$. We want to prove a Gap Lemma of the form: If $R \subsetneq V(G)$ and $|R| \geqslant b$, then $\rho(R) \geqslant c$, for some $b, c \in \mathbb{Z}^+$. As in the previous examples, we assume $R$ contradicts the Gap Lemma, find a coloring $\varphi$ of $G[R]$ by criticality, and contract $G[R]$ to some small subgraph $H$ (maintaining edges from $V(H)$ to $V(G) \setminus R$). One point of allowing precoloring is that we can simply take $H$ to consist of two non-adjacent vertices, one colored I and one colored F. But this doesn't quite work. Since vertices of $R$ colored F by $\varphi$ might have neighbors colored I, our vertex colored F in $H$ must also have a neighbor in $H$ colored I. So a natural choice for $H$ is an edge with its endpoints colored I and F. Such an $H$ would allow us to prove some Gap Lemma. But a more subtle choice for $H$ allows us to do slightly better (which will make all the difference in finishing the proof of the theorem).

We hope to prove a gap of 2, that is, to show that $\rho(R) > 2$, for each $R \subsetneq V(G)$ satisfying some mild assumptions. Clearly, if $R$ consists of one or two isolated vertices in $I_0$, then $\rho(R) \leqslant 2$. There are also a few other small examples with small potential. However, the following is true.

**Lemma 12.39.** *Fix* $R \subsetneq V(G)$ *with* $R \neq \emptyset$ *and* $|R| + \|G[R]\| \leqslant 4$. *If* $\rho(R) \leqslant 2$*, then* $G[R]$ *is one of* 5 *precolored graphs, each with* $\rho(R) \geqslant 1$. *If also* $R \cap U_0 \neq \emptyset$*, then* $|R \cap U_0| = 1$ *and* $\rho(R) \geqslant 2$.

Note that $G[R]$ above is a subgraph with at most one edge. Thus, the proof consists of a short case analysis, which we omit it.

**Lemma 12.40 (Gap Lemma).** *If* $R \subsetneq V(G)$ *and* $|R| + \|G[R]\| \geqslant 5$*, then* $\rho(R) \geqslant 3$.

*Proof.* Suppose the lemma is false, and choose $R$ to minimize $\rho(R)$; subject to this, choose $R$ to maximize $|R|$. Following our standard approach, we find a coloring $\varphi$ of $G[R]$ by criticality, and contract the color classes of $\varphi$ to single vertices, $v_F$ and $v_I$, which we precolor as U and I; as $v_F, v_I$

[11] It is enlightening to note that Theorem 12.38 is logically equivalent to its restriction to graphs that are not precolored. To see this, start with a precolored graph and replace each vertex in $I_0 \cup F_0$ with a vertex in $U_0$ identified with the corresponding gadget; the resulting graph has a coloring if and only if the original graph does. However, phrasing the theorem in terms of precolorings allows us to more simply state our partial ordering on the graphs; a precolored graph $G_1$ is smaller than a precolored graph $G_2$ if and only if $|G_1| + \|G_1\| < |G_2| + \|G_2\|$. (Without the notion of precolored graphs, it would be more complicated to stipulate that starting with an induced subgraph $H$ of a graph $G$ and adding pendent gadgets at various vertices in $G$ necessarily yields a graph that is smaller than $G$.)

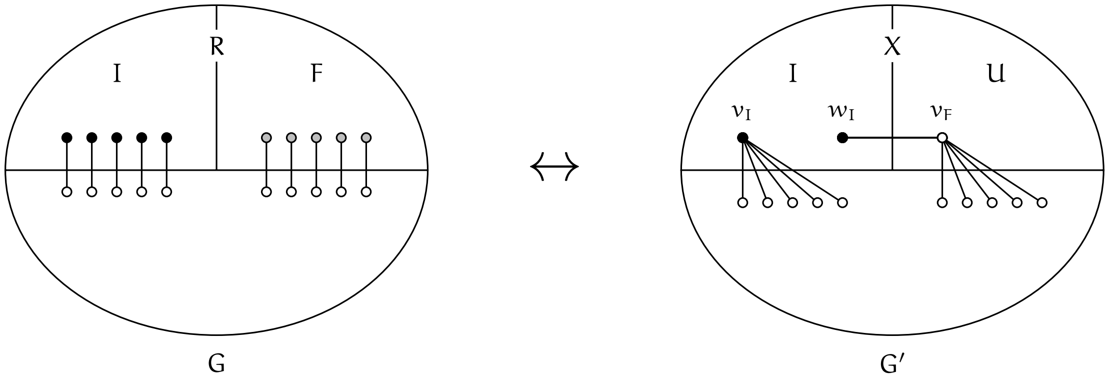



Figure 12.10: The proof of the Gap Lemma (Lemma 12.40).

always, we preserve edges from $R$ to $V(G) \setminus R$, except that we suppress all but one in each set of resulting parallel edges. We also add an edge from $v_F$ to another vertex precolored I, which we call $w_I$; this ensures both that $v_F$ will be colored F and that no neighbor of $v_F$ in $V(G) \setminus R$ is colored I. See Figure 12.10. (Of course, we could precolor $v_F$ with F, but that decreases the potential without any benefit.) Let $X := \{v_F, v_I, w_I\}$; so the subgraph induced by $X$ is an edge and an isolated vertex $v_I$. Call the resulting graph $G'$. If $G'$ is colorable, then so is $G$, a contradiction; this relies on the fact that every $v \in V(G) \setminus R$ has at most one neighbor in $R$, since otherwise[12] $\rho(R \cup \{v\}) \leqslant \rho(R) + 5 - 4(2) < \rho(R)$. So $G'$ has a critical subgraph $G''$, and we let $S := V(G'')$. As usual, we have

$w_I$ X G′ G″, S

$$\rho_G((S \setminus X) \cup R) \leqslant \rho_{G'}(S) - \rho_{G'}(S \cap X) + \rho_G(R). \tag{12.5}$$

Since $|R| + \|G[R]\| \geqslant 5$, we have $|G''| + \|G''\| < |G| + \|G\|$. Thus, the minimality of $G$ gives $\rho(S) \leqslant 0$; by assumption, $\rho(R) \leqslant 2$. If we can show that $\rho(S \cap X) \geqslant 2$, then we are done, as follows. Either $(S \setminus X) \cup R \neq V(G)$, so $\rho((S \setminus X) \cup R) \leqslant \rho(S) - \rho(S \cap X) + \rho(R) \leqslant 0 - 2 + \rho(R)$ contradicts the minimality of $R$, or else $(S\setminus X) \cup R = V(G)$, so $\rho(V(G)) \leqslant \rho(S) - \rho(S \cap X) + \rho(R) \leqslant 0 - 2 + 2 = 0$, which shows that $G$ is not a counterexample. Thus, we must handle the case that $\rho(S \cap X) \leqslant 1$.

It is easy to check that $\rho_{G'}(Y) \geqslant 1$ for each nonempty $Y \subseteq X$. In fact, since $G[S]$ is connected, and $N(w_I) = \{v_F\}$, we cannot have $S \cap X = \{w_I\}$. So $\rho_{G'}(S \cap X) = 1$ if and only if $S \cap X = \{v_I\}$ (here we use that $v_F$ is precolored U rather than F). Thus, we must ensure this is not the case. If $\rho(R) \leqslant 1$, then $\rho(R) \leqslant \rho(S \cap X)$ and we again reach a contradiction as in the previous paragraph. So assume $\rho(R) = 2$. Let $\nabla(R) := R \cap (N(V(G) \setminus R))$. If $\nabla(R) \subseteq I_0$ and $R \neq \nabla(R)$, then by criticality we can color both $G[R]$ and $G \setminus (R \setminus \nabla(R))$. Since these colorings agree on $\nabla(R)$, and $\nabla(R) \subseteq I_0$, the two colorings combine to give a coloring of $G$, a contradiction. If $\nabla(R) \subseteq I_0$ and $R = \nabla(R)$, then $G[R]$ is edgeless, so $\rho(R) = |R| \geqslant 5$, and we

[12]This argument works if $|R| \leqslant |G| - 2$. Similarly, if $|R| = |G| - 1$ and $v \notin R$ but $d_R(v) \geqslant 2$, then $\rho_G(V(G)) = \rho(R \cup \{v\}) \leqslant \rho(R) + 5 - 4(2) \leqslant 2 + 5 - 4(2) = -1$ so, contrary to our assumption, $G$ is not a counterexample.

are done. So we assume there exists $z \in \nabla(R) \setminus I_0$. If $z$ is precolored F, then we do nothing. But if $z$ is precolored U, then before we color $G[R]$ by criticality, we precolor $z$ with F. Call this modified (precolored) graph $G_1[R]$. $z$ $G_1[R]$

We must show that $G_1[R]$ has a coloring. The key observation is that precoloring $z$ with F decreases the potential of $z$ by exactly 1, and so it also decreases the potential of each vertex subset containing $z$ by 1. If $Q \subseteq R$ and $|Q| + \|G[Q]\| \geqslant 5$, then $\rho_G(Q) \geqslant \rho_G(R) \geqslant 2$ by the minimality of our choice of R. Thus, $\rho_{G_1}(Q) \geqslant \rho_G(Q) - 1 \geqslant 1$. Suppose instead that $|Q| + \|G[Q]\| \leqslant 4$. If $z \notin Q$, then $\rho_{G_1}(Q) = \rho_G(Q) \geqslant 1$, by Lemma 12.39. If $z \in Q$, then $Q \cap U_0 \neq \emptyset$. Now Lemma 12.39 gives $\rho_G(Q) \geqslant 2$, so $\rho_{G_1}(Q) \geqslant 2 - 1 = 1$. Thus, for every nonempty $Q \subseteq R$, we have $\rho_{G_1}(Q) \geqslant 1$. So $G_1[R]$ does not contain a critical subgraph, and thus has a coloring, as desired.

Pick $z' \in N(z) \setminus R$. If $v_F \in S$, then $\rho_{G'}(S \cap X) \geqslant 2$, as above, so we are done. Suppose instead that $v_F \notin S$. Now we can improve the bound in (12.5) by 4, since the edge $zz'$ is counted on the left, but not on the right. Thus, $\rho((S \setminus X) \cup R) \leqslant 0 - 1 + \rho(R) - 4 < \min\{0, \rho(R)\}$, a contradiction. □ $z'$

As we mentioned above, we omit most of the reducibility lemmas, as well as the discharging argument. However, we do prove one lemma that illustrates how the Gap Lemma is used for proving reducibility of various configurations. Its proof is similar to the case in the proof of the Gap Lemma when $z$ is precolored with U and we recolor it with F.

**Lemma 12.41.** *Fix* $R \subsetneq V(G)$. *If* $G'$ *is formed from* $G[R]$ *by moving at most two vertices from* $U_0$ *to* $F_0$*, then* $G'$ *is colorable.*

*Proof.* Define R and $G'$ as above and fix $v, w \in V(G')$ such that each vertex moved from $U_0$ to $F_0$ when forming $G'$ is in $\{v, w\}$. Let $U_1$ and $F_1$ denote the modified versions of $U_0$ and $F_0$. Since $|G'| + \|G'\| < |G| + \|G\|$, by the minimality of G it suffices to show that $\rho_{G'}(S) \geqslant 1$ for each $S \subseteq V(G')$. Fix $S \subseteq V(G')$.

First suppose that $|S| + \|S\| \geqslant 5$. Now $\rho_{G'}(S) \geqslant \rho_G(S) - 2(5) + 2(4) \geqslant 3 - 2 = 1$, where the final inequality holds by the Gap Lemma. So suppose instead that $|S| + \|S\| \leqslant 4$. If $\rho_G(S) \geqslant 3$, then the proof is nearly identical. So assume that $\rho_G(S) \leqslant 2$. Now Lemma 12.39 implies that (a) $\rho_G(S) \geqslant 1$ and (b) if $S \cap U_0 \neq \emptyset$, then $|S \cap U_0| = 1$ and $\rho_G(S) = 2$. So if $S \cap U_0 \neq \emptyset$, then $\rho_{G'}(S) = \rho_G(S) - 5(1) + 4(1) \geqslant 2 - 1 = 1$. Otherwise, $S \cap U_0 = \emptyset$, so $\rho_{G'}(S) = \rho_G(S) \geqslant 1$. Thus, $G'$ contains no critical subgraph, so is colorable. □

We now highlight a few similarities between the proof of Lemma 12.40 and those of previous Gap Lemmas. First, Lemma 12.39 should come as no surprise, both that there are small exceptions to the Gap Lemma and that we handle all small subgraphs before proving the Gap Lemma. This lemma is analogous to Lemma 12.5, Lemma 12.19, and Lemma 12.29 (as well as Lemma 12.58). Second, our choice to precolor $z$ with F is quite similar to the proof of Lemma 12.11, where we add an edge between two vertices in $\nabla(R)$ before coloring $G[R]$. In each case, the goal of our modification is to ensure that $\rho(S \cap X)$ is larger than it would be otherwise, which helps prove a stronger Gap Lemma. So Lemma 12.40 proves an analogue of both Lemmas 12.7 and 12.11 all at once, skipping right to a "strong" Gap Lemma.[13]

### 12.4.2 Triangle-free 4-critical Graphs

In Section 12.1, we proved that every 4-critical graph $G$ has $\|G\| \geqslant (5|G|-2)/3$. All of our sharpness examples had lots of triangles, so perhaps by restricting to triangle-free graphs we can prove a stronger lower bound on $\|G\|$. The following example shows that we cannot improve the coefficient on $|G|$. So we will focus on a larger constant term.

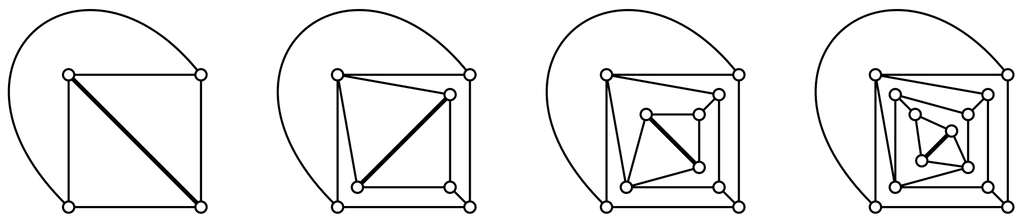


Figure 12.11: $K_4$ and the next three graphs in an infinite family of 4-critical planar graphs with exactly four triangles; see Example 12.42 (each bold edge is replaced by a copy of $J$ in the next graph).

**Example 12.42.** We construct an infinite family of 4-critical (planar) graphs with two copies of $K_4 - e$ and every cycle that is edge-disjoint from these two copies having length at least 4; see
$J$ Figure 12.11. (Further, every face disjoint from these two copies of $K_4 - e$ has length 5.) Let $J$ denote the graph formed from $K_4 - e$ by adding a pendent edge at a vertex of degree 2. Denote by $x$ and $y$ the vertices of degrees 1 and 2 in $J$. To form $G_1$, start from $K_4$, remove an edge
$x, y$ $xy$, and identify vertices $x$ and $y$ in the $K_4$ with the vertices of the same names in $J$. For each
$G_i$ $i \geqslant 2$, form $G_i$ from $G_{i-1}$ by choosing some edge $xy$ that lies on two 3-cycles and replacing it with a copy of $J$, as explained above.

By induction, we prove that $G_i$ is 4-critical. First, we show that it is not 4-colorable. The copy of $J$ ensures that $G_i$ has no 3-coloring $\varphi$ with $\varphi(x) = \varphi(y)$. Since $G_{i-1}$ is not 3-colorable, the copy of $G_{i-1} - xy$ ensures that $G_i$ has no 3-coloring $\varphi$ with $\varphi(x) \neq \varphi(y)$. Thus, $G_i$ has no 3-coloring. Now we show that $G_i - e$ is 3-colorable, for each edge $e$. If $e \in E(J)$, then $G_i - e$ has a 3-coloring $\varphi$ with $\varphi(x) = \varphi(y)$. And if $e \in E(G_i) \setminus E(J)$, then $G_i - e$ has a 3-coloring $\varphi$ with $\varphi(x) \neq \varphi(y)$; this uses that $G_{i-1}$ is 4-critical.

We omit a formal proof that $G_i$ has no 3-cycles disjoint from the two copies of $K_4 - e$. But this can also be proved easily by induction on $i$. With each inductive step we destroy the two 3-cycles in one copy of $K_4 - e$ and create two new 3-cycles in a new copy of $K_4 - e$. Similarly,

[13] An astute reader might ask why we were able to do this all at once in Lemma 12.40 but needed two steps in Lemmas 12.7 and 12.11. In fact, the difference is purely cosmetic. We proved Lemmas 12.7 and 12.11 separately to keep the argument as accessible as possible, particularly since it was our first example. However, it is an instructive exercise to reprove those lemmas all at once, following the model in the proof of Lemma 12.40.

we omit the proof by induction that $\|G_i\| = (5|G_i| - 2)/3$. Intuitively, this is true for $K_4$ and with each inductive step we add 3 vertices and 5 edges.

It is worth noting that we can slightly modify the $G_i$ to get triangle-free graphs $G'_i$ with $\|G'_i\| = \frac{5}{3}|G'_i| + M$, for some constant $M$. The idea is to replace each of the two copies of $J$ in $G_i$ with $H - e$ for some triangle-free 4-critical graph $H$. A reasonable choice for $H$ is the 11-vertex Grötzsch graph, formed from $C_5$ by a single application of Mycielski's construction (see Exercise 2(b)). ♢

Example 12.42 tell us that we should probably still use $\rho(R) := 5|R| - 3\|G[R]\|$. But now we encounter a significant problem. To prove our Gap Lemma, we form a new graph $G'$ by contracting some subgraph $G[R]$ down to a copy of $K_3$. Now clearly $G'$ will no longer be triangle-free. In fact, $G'$ may have arbitrarily many triangles! But not all is lost.

What if we prove a more general result that accounts for the number of triangles? To get some control on the number of triangles in $G'$, we count only *vertex disjoint* triangles. Specifically, for a graph $H$, let $\mathcal{T}(H)$ denote the maximum number of vertex-disjoint triangles in $G$. For each $R \subseteq V(G)$, we also write $\mathcal{T}(R)$ to denote $\mathcal{T}(G[R])$. We briefly discuss the graph $W_5$ and the graph class $\mathcal{B}$ after stating the theorem. $\mathcal{T}(H), \mathcal{T}(R)$

The class of 4-*Ore* graphs plays a key role in the statement of our next result, as well as the corresponding Gap Lemma. We formally define k-Ore graphs in Definition A.11, and discuss some of their properties in Section A.11.1. Informally, $K_k$ is a k-Ore graph and each larger k-Ore graph is formed by "gluing" together two smaller k-Ore graphs in a specified way. As an example, the *Moser spindle* is the unique 4-Ore graph of order 7; see the top center of Figure 12.12. It is formed from two copies of $K_4 - e$ by identifying one 3-vertex from each copy and adding an edge between the two remaining 3-vertices. All k-Ore graphs are k-critical, and these are the sparsest k-critical graphs. Further, k-Ore graphs have many nice properties, and nearly all of them can be proved by induction on the order of the graph. 4-Ore Moser spindle

**Theorem 12.43.** *Let $G$ be a graph and let $\rho(R) := 5|R| - 3\|G[R]\| - \mathcal{T}(R)$ for each $R \subseteq V(G)$. (For a subgraph $J$ of $G$, we often write $\rho(J)$ to denote $\rho(V(J))$.) If $G$ is 4-critical, then* $\rho(R)$

- $\rho(G) = 1$ *if* $G = K_4$,
- $\rho(G) = 0$ *if* $\mathcal{T}(G) = 2$ *and* $G$ *is 4-Ore,*
- $\rho(G) = -1$ *if* $G = W_5$, *or* $G \in \mathcal{B}$, *or* $G$ *is 4-Ore with* $\mathcal{T}(G) = 3$, *and*
- $\rho(G) \leqslant -2$ *otherwise.*

Here $\mathcal{B}$ is a particular class of graphs with $\mathcal{T}(G) = 2$, and $W_5$ is formed from $C_5$ by adding a dominating vertex; see the top right of Figure 12.12. This theorem immediately implies the following corollary (which is actually what we wanted to prove). $\mathcal{B}, W_5$

**Corollary 12.44.** *If $G$ is 4-critical and triangle-free, then $\|G\| \geqslant (5|G| + 2)/3$.*

*Proof.* If $\rho(G) \geqslant -1$ then $G$ has triangles, so $\rho(V(G)) \leqslant -2$; algebra yields the result. □

A complete proof of Theorem 12.43 is long. However, most of the work goes into (a) proving lemmas about 4-Ore graphs and (b) proving structural lemmas that help with the discharging. Here, we simply prove the Gap Lemma. As usual, we assume the theorem is false, and choose $G$ to be a counterexample minimizing $|G|$. It easy to check that the theorem is true for the graphs in the first three bullet points.[14] Our counterexample $G$ has $\rho(G) \geqslant -1$.

For reference, we record the following facts, which are proved by simple calculations.

**Lemma 12.45.** *We have* $\rho(K_1) = 5$*,* $\rho(K_2) = 7$*, and* $\rho(K_3) = 5$.

We will also need the following lemmas about the structure of 4-Ore graphs. They are straightforward to prove by induction on $|G|$, so we defer the proofs to the exercises.

**Lemma 12.46.** *If* $J$ *is a* 4*-Ore graph and* $J \neq K_4$*, then for any copy* $K$ *of* $K_3$*, the subgraph* $J - K$ *also contains a copy of* $K_3$.

kite

A *kite* in $G$ is a copy $J$ of $K_4 - e$, such that each 3-vertex in $J$ is also a 3-vertex in $G$. (This departs from common terminology, by requiring the 3-vertices in $J$ to also be 3-vertices in $G$.)

**Lemma 12.47.** *If* $G$ *is a* 4*-Ore graph with* $\mathcal{T}(G) = 2$*, then either* $G$ *is the Moser spindle or* $G$ *contains two disjoint kites.*

**Lemma 12.48 (Gap Lemma).** *If* $R \subsetneq V(G)$ *and* $R \neq \emptyset$*, then* $\rho(R) \geqslant 3$*. Furthermore,* $\rho(R) \geqslant 4$ *unless one of the following holds: (i)* $G - R$ *is a triangle of* 3*-vertices (in* $G$*), (ii)* $V(G) \setminus R$ *is a single* 3*-vertex, or (iii)* $G$ *contains a kite.*

R, G′, X, H, S

*Proof.* Suppose the lemma is false. Among all subsets witnessing this, choose $R$ to maximize $|R|$ and, subject to this, to minimize $\rho(R)$. Let $\varphi$ be a 3-coloring of $G[R]$ and form $G'$ from $G$ by contracting each color class of $\varphi$ to a single vertex and adding edges between each of these new vertices (suppressing all but one in any set of parallel edges). As usual, let $X$ denote the set of these three new vertices. If $G'$ has a coloring, then so does $G$, so assume that $G'$ contains a critical subgraph $H$, and let $S := V(H)$. Now we have

$$\rho((S \setminus X) \cup R) \leqslant \rho(R) + \rho(S) - \rho(S \cap X) - \mathcal{T}(S \cap X) + \mathcal{T}(S) - \mathcal{T}(S \setminus X) \tag{12.6}$$

$$\leqslant \rho(R) + \rho(S) - \rho(S \cap X) - \mathcal{T}(S \cap X) + |S \cap X|. \tag{12.7}$$

The proof of (12.6) and (12.7) is similar to previous examples. The main difference is the $\mathcal{T}$ terms. Here, we simply observe that $\mathcal{T}((S \setminus X) \cup R) \geqslant \mathcal{T}(S \setminus X) + \mathcal{T}(R)$ and that $\mathcal{T}(S) - \mathcal{T}(S \setminus X) \leqslant |S \cap X|$, since every triangle has vertex set either contained in $S \setminus X$ or intersecting $X$. The rest is algebra.

One useful consequence of inequality (12.7) is that $(S \setminus X) \cup R = V(G)$. Suppose it is not. Now we show that $\rho((S \setminus X) \cup R) \leqslant \rho(R)$, which contradicts our choice of $R$. This follows immediately from (12.7) and the facts that (a) $\rho(S) \leqslant 1$, (b) $\rho(S \cap X) \geqslant 5$, (c) $\mathcal{T}(S \cap X) \geqslant 0$, and (d) $|S \cap X| \leqslant 3$. Here (a) holds because $|S| < |G|$ and $G$ is a minimal counterexample, (b) holds by Lemma 12.45,

[14] Every $n$-vertex 4-Ore graph has $(5n-2)/3$ edges, and every $n$-vertex graph in $\mathcal{B}$ has $(5n-1)/3$ edges.

since $G[X] = K_3$, and (c) and (d) hold trivially. So $\rho((S \setminus X) \cup R) \leqslant \rho(R) + 1 - 5 - 0 + 3 < \rho(R)$. Thus, $(S \setminus X) \cup R = V(G)$. So $\rho((S \setminus X) \cup R) = \rho(V(G)) \geqslant -1$, since G is a counterexample. Now we consider 6 possibilities for H. Throughout, let $s := |S \cap X|$. s

**Case 1: $H = K_4$.** Suppose that $s = 1$. Now $\mathcal{T}(S) = \mathcal{T}(K_4) = 1 = \mathcal{T}(S \setminus X)$, so (12.6) simplifies to $-1 \leqslant \rho(R) + 1 - 5 - 0 + 0 = \rho(R) - 4$. This immediately gives $\rho(R) \geqslant 3$. If $\rho(R) \geqslant 4$, then we are done, so assume that $\rho(R) = 3$. Now $\rho(S \setminus X) = \rho(K_3) = 5$. So $-1 \leqslant \rho(G) \leqslant \rho(S \setminus X) + \rho(R) - 3|E_G(R, \overline{R})|$. If $|E_G(R, \overline{R})| \geqslant 4$, then $\rho(G) \leqslant 5 + 3 - 3(4) = -4$, a contradiction. So instead $|E(R, V(G) \setminus R)| \leqslant 3$. Since $\delta(K_4) = 3$, each vertex of $V(G) \setminus R$ has a single neighbor in R. So we are in case (i) of the lemma.

Suppose instead that $s = 2$. Now $\rho(S \cap X) = 7$, $\mathcal{T}(S) = \mathcal{T}(K_4) = 1$, and $\mathcal{T}(S \setminus X) = 0$. So (12.6) simplifies to $-1 \leqslant \rho(G) \leqslant \rho(R) + 1 - 7 + 1$. Thus, $\rho(R) \geqslant 4$.

Finally, assume that $s = 3$. Now $\rho(S \cap X) = 5$, $\mathcal{T}(S) = \mathcal{T}(K_4) = 1$, and $\mathcal{T}(S \setminus X) = 0$. So (12.6) simplifies to $-1 \leqslant \rho(G) \leqslant \rho(R) + 1 - 5 - 1 + 1 - 0 = \rho(R) - 4$. Thus, $\rho(R) \geqslant 3$. If $\rho(R) = 4$, then we are done, so assume $\rho(R) = 3$. As in the case $s = 1$ above, we must have $|E_G(R, \overline{R})| = 3$. But now we are in case (ii) of the lemma.

**Case 2: H is 4-Ore with $\mathcal{T}(H) = 2$.** Note that $\rho(S) = 0$.

Suppose $s = 1$. First, $-1 \leqslant \rho(R) + 0 - 5 - 0 + 1$, so $\rho(R) \geqslant 3$. If H is the Moser spindle and $S \cap X$ is the unique 4-vertex in H, then $\mathcal{T}(S \setminus X) = \mathcal{T}(S) = 2$. So $-1 \leqslant \rho(R) + 0 - 5 - 0 + 0$, and $\rho(R) \geqslant 4$. If H is the Moser spindle and $S \cap X$ is not the unique 4-vertex in H, then G contains a kite, and we are in case (iii) of the lemma. So assume that H is not the Moser spindle. Now, by Lemma 12.47, G contains a kite and we are again in case (iii) of the lemma.

Suppose $s = 2$. Now $\rho(S \cap X) = 7$, so So $-1 \leqslant \rho(R) + 0 - 7 - 0 + 2$; thus, $\rho(R) \geqslant 4$.

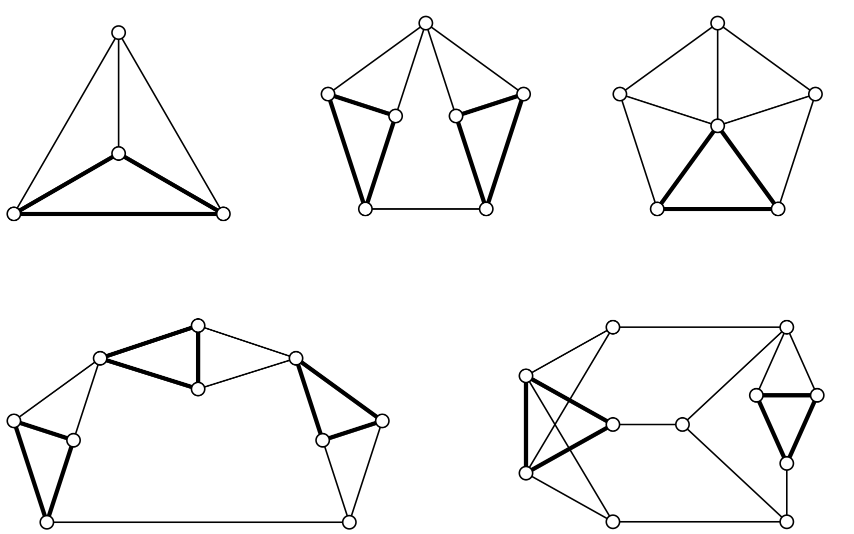


Figure 12.12: One graph in each of Cases 1–5 of Lemma 12.48, clockwise from top left. Bold edges indicate a maximum set of vertex disjoint triangles in each graph.

Suppose $s = 3$. Now $\rho(S \cap X) = 5$ and $\mathcal{T}(S \cap X) = 1$. By Lemma 12.47, either H is the Moser spindle or H contains two vertex-disjoint kites. In both case, $\mathcal{T}(S \setminus X) \geqslant 1$. Since $\mathcal{T}(H) = 2$, we get $-1 \leqslant \rho(R) + 0 - 5 - 1 + (2 - 1)$. Thus, $\rho(R) \geqslant 4$.

**Case 3: $H = W_5$.** Now $\rho(H) = -1$ and $\rho(S \cap X) \geqslant 5$. Since $\mathcal{T}(S) = \mathcal{T}(H) = 1$, we have $-1 \leqslant \rho(R) + (-1) - 5 - 0 + 1$. Thus, $\rho(R) \geqslant 4$.

**Case 4: $H \in \mathcal{B}$.** If $s = 1$, then $-1 \leqslant \rho(R) + (-1) - 5 - 0 + 1$; so $\rho(R) \geqslant 4$. If $s \in \{2, 3\}$, then $\mathcal{T}(S) - \mathcal{T}(S \setminus X) \leqslant 2$ (since $\mathcal{T}(S) = 2$) and $\rho(S \cap X) + \mathcal{T}(S \cap X) \geqslant 6$. Thus, $-1 \leqslant \rho(R) + (-1) - 6 + 2 = \rho(R) - 5$; so $\rho(R) \geqslant 4$.

**Case 5: H is 4-Ore with $\mathcal{T}(H) = 3$.** This case is like the previous one. Again, $\rho(H) = -1$. If $s = 1$, then $-1 \leqslant \rho(R)+(-1)-5-0+1$; so $\rho(R) \geqslant 4$. If $s = 2$, then $-1 \leqslant \rho(R)+(-1)-7-0+2$; so $\rho(R) \geqslant 5$. Now assume that $s = 3$. By Lemma 12.46, $\mathcal{T}(S \setminus X) \geqslant 1$. So $\mathcal{T}(S) - \mathcal{T}(S \setminus X) \leqslant 2$. Also $\rho(S \cap X) + \mathcal{T}(S \cap X) \geqslant 6$. Thus, $-1 \leqslant \rho(R) + (-1) - 6 + 2 = \rho(R) - 5$; so $\rho(R) \geqslant 4$.

**Case 6: H is not in one of the five cases above.** Now $\rho(H) \leqslant -2$, so (12.7) implies that

$$-1 \leqslant \rho(R) + (-2) - \rho(S \cap X) - \mathcal{T}(S \cap X) + |S \cap X|.$$

If $s = 1$, then $\rho(S \cap X) = 5$, $\mathcal{T}(S \cap X) = 0$, and $|S \cap X| = s = 1$, so $-1 \leqslant \rho(R)+(-2)-5-0+1 = \rho(R) - 6$; thus, $\rho(R) \geqslant 5$. If $s = 2$, then $\rho(S \cap X) = 7$, $\mathcal{T}(S \cap X) = 0$, and $|S \cap X| = s = 2$, so $-1 \leqslant \rho(R) + (-2) - 7 - 0 + 2 = \rho(R) - 7$; thus, $\rho(R) \geqslant 6$. If $s = 3$, then $\rho(S \cap X) = 5$, $\mathcal{T}(S \cap X) = 1$, and $|S \cap X| = s = 3$, so $-1 \leqslant \rho(R) + (-2) - 5 - 1 + 3 = \rho(R) - 5$; thus, again $\rho(R) \geqslant 4$, as desired. □

## 12.5 Ore's Conjecture is Nearly True for all k

### 12.5.1 An Overview

Hajós join

$H_{k,t}$

Let $G_1$ and $G_2$ be graphs with $v_1w_1 \in E(G_1)$ and $v_2w_2 \in E(G_2)$. A *Hajós join* of $G_1$ and $G_2$ is formed by deleting $v_1w_1$ and $v_2w_2$, identifying $v_1$ and $v_2$, and adding the edge $w_1w_2$; see Figure 12.13. It is straightforward to check (see Exercise 9) that if $G_1$ and $G_2$ are k-critical, then so is this Hajós join. Let $H_{k,1} := K_k$ and when $t \geqslant 2$ let $H_{k,t}$ denote a Hajós join of $H_{k,t-1}$ and $K_k$. By induction, each $H_{k,t}$ is k-critical. The average degree of these graphs tends to $2(\binom{k}{2} - 1)/(k - 1) = k - \frac{2}{k-1}$, since at each step we add $\binom{k}{2} - 1$ edges and $k - 1$ vertices.

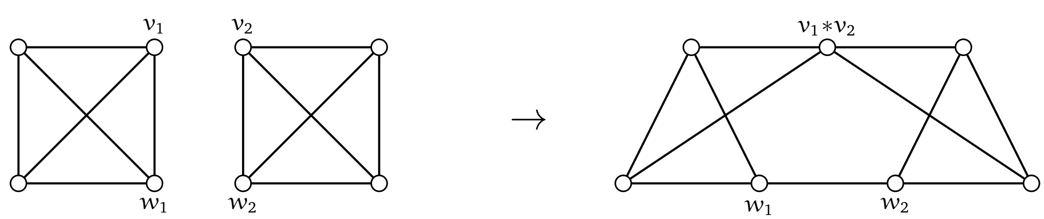



Figure 12.13: A Hajós join of two copies of $K_4$.

Gallai conjectured that no $n$-vertex $k$-critical graph has fewer edges than these graphs $H_{k,t}$. In other words, he conjectured the truth of Theorem 12.49 below when $n \equiv 1 \pmod{k-1}$. Ore went further and conjectured that the minimum number $f_k(n)$ of edges in an $n$-vertex $k$-critical graph always satisfies $f_k(n+k-1) = f_k(n) + \frac{k-1}{2}(k - \frac{2}{k-1})$, unless $n \leqslant 2k-2$ (in which case Gallai exactly determined $f_k(n)$). We observe that the left side is at most the right, since we can take a Hajós join of $k$-critical graphs on $n$ vertices and on $k$ vertices.

Theorem 12.49 proves Gallai's conjecture and goes a long way toward proving Ore's conjecture. In fact, with a bit more work, we can use Theorem 12.49 to show that, for each fixed $k$, Ore's conjecture holds for all but at most $k^3/12$ values of $n$. (But this argument does not use the potential method, so we omit it.) As noted above, Theorem 12.49 is sharp for every $k \geqslant 4$ when $n \equiv 1 \pmod{k-1}$. Furthermore, when $k = 4$, it is sharp for every $n \geqslant 6$ (see Exercise 9).

**Theorem 12.49.** *Fix an integer $k \geqslant 4$. If $G$ is $k$-critical and $|G| \geqslant k+2$, then*

$$2\|G\| \geqslant \left(k - \frac{2}{k-1}\right)|G| - \frac{k(k-3)}{k-1}. \tag{12.8}$$

In this section we outline the proof of Theorem 12.49, deferring most details to later sections. For a graph $G$ and each nonempty $R \subseteq V(G)$, let

$$\rho_{k,G}(R) := (k-2)(k+1)\,|R| - 2(k-1)\|G[R]\|.$$ $\rho_G(R)$

Since $k$ is fixed throughout the proof, we typically write only $\rho_G$, or occasionally $\rho$, when the context is clear. Easy algebra shows that the conclusion of the theorem is equivalent to $\rho_G(V(G)) \leqslant k(k-3)$; this is the form we will work with. In fact, it is this equivalence that motivates the definition of $\rho_G$.

**Theorem 12.50.** *Fix an integer $k \geqslant 4$. If $G$ is $k$-critical and $|G| \geqslant k+2$, then*

$$\rho_G(V(G)) \leqslant k(k-3). \tag{12.9}$$

The proof is by induction, essentially on $\|G\|$. More precisely, a graph $H$ is *smaller than* $G$ if either $\|H\| < \|G\|$ or $\|H\| = \|G\|$ and $H$ has more pairs of vertices with the same closed neighborhood (the second case is used only once in the proof, and can largely be ignored on a first reading). It is easy to check from the definitions that $\|H\| < \|G\|$ when either (i) $|H| < |G|$ and $\rho_H(V(H)) \geqslant \rho_G(V(G))$ or (ii) $|H| = |G|$ and $\rho_H(V(H)) > \rho_G(V(G))$; we will frequently invoke the induction hypothesis in both ways. The following observation is crucial in the proof. smaller

**Observation 12.51.** Assume the theorem holds for all graphs smaller than $G$. If $H$ is smaller than $G$ and $\rho_H(U) > k(k-3)$ for every nonempty $U \subseteq V(H)$, then $H$ is $(k-1)$-colorable.

*Proof.* Suppose to the contrary that $\chi(H) \geqslant k$, and let $U$ be the vertex set of a $k$-critical subgraph $H'$ of $H$. Now $\rho_{H'}(U) \geqslant \rho_H(U) > k(k-3)$ by assumption. Since $\|H[U]\| \leqslant \|H\|$, graph $H[U]$ is smaller than $G$, so by induction $H[U]$ is *not* $k$-critical, a contradiction. □

As usual, the heart of the potential method is proving a powerful Gap Lemma. Our first step in this direction is Lemma 12.59 (our "Weak" Gap Lemma), which implies that we can add any single edge to any proper induced subgraph $H$ of $G$, and $(k-1)$-color the resulting graph by induction. We then bootstrap this result to prove a stronger bound in Lemma 12.61 (our "Strong" Gap Lemma).

For the induction step we will assume that $\chi(G) \geqslant k$ and show that, in fact, $\rho(V(G)) \leqslant 0$. Recall that $\rho(V(G)) \leqslant k(k-3)$ if and only if $2\|G\| \geqslant (k - \frac{2}{k-1})|G| - \frac{k(k-3)}{k-1}$. Similarly, $\rho(V(G)) \leqslant 0$ is equivalent to $\overline{d}(G) \geqslant k - \frac{2}{k-1}$ (here $\overline{d}(G)$ is defined as $2\|G\|/|G|$). We prove this lower bound on $\overline{d}(G)$ via discharging. In this context, Lemma 12.61 allows us to prove stronger reducibility lemmas, which better facilitate the discharging proof.

Below we sketch the discharging proof for $k \geqslant 6$. This motivates much of the work we do bounding the potentials of subgraphs, and ultimately proving that certain subgraphs are reducible, i.e., their presence allows us to succeed in the induction step. We assume that $G$ has no $(k-1)$-coloring, and aim to show that $\overline{d}(G) \geqslant k - \frac{2}{k-1}$. To prove this bound on $\overline{d}(G)$, we give each vertex $v$ initial charge $d(v)$ and redistribute charge so that $ch^*(v) \geqslant k - \frac{2}{k-1}$ for each $v$. Recall that $\delta(G) \geqslant k-1$.

Clearly, each $(k-1)$-vertex $v$ needs charge. The most natural way for $v$ to get charge is to take some from each neighbor of higher degree. However, $k$-vertices have very little charge to spare. For example, $\overline{d}(K_{k,k-1}) = k - \frac{k}{2k-1} < k - \frac{2}{k-1}$. Fortunately, this configuration and

$\mathcal{H}_0, \mathcal{H}_1$ similar ones are reducible. Let $\mathcal{L}$, $\mathcal{H}_0$, and $\mathcal{H}_1$ denote the sets of vertices with degrees $k-1$, $k$, and at least $k+1$. (Intuitively, $\mathcal{L}$ stands for *low*, and $\mathcal{H}_0$ and $\mathcal{H}_1$ stand for *high* and *higher*.)

$\mathcal{L}_0, \mathcal{L}_1$ We further partition $\mathcal{L}$ into $\mathcal{L}_0$, its vertices with no $(k-1)$-neighbor, and $\mathcal{L}_1$, those vertices with at least one $(k-1)$-neighbor.

Our first class of reducible configurations shows that $|E(\mathcal{L}_0, \mathcal{H}_0)| \leqslant 2(|\mathcal{L}_0| + |\mathcal{H}_0|)$. Since $\mathcal{L}_0$ induces no edges, this implies that $\mathcal{L}_0$ has many edges to $\mathcal{H}_1$, along which to receive charge. This motivates rules (R1) and (R3) below. It turns out that we need one more rule, (R2), which we will motivate shortly. We use discharging, with the following 3 rules.

(R1) Each $(k+1)^+$-vertex splits its excess charge (that exceeding $k - \frac{2}{k-1}$) equally among its $(k-1)$-neighbors.

(R2) If a copy $J$ of $K_{k-1}$ contains $s$ $(k-1)$-vertices adjacent to a $(k-1)$-vertex $v$ outside of $J$, and $v$ is not in a $K_{k-1}$, then each of these $s$ vertices gives $\frac{k-3}{s(k-1)}$ to $v$.

(R3) After applying (R1) and (R2), the vertices of $\mathcal{L}_0 \cup \mathcal{H}_0$ average their charge.

Now we analyze $ch^*(v)$ for each vertex $v$. Clearly (R1) implies that $ch^*(v) \geqslant k - \frac{2}{k-1}$ for each $(k+1)^+$-vertex $v$ (with equality when $v$ has at least one $(k-1)$-neighbor). It is also easy to check that each $(k+1)^+$-vertex sends at least $\frac{1}{k-1}$ to each $(k-1)$-neighbor. The expression $\frac{d(v)-(k-\frac{2}{k-1})}{d(v)}$ increases as a function of $d(v)$, so it suffices to check that each $(k+1)$-vertex sends the desired charge (which it does). Now consider the total charge of $\mathcal{H}_0 \cup \mathcal{L}_0$ after (R1) and (R2). Since $|E(\mathcal{L}_0, \mathcal{H}_0)| \leqslant 2(|\mathcal{L}_0| + |\mathcal{H}_0|)$, we have:

$$\begin{aligned} ch^*(\mathcal{L}_0 \cup \mathcal{H}_0) &\geqslant (k-1)\,|\mathcal{L}_0| + k\,|\mathcal{H}_0| + \frac{1}{k-1}\,|E(\mathcal{L}_0, \mathcal{H}_1)| \\ &\geqslant (k-1)\,|\mathcal{L}_0| + k\,|\mathcal{H}_0| + \frac{1}{k-1}\,((k-1)\,|\mathcal{L}_0| - (2(|\mathcal{L}_0| + |\mathcal{H}_0|)) \\ &= (k - \frac{2}{k-1})(|\mathcal{L}_0| + |\mathcal{H}_0|). \end{aligned}$$

So vertices in $\mathcal{H}_1$ are happy, and vertices in $\mathcal{L}_0 \cup \mathcal{H}_0$ are happy. Now we need only show that vertices in $\mathcal{L}_1$ are also happy. Since such a vertex gets no charge from neighbors in $\mathcal{H}_0$, it would be ideal to show that $v$ must receive lots of charge from neighbors in $\mathcal{H}_1$. However, we use a slight variation on this idea. We show that either (i) $v$ does receives lots of charge from neighbors in $\mathcal{H}_1$ or (ii) $v$ has some $(k-1)$-neighbors that receive lots of charge from their neighbors in $\mathcal{H}_1$, and they can afford to pass some of this charge to $v$. This motivates (R2) above. In fact, if a $(k-1)$-vertex $v$ receives charge from (R2), then $ch^*(v) \geqslant (k-1) + s\frac{k-3}{s(k-1)} = k - \frac{2}{k-1}$.

The remaining details are split into four sections.

- The focus of Section 12.5.2 is proving the bound $|E(\mathcal{L}_0, \mathcal{H}_0)| \leqslant 2(|\mathcal{L}_0| + |\mathcal{H}_0|)$, which we used above to show that all vertices in $\mathcal{L}_0 \cup \mathcal{H}_0$ end happy.
- In Section 12.5.3, we first prove our Weak Gap Lemma (Lemma 12.59) and later leverage this to prove our Strong Gap Lemma (Lemma 12.61). A cluster is a maximal subset of $(k-1)$-vertices such that all vertices of the cluster have the same neighbors.
- In Section 12.5.4, we use the Strong Gap Lemma to prove results about how clusters and copies of $K_{k-1}$ can interact.
- Finally, in Section 12.5.5, we finish the discharging argument, by providing the details for the vertices in $\mathcal{L}_1$.

### 12.5.2 Reducible Configurations in $G[\mathcal{L}_0, \mathcal{H}_0]$

We use the kernel method to show that many induced subgraphs cannot appear in $G[\mathcal{L}_0, \mathcal{H}_0]$.

**Lemma 12.52.** *Let $U$ be an independent set in a graph $G$ and let $W := V(G) \setminus U$. Form a digraph $D$ from $G$ by directing each edge of $G[W]$ in both directions and directing each edge between $U$ and $W$ arbitrarily. Now $D$ is kernel-perfect.*[15]

[15]Recall that a *kernel* in a digraph $D$ is an independent set $I$ such that each vertex of $V(D) \setminus I$ has an outneighbor in $I$. A digraph is *kernel-perfect* if each of its induced subgraphs has a kernel. Richardson's Theorem (Lemma 5.3) considers the special case when $G$ is bipartite and $U$ is one of the parts; so $G[W]$ is edgeless. The proofs of Lemmas 5.3 and 12.52 are similar, but the former is longer; most of the extra work comes from using the hypothesis "G has no directed odd cycle" rather than "G has an underlying graph that is bipartite".

*Proof.* Since the class of resulting digraphs D is hereditary, it suffices to show that D has a kernel. We use induction on $|G|$; the base case $|G| = 1$ is trivial. If each $w \in W$ has an outneighbor in U, then U is a kernel. Suppose instead that some $w \in W$ has no outneighbor in U. Now every neighbor of $w$ in G is an inneighbor of $w$ in D. Let $D' := D \setminus N_G[w]$. By induction, $D'$ has a kernel I. Now $I \cup \{w\}$ is a kernel of D. □

$G(U, W)$ For a graph G and disjoint vertex subsets U and W, let $G(U, W)$ denote the bipartite subgraph of G induced by the set of edges with one endpoint in each of U and W. Case (i) of the following lemma immediately implies that $\|G(U, W)\| \leqslant 2(|U| + |W|)$, which is what we used to show that all vertices in $\mathcal{L}_0 \cup \mathcal{H}_0$ finish happy. Case (ii) is similar, and is needed for the discharging when $k = 5$.

**Lemma 12.53.** *Let* G *be a* k*-critical graph. If* $U \subseteq \mathcal{L}_0$ *and* $W \subseteq \mathcal{H}_0$ *and* $U \cup W \neq \emptyset$*, then*

1. $\delta(G(U, W)) \leqslant 2$ *and*
2. *either there exists* $u \in U$ *with at most* 1 *neighbor in* W *or there exist* $w \in W$ *with at most* 3 *neighbors in* U.

*Proof.* Since G is k-critical, $G \setminus (U \cup W)$ has a $(k-1)$-coloring $\varphi$. For each $x \in U \cup W$, let
$L(x)$ $L(x)$ be the colors of $[k-1]$ unused on $N(x)$ by $\varphi$. We prove (i); the proof of (ii) is similar.

We assume to the contrary that $\delta(G(U, W)) \geqslant 3$ and show that $G(U, W)$ has an L-coloring $\varphi'$. Together $\varphi$ and $\varphi'$ give a $(k-1)$-coloring of G, a contradiction. To L-color $G(U, W)$, we use the Kernel Lemma (Lemma 5.2). Recall that U induces no edges, by the definition of $\mathcal{L}_0$, so any digraph D formed as in Lemma 12.52 is kernel-perfect. Thus, we only need to direct the edges of $G(U, W)$ so that each vertex $x$ has $|L(x)| > d^+_D(x)$. Each $u \in U$ needs at least one inneighbor in W (since $u \in \mathcal{L}_0$) and each $w \in W$ needs at least two inneighbors in U (since $w \in \mathcal{H}_0$). To find such an orientation, we apply Hall's Theorem to an auxiliary graph.

Form $G'$ from $G(U, W)$ by splitting (arbitrarily) each vertex $w \in W$ into $\lceil d_{G(U,W)}(w)/3 \rceil$ vertices, each of degree 2 or 3; call the resulting set of new vertices $W'$. Since $d_{G'}(u) \geqslant 3$ for each $u \in U$ and $d_{G'}(w) \leqslant 3$ for each $w \in W'$, each $S \subseteq U$ has $|N_{W'}(S)| \geqslant |S|$. By Hall's Theorem, $G'$ has a matching $M'$ that saturates U. Let M be the edge set in G corresponding to the edges of $M'$ (note that M need not be a matching).

To form D from G, direct each edge of M towards its endpoint in U, direct all other edges of $G(U, W)$ toward W, and direct edges induced by W in both directions. For all $u$, clearly $d^-_D(u) \geqslant 1$, so $|L(u)| > d^+_D(u)$. Now consider $w \in W$. If $w$ is split into two or more vertices in $W'$, then each has an incident edge not in $M'$, so $d^-_D(w) \geqslant 2$. Otherwise $w$ has degree 3 in $G'$, and at most one incident edge in $M'$, so again $d^-_D(w) \geqslant 2$, as desired.

The proof of (2) is similar, but to form $G'$ we split each $w \in W$ into $\lceil d_{G(U,W)}(w)/2 \rceil$ vertices, each of degree 1 or 2. Again $G'$ has a matching saturating U, which yields the desired orientation D in our application of Lemma 12.52. □

Lemma 12.53 immediately implies the following, by induction on $|U| + |W|$.

**Lemma 12.54.** *Let* $G$ *be a* $k$*-critical graph. If* $U \subseteq \mathcal{L}_0$ *and* $W \subseteq \mathcal{H}_0$ *then*

1. $\|G(U, W)\| \leqslant 2(|U| + |W|)$ *and*
2. $\|G(U, W)\| \leqslant |U| + 3\,|W|$.

### 12.5.3 The Gap Lemmas

Our work in this section closely mirrors that in Section 12.1. By now our approach should feel standard. Our first main result is our Weak Gap Lemma (Lemma 12.59). Our second main result is the most important in the section. It is our Strong Gap Lemma (Lemma 12.61), and its proof requires the Weak Gap Lemma, which allows us to modify $G[R]$ before coloring it.

Before proving either of these main results, we explicitly compute the potential of cliques of order at most $k - 1$. Since deleting edges only increases potential, this computation gives a lower bound on the potential of every graph with order at most $k - 1$. Our next definition, observation, and lemma will perhaps seem obvious to the reader. But we state them explicitly to highlight the parallels with Section 12.1.

**Definition 12.55.** Given a graph $G$, a set $R \subsetneq V(G)$, and a $(k-1)$-coloring $\varphi$ of $G[R]$, form the graph $H(G, R, \varphi)$ from $G$ by contracting each color class $i$ of $\varphi$ to a single vertex $x_i$ (adding an isolated $x_i$ if color class $i$ is empty), and making vertices $x_1, \ldots, x_{k-1}$ pairwise adjacent. Also, delete any multiple edges formed in the process. Let $X := \{x_1, \ldots, x_{k-1}\}$. $H(G, R, \varphi)$ $X$

**Observation 12.56.** Fix a vertex set $R \subsetneq V(G)$ and $(k-1)$-coloring $\varphi$ of $R$, and let $G' := H(G, R, \varphi)$. For any $S \subseteq V(G')$, simply counting edges and vertices gives

$$\rho_G((S \setminus X) \cup R) \leqslant \rho_{G'}(S) - \rho_{G'}(S \cap X) + \rho_G(R). \tag{12.10}$$

**Lemma 12.57.** *If* $G$ *has no* $(k-1)$*-coloring, then for each set* $R \subsetneq V(G)$ *and each* $(k-1)$*-coloring* $\varphi$ *of* $G[R]$*, the graph* $H(G, R, \varphi)$ *also has no* $(k-1)$*-coloring.*

*Proof.* Given a $(k-1)$-coloring $\varphi'$ of $H(G, R, \varphi)$, we get a $(k-1)$-coloring of $G$ by "uncontracting" $H(G, R, \varphi)$. That is, for each color class $j$ of $\varphi$, we give to its vertices, $\varphi^{-1}(j)$, the color $\varphi'(x_j)$ given to their image in $\varphi'$; each vertex outside $R$ keeps its color from $\varphi'$. □

It is convenient to explicitly compute the potential of small cliques, which gives a bound on the potential of all small subgraphs. The computation is simple algebra, but we do it once and record it, for the sake of having an easy reference in what follows.

**Lemma 12.58.** *If* $\ell \in [k]$*, then* $\rho_{K_\ell}(V(K_\ell)) = \ell(k^2 - k\ell + \ell - 3)$*. In particular,* $\rho_{K_k}(V(K_k)) = k(k-3)$*,* $\rho_{K_1}(V(K_1)) = k^2 - k - 2$*,* $\rho_{K_{k-1}}(V(K_{k-1})) = 2(k^2 - 3k + 2)$*, and* $\rho_{K_2}(V(K_2)) = 2(k^2 - 2k - 1)$*. Further, every graph* $H$ *with* $2 \leqslant |H| \leqslant k-1$ *satisfies* $\rho_H(V(H)) \geqslant 2(k^2 - 3k + 2)$*.*

*Proof.* Direct computation gives $\rho_{K_\ell}(V(K_\ell)) = (k-2)(k+1)\ell - 2(k-1)\binom{\ell}{2} = \ell(k^2 - k\ell + \ell - 3)$, as well as the expressions for specific values of $\ell$. Since deleting edges increases $\rho$, we need only verify the final statement for complete graphs. Note that $\rho_{K_\ell}(V(K_\ell))$ is quadratic in $\ell$ with leading coefficient negative. So it suffices to check that the bound holds for $K_2$ and $K_{k-1}$, which it does. □

Now we can prove our Weak Gap Lemma.

**Lemma 12.59 (Weak Gap Lemma).** *If* $R \subsetneq V(G)$ *and* $|R| \geqslant 2$*, then* $\rho_G(R) > k^2 - k - 2$*.*

$R$ *Proof.* Choose $R \subsetneq V(G)$ to minimize $\rho_G(R)$, such that $|R| \geqslant 2$. If $|R| \leqslant k-1$, then we are done by Lemma 12.58. So assume $|R| \geqslant k$. Assume, for a contradiction, that $\rho_G(R) \leqslant k^2 - k - 2$. Since
$G'$ $G$ is $k$-critical and $G[R]$ is a proper subgraph, it has a $(k-1)$-coloring $\varphi$. Let $G' := H(G, R, \varphi)$. Note that $|G'| < |G|$, since $|R| \geqslant k > |X|$.

Letting $S := V(G')$ in (12.10) we get $\rho_G(V(G)) = \rho_G((V(G') \setminus X) \cup R) \leqslant \rho_{G'}(V(G')) - \rho_{G'}(X) + \rho_G(R) < \rho_{G'}(V(G'))$, since $\rho_{G'}(X) > k^2 - k - 2 \geqslant \rho_G(R)$. So, as desired, $G'$ is smaller than $G$, since $|G'| < |G|$ and $\rho_{G'}(V(G')) > \rho_G(V(G))$.

By Lemma 12.57, $G'$ has no $(k-1)$-coloring, so it has some $k$-critical subgraph $G''$; let $S := V(G'')$. If $G'' \neq G'$, then $G''$ is smaller than $G'$, since it has fewer edges. Thus $G''$ is smaller than $G$, regardless of whether or not $G'' = G'$. Since $G''$ is $k$-critical, by induction $\rho_{G'}(S) \leqslant \rho_{G''}(S) \leqslant k^2 - 3k$. Since $G$ is $k$-critical itself, $S \cap X \neq \emptyset$. By Lemma 12.58, every nonempty subset of $X$ has potential at least $k^2 - k - 2$, so (12.10) gives

$$\begin{aligned}\rho_G((S \setminus X) \cup R) &\leqslant \rho_{G'}(S) - \rho_{G'}(S \cap X) + \rho_G(R) \\ &\leqslant k^2 - 3k - (k^2 - k - 2) + \rho_G(R) \\ &\leqslant \rho_G(R) - 2k + 2.\end{aligned}$$

Now $|(S \setminus X) \cup R| \geqslant 2$, since $|R| \geqslant 2$. Because $R$ was chosen to minimize $\rho_G$ and $\rho_G((S \setminus X) \cup R) < \rho_G(R)$, we conclude that $(S \setminus X) \cup R = V(G)$. However, now $\rho_G(V(G)) \leqslant \rho_G(R) - 2k + 2 \leqslant (k^2 - k - 2) - 2k + 2 = k^2 - 3k$, so $G$ is not a counterexample at all, which is a contradiction. □

We now want to strengthen the inequality in our Weak Gap Lemma. In the proof above, the obstacle was the fact that $\rho_{K_1}(V(K_1)) = k^2 - k - 2$. But if we can ensure that $|S \cap X| \geqslant 2$, then we can significantly improve this, by Lemma 12.58. We cannot always guarantee that this inequality holds; but we can construct our coloring of $G[R]$ so that if the inequality fails, then we can improve on (12.10) in another way.

Our next lemma ensures that in our $(k-1)$-coloring $\varphi$ of $G[R]$ no single color class of $\varphi$ contains the endpoints of too large a fraction of the edges from $V(G) \setminus R$ to $R$. If it happens that $|S \cap X| = 1$, then for some color class $\varphi^{-1}(j)$, the edges from $V(G) \setminus R$ to $R \setminus \varphi^{-1}(j)$ are not counted by (12.10), and when we also count these extra edges we are able to substantially improve this bound.

**Lemma 12.60.** *Choose an integer* $k$ *such that* $k-1 \geqslant 2$. *Let* $R_* := \{u_1, \ldots, u_s\}$, *and let* $\sigma : R_* \to \mathbb{Z}^+$ *be a positive integral weight function on* $R_*$ *such that* $\sigma(u_1) + \cdots + \sigma(u_s) \geqslant k-1$. *Now for each* $i$ *such that* $1 \leqslant i \leqslant \frac{k-1}{2}$, *there exists a graph* $H$ *with* $V(H) = R_*$ *and* $\|H\| \leqslant i$ *such that every independent set* $I$ *in* $H$ *with* $|I| \geqslant 2$ *satisfies* $R_*$ $\sigma(u_i)$

$$\sum_{u \in R_* \setminus I} \sigma(u) \geqslant i. \qquad (12.11)$$

Figure 12.14 shows 3 examples of the lemma.

*Proof.* By symmetry, we assume that $\sigma(u_1) \geqslant \sigma(u_2) \geqslant \cdots \geqslant \sigma(u_s)$.

Suppose first that $\sigma(u_2) + \cdots + \sigma(u_s) \leqslant i$. Let $H$ be the graph with an edge from $u_1$ to every other vertex of $R_*$. If $I$ is an independent set with $|I| \geqslant 2$, then $u_1 \notin I$, so $\sum_{u \in R_* \setminus I} \sigma(u) \geqslant \sigma(u_1) \geqslant (k-1) - i \geqslant i$.

Assume instead that $\sigma(u_2) + \cdots + \sigma(u_s) \geqslant i+1$. To form $H$ from $R_*$ we will add $i$ edges such that $d_H(u) \leqslant \sigma(u)$ for all $u \in R_*$. Because $I$ is independent, this implies the lemma, since we have

$$\sum_{u \in R_* \setminus I} \sigma(u) \geqslant \sum_{u \in R_* \setminus I} d_H(u) \geqslant \frac{1}{2} \sum_{u \in R_*} d_H(u) = i.$$

To add these $i$ edges as desired, choose the largest $j$ such that $\sigma(u_j) + \cdots + \sigma(u_s) \geqslant i$. Let $\ell := i - (\sigma(u_{j+1}) + \cdots + \sigma(u_s))$. We add $\ell$ edges between $u_1$ and $u_j$ and we add $i - \ell$ edges between $\{u_1, \ldots, u_j\}$ and $\{u_{j+1}, \ldots, u_s\}$. This is possible because $\ell \leqslant \sigma(u_j) \leqslant \sigma(u_1)$, and also $i \leqslant \frac{k-1}{2}$ and $\sigma(u_1) + \cdots + \sigma(u_s) \geqslant k-1 \geqslant 2i$, so $\sigma(u_1) + \cdots + \sigma(u_j) \geqslant 2i - (i - \ell) = i + \ell$. Finally, if $H$ has parallel edges, replace each set of them with a single edge. □

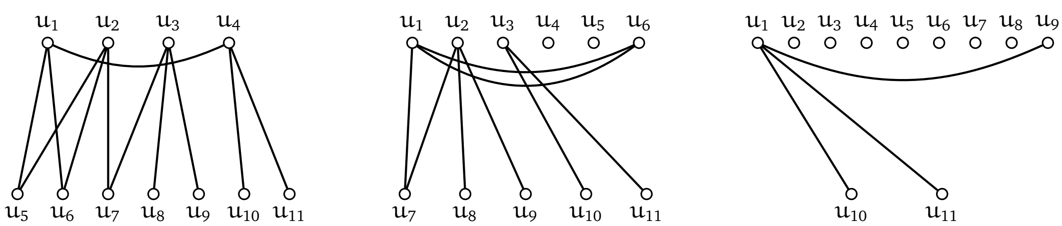



Figure 12.14: Three examples of Lemma 12.60, when $\sigma(u_1) = \cdots = \sigma(u_4) = 3$, $\sigma(u_5) = \cdots = \sigma(u_7) = 2$, and $\sigma(u_8) = \cdots = \sigma(u_{11}) = 1$, so $\sum_{i=1}^{11} \sigma(u_i) = 22$. Left: $i = 11$, $j = 4$, $\ell = 1$. Center: $i = 8$, $j = 6$, $\ell = 2$. Right: $i = 3$, $j = 9$, $\ell = 1$.

The next lemma is the most important result in this section, and we use it repeatedly in the section that follows.

**Lemma 12.61 (Strong Gap Lemma).** *If* $R \subsetneq V(G)$*,* $|R| \geqslant 2$*, and* $\rho_G(R) \leqslant 2(k-2)(k-1)$*, then* $G[R] = K_{k-1}$*. Further,* $\rho_G(R) = 2(k-2)(k-1)$*.*

*Proof.* We first sketch the proof, and give more details thereafter. Choose $R \subsetneq V(G)$ to minimize $\rho_G(R)$, such that $|R| \geqslant 2$ and $G[R] \neq K_{k-1}$. Suppose $\rho_G(R) \leqslant 2(k-2)(k-1)$. Lemma 12.58 implies that $|R| \geqslant k$. Choose $i$ as large as possible such that $\rho_G(R) - 2(k-1)i > k(k-3)$. In particular

$$\rho_G(R) - 2(i+1)(k-1) \leqslant k(k-3). \tag{12.12}$$

This is the largest $i$ such that adding any $i$ edges to $G[R]$ results in a graph, call it $G^+[R]$, where $\rho_{G^+[R]}(R) > k(k-3)$. We will show that $G^+[R]$ is smaller than $G$, and $\rho_{G^+[R]}(U) > k(k-3)$ for every $U \subset V(G^+[R])$. So $G^+[R]$ has a $(k-1)$-coloring, by Observation 12.51. Lemma 12.59 shows that $\rho_G(R) > k^2 - k - 2 = k^2 - 3k + 2(k-1)$, so $i \geqslant 1$. Also $i \leqslant \frac{k-2}{2}$, since taking $i \geqslant \frac{k-1}{2}$ gives

$$\rho_G(R) - 2(k-1)i \leqslant \rho_G(R) - 2(k-1)\frac{k-1}{2} \leqslant 2(k-2)(k-1) - (k-1)^2 < (k-3)k,$$

which contradicts the definition of $i$. Using Lemma 12.60, we add edges to form $G^+[R]$ so that for each independent set $I$ in $G^+[R]$ at least $i$ edges of $G$ go between $V(G) \setminus R$ and $R \setminus I$; these
$G'$
edges are counted by $\sum_{u \in R_* \setminus I} \sigma(u)$. Given a $(k-1)$-coloring $\varphi$ of $G^+[R]$, let $G' := H(G, R, \varphi)$. Lemma 12.57 shows that $G'$ has no $(k-1)$-coloring. Since $G'$ is smaller than $G$, there exists
$R'$
$R' \subseteq V(G')$ with $\rho_{G'}(R') \leqslant k(k-3)$. We show that $|R' \cap X| = 1$. By symmetry, we assume that $R' \cap X = \{x_1\}$ and let $I := \varphi^{-1}(1)$. Since $I$ is independent, at least $i$ edges in $G$ go from $V(G) \setminus R$ to $R \setminus I$. Now revising (12.10) to account for these extra $i$ edges gives

$$\begin{aligned}\rho_G((R' \setminus X) \cup R) &\leqslant \rho_{G'}(R') - \rho_{G'}(X \cap R') + \rho_G(R) - i2(k-1)\\ &\leqslant \rho_G(R) - 2(i+1)(k-1)\\ &\leqslant k(k-3),\end{aligned}$$

where the second inequality holds because $\rho_{G'}(R') \leqslant k(k-3)$ and $\rho_{G'}(\{x_1\}) = (k-2)(k+1)$, and the third holds by (12.12). But $\rho_G((R' \setminus X) \cup R) \leqslant k(k-3)$ contradicts Lemma 12.59, and this contradiction finishes the proof.

Now we provide the remaining details. To apply Lemma 12.60, we must specify a vertex set $R_* \subseteq R$ and positive integer weights $\sigma(u)$ for each $u \in R_*$. For each $u \in R$, let $\sigma(u) :=$
$\sigma(u)$
$|N(u) \cap (V(G) \setminus R)|$; so the weight of $u$ is its number of neighbors in $V(G) \setminus R$. Let $R_* := \{u \in$
$R_*$
$R : \sigma(u) \geqslant 1\}$. Since $G$ is $k$-critical, it has no cut-vertex, so $|R_*| \geqslant 2$. Further, $G$ is $(k-1)$-edge-
$G^+[R]$
connected[16], so $\sum_{u \in R} \sigma(u) \geqslant k-1$. Now let $G^+[R] := G[R] \cup E(H)$, where $H$ is the graph

[16]This is a classic result of Dirac. It can be proved by assuming that $G$ has an edge-cut $S$ that is smaller, $(k-1)$-coloring the components of $G \setminus S$, then using Hall's Theorem to permute color classes for each component to avoid conflicts on $S$.

guaranteed by Lemma 12.60. We chose $i$ so that $\rho_{G^+[R]}(R) > k(k-3)$. As noted above $G$ is $(k-1)$-edge-connected, so $|E_G(V(G) \setminus R, R)| \geqslant k-1$. Since $\|G\| - \|G^+[R]\| \geqslant k-1-i > 0$, graph $G^+[R]$ is smaller than $G$. Since $\rho_G(R) \leqslant \rho_G(U)$ for all nonempty $U \subseteq R$, we have

$$\rho_{G^+[R]}(U) \geqslant \rho_G(U) - 2i(k-1) \geqslant \rho_G(R) - 2i(k-1) > k(k-3).$$

By Observation 12.51, $G^+[R]$ has a $(k-1)$-coloring $\varphi$. Recall that $G' = H(G, R, \varphi)$ and $R' \subseteq V(G')$ with $\rho_{G'}(R') \leqslant k(k-3)$. Since $G$ is $k$-critical, $R' \cap X \neq \emptyset$. If $|R' \cap X| \geqslant 2$, then $\rho_{G'}(R' \cap X) \geqslant 2(k-1)(k-2)$, by Lemma 12.58. So (12.10) gives $\rho_G((R' \setminus X) \cup R) \leqslant \rho_{G'}(R') - \rho_{G'}(R' \cap X) + \rho_G(R) \leqslant k(k-3) - 2(k-1)(k-2) + 2(k-2)(k-1) = k(k-3)$, which contradicts Lemma 12.59 (or that $G$ is a counterexample, if $(R' \setminus X) \cup R = V(G)$).

Now assume $|R' \cap X| = 1$. By symmetry, say $R' \cap X = \{x_1\}$. Now again (12.10) gives $\rho_G((R' \setminus \{x_1\}) \cup R) \leqslant \rho_{G'}(R') - \rho_{G'}(\{x_1\}) + \rho_G(R) \leqslant k(k-3) - (k-2)(k+1) + \rho_G(R) < \rho_G(R)$. Since we chose $R$ to minimize $\rho_G(R)$, we must have $(R' \setminus \{x_1\}) \cup R = V(G)$. Let $R_1 := \{u \in R_* : \varphi(u) = \varphi(x_1)\}$. If $|R_1| = 1$, then $\rho_G((R' \setminus \{x_1\}) \cup R_1) = \rho_{G'}(R') \leqslant k(k-3)$, which is a contradiction. So $|R_1| \geqslant 2$. Now, since $G^+[R]$ was constructed using Lemma 12.60, and $R_1$ is an independent set, at least $i$ edges in $G$ connect $V(G) \setminus R$ to $R_* \setminus R_1$, as desired. (This follows directly from (12.11) since, for each $u \in R_*$, we defined $\sigma(u)$ as the number of neighbors of $u$ in $V(G) \setminus R$.) □

### 12.5.4 Structure of $(k-1)$-cliques and Clusters

In this section we prove that a variety of subgraphs $H$ cannot appear in $G$. One simple way to do this is to show that $\rho(H)$ is small enough to contradict Lemma 12.61. Another approach is to assume the presence of $H$ and use it to form a graph $G'$ from $G$. If $G'$ has a $(k-1)$-coloring $\varphi$, then we modify $\varphi$ to get a $(k-1)$-coloring of $G$. If $G'$ has no $(k-1)$-coloring, then, since $G'$ is smaller than $G$, we have some $R' \subseteq V(G')$ with $\rho_{G'}(R')$ small. We modify $R'$ to form some $R \subseteq V(G)$ with $\rho_G(R)$ small, again contradicting Lemma 12.61.

A *cluster* $R$ is a maximal set of $(k-1)$-vertices in $G$ with $N[u] = N[v]$ for all $u, v \in R$. cluster

**Lemma 12.62.** *All $(k-1)$-vertices in the same $K_{k-1}$ are in the same cluster.*

*Proof.* Assume, contrary to the lemma, that $v$ and $w$ are $(k-1)$-vertices in the same $K_{k-1}$, call it $K$, but $N[v] \neq N[w]$. So there exist distinct vertices $y$ and $z$ such that $N(v) = K - v + y$ and $N(w) = K - w + z$. Let $G' := G \setminus \{v, w\} + yz$ (and $G' := G \setminus \{v, w\}$ if $yz \in E(G)$). Lemma 12.61 implies that $\rho_G(R) \geqslant 2(k-2)(k-1)$ for all $R \subseteq V(G')$ with $|R| \geqslant 2$. Recall that $\rho_{K_1}(V(K_1)) = (k-2)(k+1)$. Since adding an edge decreases potential by $2(k-1)$, for all $S \subseteq V(G')$

$$\rho_{G'}(S) \geqslant \min\{(k-2)(k+1), 2(k-2)(k-1) - 2(k-1)\} > k(k-3).$$

Now Observation 12.51 implies that $G'$ has a $(k-1)$-coloring $\varphi$ with $\varphi(x) \neq \varphi(y)$, and this easily extends to a $(k-1)$-coloring of $G$, contradicting that $G$ is $(k-1)$-critical. □

**Lemma 12.63.** *If* $S$ *is a cluster, then* $|S| \leqslant k-3$. *If* $S$ *is also in a* $K_{k-1}$*, then* $|S| \leqslant \frac{k-1}{2}$.

*Proof.* Since $G$ is $k$-critical, it contains no $K_k$. So a cluster with $k-2$ vertices, together with its two neighbors, induces $K_k - e$. But $\rho(K_k - e) = k(k-3) + 2(k-1) < 2(k-2)(k-1)$, contradicting Lemma 12.61. For the second statement, let $K$ be a $K_{k-1}$ containing a cluster $S$. Let $v$ be the vertex in $N(S) \setminus K$. If $|S| \geqslant \lceil \frac{k}{2} \rceil$, then $\rho(K+v) \leqslant 2(k-2)(k-1) + (k-2)(k+1) - 2(k-1)\frac{k}{2} \leqslant 2(k-2)(k-1) - 2$, again contradicting Lemma 12.61. □

**Lemma 12.64.** *Let* $v$ *and* $w$ *be in distinct clusters of sizes* $s$ *and* $t$*, respectively, and assume* $s \geqslant t$. *If* $v \leftrightarrow w$*, then* $v$ *is in a* $K_{k-1}$. *If also* $w$ *is not in a* $K_{k-1}$*, then* $s > t$.

*Proof.* The second statement follows from the first by assuming that $s = t$ and swapping the roles of $v$ and $w$. Now we prove the first statement.

$G', v'$ Assume that $v$ and $w$ are adjacent, but $v$ is not in a $K_{k-1}$. Let $G' := G - w + v'$, where $N[v'] = N[v]$. Clearly $\|G'\| = \|G\|$. If two vertices, excluding $w$, have the same closed neighborhood in $G$, then they also do in $G'$. Since the cluster containing $v'$ in $G'$ is larger than that containing $w$ in $G$, graph $G'$ is smaller than $G$ in our ordering (the comment following this proof recalls the definition of *smaller*). If $G'$ has a $(k-1)$-coloring, then it induces a $(k-1)$-coloring $\varphi$ of $G - \{w\}$. To extend $\varphi$ to $G$, uncolor $v$ and greedily color $w$, followed by $v$. This $(k-1)$-coloring of $G$ gives a contradiction.

So assume $G'$ has no $(k-1)$-coloring. Let $G''$ be a $k$-critical subgraph of $G'$, and let $R := V(G'')$. Since $G''$ is smaller than $G$, by induction $k(k-3) \geqslant \rho_{G''}(R) \geqslant \rho_{G'}(R)$. Since $G$ is $k$-critical, $G''$ is not a subgraph of $G$, so $v' \in R$. Removing $v'$ from $R$ removes a single vertex and $(k-1)$ edges, so $\rho_G(R \setminus \{v'\}) \leqslant k(k-3) - (k-2)(k+1) + 2(k-1)(k-1) = 2(k-2)(k-1)$. But this inequality contradicts Lemma 12.61, since $G[R \setminus \{v'\}] \neq K_{k-1}$, and also $R \setminus \{v'\} \neq V(G)$, since $w \notin R \setminus \{v'\}$. □

The proof of Lemma 12.64 is the only place in the proof of Theorem 12.49 where we use that $H$ is defined to be smaller than $G$ if $\|H\| = \|G\|$ and $H$ has more pairs of vertices with the same closed neighborhood.

**Lemma 12.65.** *If* $k = 6$ *and a cluster* $S$ *is contained in a* $K_5$*, then* $|S| = 1$.

*Proof.* Lemma 12.63 gives $|S| \leqslant \frac{6-1}{2}$, so assume that $|S| = 2$ and let $S = \{u_1, u_2\}$. Let $K$ be the $K_5$ containing $S$. Let $v_1, v_2, v_3$ be the other vertices of $K$, and $w$ be the neighbor of $S$ outside of $K$. Form $G'$ from $G - S$ by identifying $w$ and $v_1$; call the new vertex $w{*}v_1$; see Figure 12.15. If $G'$ has a 5-coloring $\varphi$, then we can extend $\varphi$ to $G$ by coloring $u_1$ and $u_2$ greedily, since they are 5-vertices and $\varphi(w) = \varphi(v_1)$. So assume $G'$ has no 5-coloring. Let $G''$ be a 6-critical subgraph of $G'$, and let $R := V(G'')$. Since $\|G''\| \leqslant \|G'\| < \|G\|$, graph $G''$ is smaller than $G$. So by induction $18 = k(k-3) \geqslant \rho_{G''}(R) \geqslant \rho_{G'}(R)$. Since $G$ itself is $k$-critical, $w{*}v_1 \in R$.
$s$ Let $s := |\{v_2, v_3\} \cap R|$. We have either $s = 0$, $s = 1$, or $s = 2$, and we consider these cases separately, though the analysis is similar in each case.

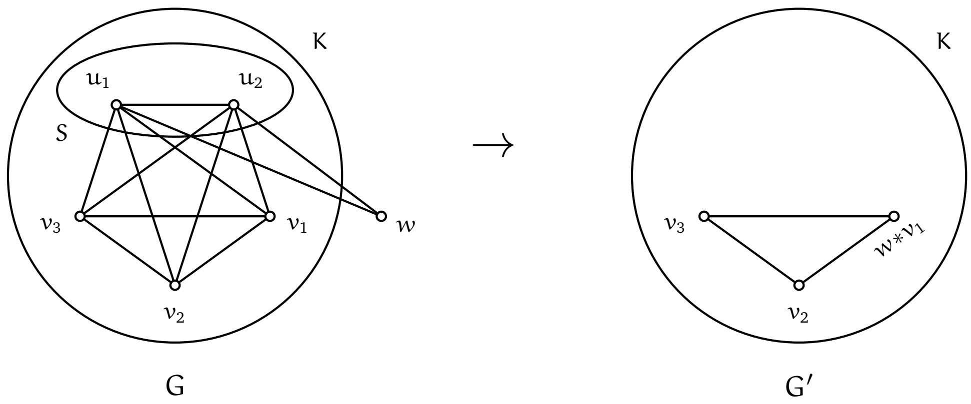



Figure 12.15: Forming $G'$ from $G$ in the proof of Lemma 12.65.

If $s = 0$, then $\rho_G((R - w{*}v_1 + w) \cup K) \leqslant \rho_{G'}(R) + 28(|K \cup \{w\}| - |\{w{*}v_1\}|) - 10\|G[K \cup \{w\}\| \leqslant 18 + 28(5) - 10(12) = 38$. So Lemma 12.61 implies that $(R - w{*}v_1 + w) \cup K) = V(G)$. But now we have at least one more edge incident to each of $v_2$ and $v_3$, that we have not accounted for. So $\rho_G(V(G)) \leqslant 38 - 10(2) = 18$, contradicting Lemma 12.61.

If $s = 1$, then $\rho_G((R - w{*}v_1 + w + v_1) \cup S) \leqslant \rho_{G'}(R) + 28(|S \cup \{w, v_1\}| - |w{*}v_1|) - 10(\|G[S \cup \{v_1, w\} \cup (R \cap \{v_2, v_3\})]\| - 1) \leqslant 18 + 28(3) - 10(7) = 32$. This inequality contradicts Lemma 12.61, since $((R - w{*}v_1 + w + v_1) \cup S) \neq V(G)$.

If $s = 2$, then $\rho_G((R - w{*}v_1 + w + v_1) \cup S) \leqslant \rho_{G'}(R) + 28(|S \cup \{w, v_1\}| - |w{*}v_1|) - 10(\|G[S \cup \{v_1, w\} \cup (R \cap \{v_2, v_3\})]\| - 1) \leqslant 18 + 28(3) - 10(9) = 12$. This inequality again contradicts Lemma 12.61. □

**Lemma 12.66.** *If* $k \geqslant 6$ *and* $v$ *is the unique* $(k-1)$*-vertex in a copy* $K$ *of* $K_{k-1}$*, then the number of* $(k+1)^+$*-vertices in* $K$ *is at least* $\frac{k-1}{2}$.

*Proof.* Let $u$ be the vertex in $N(v) \setminus K$; see Figure 12.16. Assume to the contrary that the number of $k$-vertices in $K$ is at most $\frac{k}{2} - 1$. Now $|N(u) \cap K| < \frac{k}{2}$, as in Lemma 12.63, since otherwise $\rho_G(K \cup \{u\}) \leqslant \rho(K_{k-1}) + (k-2)(k+1) - 2(k-1)|N(u) \cap K| \leqslant \rho(K_{k-1}) - 2$, contradicting Lemma 12.61. Since $u \not\leftrightarrow v$, there exists $w \in K \setminus N(u)$ with $d(w) \leqslant k$; in fact $d(w) = k$ since, by hypothesis, $v$ is the unique $(k-1)$-vertex in $K$. Let $y$ and $z$ be the vertices of $N(w) \setminus K$, and form $G'$ from $G - v$ by adding edges $uy$ and $uz$.

$u$ $w$ $G', y, z$ $G'$

Suppose $G'$ has a $(k-1)$-coloring $\varphi$. If $\varphi(u)$ is not used on $K \setminus \{w, v\}$, then recolor $w$ with $\varphi(u)$. (This is the point of adding edges $uy$ and $uz$.) Whether we recolored $w$ or not, $v$ will have two neighbors colored with $\varphi(u)$, so we can extend the $(k-1)$-coloring to $v$.

So instead $G'$ must have no $(k-1)$-coloring; let $G''$ be a $k$-critical subgraph of $G'$, and let $R := V(G'')$. Since $\|G'\| < \|G\|$, we know $G''$ is smaller than $G$, so by induction $\rho_{G'}(R) \leqslant k(k-3)$. If $R \neq V(G')$, then $\rho_G(R) \leqslant k(k-3) + 2(2)(k-1) < 2(k-2)(k-1)$, contradicting Lemma 12.61.

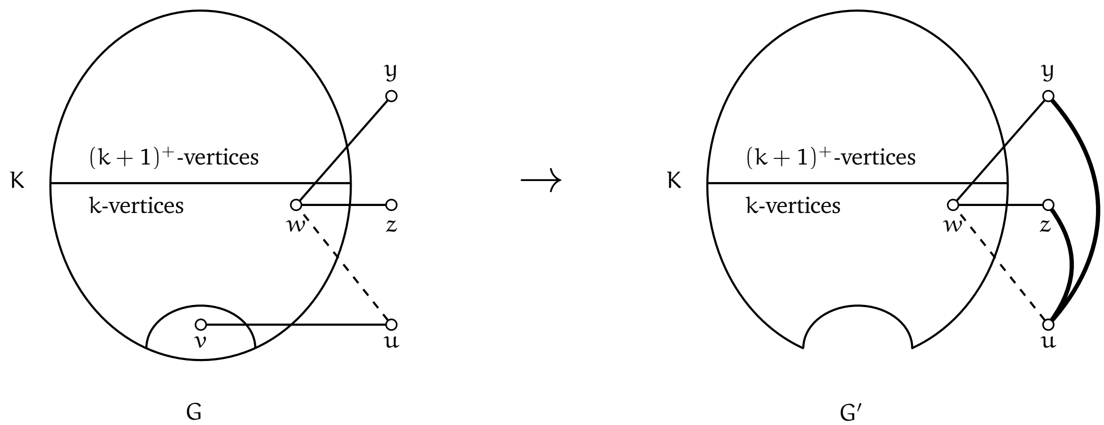



Figure 12.16: Forming $G'$ from $G$ in the proof of Lemma 12.66.

If $R = V(G')$, then $\rho_G(V(G)) = \rho_G(R \cup \{v\}) \leqslant k(k-3)+1(k-2)(k+1)-(k-3)(2)(k-1) < k(k-3)$ since $k \geqslant 6$, again contradicting Lemma 12.61. □

**Lemma 12.67.** *Choose $k \geqslant 6$. Let $S$ be a cluster in $G$, and let $s := |S|$. If $s \geqslant 2$, then both of the following 2 statements hold:*

*(a) If $N(S) \cup S$ contains no $K_{k-1}$, then $d_G(v) \geqslant k-1+s$ for every $v \in N(S) \setminus S$.*

*(b) If $N(S) \cup S$ contains a $K_{k-1}$ with vertex set $K$, then $d_G(v) \geqslant k-1+s$ for every $v \in K \setminus S$.*

*Proof.* We prove (a) and (b) together. Suppose the contrary, and choose $v \in N(S) \setminus S$ such that $d(v) \leqslant k-2+s$, and if $N(S) \cup S$ contains a $K_{k-1}$ then $v$ is in the $K_{k-1}$. Note that $d(v) \geqslant k$, by Lemmas 12.62 and 12.64. Also, $S$ is in at most one $K_{k-1}$, since the union of two copies would induce $K_k - e$, and $\rho(K_k - e) = (k-2)(k+1)$, which contradicts Lemma 12.61. As a result, $N(S) \cup S - v$ contains no $K_{k-1}$. Since $d(v) \leqslant k-2+s$, we have $|N(v) \setminus S| \leqslant k-2$. Choose $w \in S$ and let $G' := G - v + w'$, where $N[w'] = N[w]$. Suppose $G'$ has a $(k-1)$-coloring $\varphi'$. Now we can extend $\varphi'$ to $G$ as follows. We can color $v$, possibly using a color used on $S$, since $|N(v) \setminus S| \leqslant k-2$. Now each $w \in S$ keeps its color unless it was used to color $v$; in that case we recolor $w$ with $\varphi'(w')$. So $G'$ must have no $(k-1)$-coloring. Let $G''$ be a $k$-critical subgraph of $G'$, and let $R := V(G'')$.

$G', w'$

$G'', R$

Note that $\|G'\| < \|G\|$, since $d_G(v) \geqslant k$ and $d_{G'}(w') = k-1$. So $G''$ is smaller than $G$; by induction $\rho_{G'}(R) \leqslant k(k-3)$. Since $G$ is $k$-critical, $G''$ is not a subgraph of $G$, so $w' \in R$. Similarly, for any $x \in S$, graph $G'' - x$ is isomorphic to a subgraph of $G$. So $S \cup \{w'\} \subseteq R$. Now $\rho_G(R - w') \leqslant k(k-3)-(k-2)(k+1)+(k-1)2(k-1) = 2(k-2)(k-1)$. By Lemma 12.61, $W \setminus \{w'\}$ induces $K_{k-1}$. But this contradicts that, as noted above, $N(S) \cup S \setminus \{v\}$ contains no $K_{k-1}$; this contradiction proves the lemma. □

### 12.5.5 The Details of the Discharging for $\mathcal{L}_1$

We omit the proof for $k = 4$, since it is in Section 12.1. However, it is easy to check that Lemmas 12.10, 12.12, and 12.13 in the proof of Theorem 12.2 follow easily from the results of Section 12.5.3. Using these, the discharging argument here is the same as in that proof. For $k = 5$, we leave the discharging as an exercise. It is similar to, but simpler than, the case when $k \geqslant 6$, which we present below.

Recall that we use the following 3 discharging rules.

(R1) Each $(k+1)^+$-vertex splits its excess charge (that exceeding $k - \frac{2}{k-1}$) equally among its $(k-1)$-neighbors.

(R2) If a copy $J$ of $K_{k-1}$ contains $s$ $(k-1)$-vertices adjacent to a $(k-1)$-vertex $v$ outside of $J$, and $v$ is not in a $K_{k-1}$, then each of these $s$ vertices gives $\frac{k-3}{s(k-1)}$ to $v$.

(R3) After applying (R1) and (R2), the vertices of $\mathcal{L}_0 \cup \mathcal{H}_0$ average their charge.

To denote the charge at a vertex $v$ after applying the discharging rules, we write *ch*$^*(v)$. $ch^*(v)$
Near the end of Section 12.5.1 we showed that $ch^*(v) \geqslant k - \frac{2}{k-1}$, whenever $v \notin \mathcal{L}_1$. So here we show that also $ch^*(v) \geqslant k - \frac{2}{k-1}$ for each vertex $v \in \mathcal{L}_1$. We consider three cases, based on whether or not $v$ is in a $K_{k-1}$ and on the size of its cluster.

**Lemma 12.68.** *Choose $v \in \mathcal{L}_1$. If $v$ is in a cluster of size 1, then ch$^*(v) \geqslant k - \frac{2}{k-1}$.*

*Proof.* By the definition of $\mathcal{L}_1$, vertex $v$ has a $(k-1)$-neighbor $w$. If $v$ is in no $K_{k-1}$, then Lemma 12.64 implies that $w$ is in a $K_{k-1}$ and in a cluster of size at least 2. So $v$ receives $\frac{k-3}{k-1}$ from that cluster by (R2), and $ch^*(v) \geqslant k - 1 + \frac{k-3}{k-1} = k - \frac{2}{k-1}$.

So assume instead that $v$ is in a $K_{k-1}$; call it $K$. Let $L$ be the set of $(k+1)^+$-vertices in $K$. Lemma 12.66 implies $|L| \geqslant \frac{k-1}{2}$. By Lemma 12.61, all $(k-1)$-vertices in $K$ are in the same cluster. Since $v \in K$ and its cluster has size 1, all vertices of $K \setminus \{v\}$ are $k^+$-vertices. Thus, each $y \in L$ has at least $k-3$ neighbors of degree at least $k$. So by (R1) $y$ sends $v$ at least $\frac{d(y)-(k-\frac{2}{k-1})}{d(y)-(k-3)}$. This function increases with $d(y)$, so it is minimized at $d(y) = k+1$, where it equals $\frac{1+\frac{2}{k-1}}{4}$. Thus,

$$ch^*(v) \geqslant (k-1) + \frac{k-1}{2}\left(\frac{1+\frac{2}{k-1}}{4}\right) = k + \frac{k-7}{8}.$$

So $ch^*(v) \geqslant k - \frac{2}{k-1}$ when $k \geqslant 6$. $\square$

**Lemma 12.69.** *Choose $v \in \mathcal{L}_1$. If $v$ is in a cluster of size at least 2 and $v$ is not in a $K_{k-1}$, then ch$^*(v) \geqslant k - \frac{2}{k-1}$.*

*Proof.* Let $S$ be the cluster containing $v$, and let $s = |S|$; so $s \geqslant 2$. By Lemma 12.67(a), each vertex $w \in N(v) \setminus S$ has degree at least $k-1+s \geqslant k+1$. So by (R1) each such $w$ gives $v$ at least $\frac{d(w)-(k-\frac{2}{k-1})}{d(w)}$. This expression is minimized at $d(w) = k+s-1$, where it equals $\frac{s-1+\frac{2}{k-1}}{k+s-1}$. Thus, $ch^*(v) \geqslant k-1+(k-s)\left(\frac{s-1+\frac{2}{k-1}}{k+s-1}\right)$. We must show that $ch^*(v) \geqslant k - \frac{2}{k-1}$. So we let $ch^{**}(s) := \left[k-1+(k-s)\left(\frac{s-1+\frac{2}{k-1}}{k+s-1}\right) - (k-\frac{2}{k-1})\right](k+s-1)$. Since $ch^*(v)-(k-\frac{2}{k-1}) \geqslant \frac{1}{k+s-1}ch^{**}(s)$, it suffices to show that $ch^{**}(s) \geqslant 0$ for all possible $s$. By assumption, $s \geqslant 2$, and Lemma 12.63 shows $s \leqslant k-3$. Since $ch^{**}(s)$ is quadratic in $s$ with leading coefficient negative, we need only check that $ch^{**}(2) \geqslant 0$ and $ch^{**}(k-3) \geqslant 0$. Now we get

$$\begin{aligned}
ch^{**}(s) &= -(1-\frac{2}{k-1})(k+s-1)+(k-s)(s-1+\frac{2}{k-1}) \\
&= -(1-\frac{2}{k-1})(2k-1)+(k-s)s \\
&= -(1-\frac{2}{k-1})2(k-1)-(1-\frac{2}{k-1})+(k-s)s \\
&= -2(k-3)-(1-\frac{2}{k-1})+(k-s)s \\
&> s(k-s)-2k+5.
\end{aligned}$$

So $ch^{**}(2) > 2(k-2)-2k+5 = 1$, and $ch^{**}(k-3) > (k-3)3-2k+5 = k-4$. □

The proof of our final lemma is similar to the previous one. The main difference is that now $v$ is in a $K_{k-1}$, so it may give away charge by (R2). To allow for this, we also use an idea from the proof of Lemma 12.68 to show that each neighbor of $v$ in $\mathcal{H}_1$ must give it even more charge than in the proof of Lemma 12.69.

**Lemma 12.70.** *Choose $v \in \mathcal{L}_1$. If $v$ is in a cluster of size at least 2 and $v$ is in a $K_{k-1}$, then $ch^*(v) \geqslant k - \frac{2}{k-1}$.*

*Proof.* Lemma 12.65 shows that $k \geqslant 7$. Let $S$ be the cluster containing $v$, and let $K$ be the $K_{k-1}$ containing $v$. By Lemma 12.67(b), each $w \in K \setminus S$ has degree at least $k+s-1 \geqslant k+1$, so $w$ sends charge to $v$. Further, each of the $k-2-s$ neighbors of $w$ in $K \setminus S$ gets no charge from $w$, so $w$ gives $v$ at least $\frac{d(w)-(k-\frac{2}{k-1})}{d(w)-(k-2-s)}$. This expression increases with $d(w)$, so is minimized at $d(w) = k+s-1$, where it equals $\frac{s-1+\frac{2}{k-1}}{2s+1}$. So $ch^*(v) \geqslant k-1+(k-1-s)\left(\frac{s-1+\frac{2}{k-1}}{2s+1}\right) - \frac{k-3}{s(k-1)}$.

The rest of the proof is simply algebra showing that this expression is nonnegative. The trick is to know how to lower bound the expression, to make it simpler to evaluate. We begin by clearing denominators, and afterwards give away a bit in the bound to simplify. To show that $ch^*(v) \geqslant k - \frac{2}{k-1}$, note that $ch^*(v) - (k - \frac{2}{k-1}) \geqslant \frac{1}{(2s+1)(k-1)} ch^{**}(s)$, where

$$\begin{aligned} ch^{**}(s) &:= \left[k - 1 + (k-1-s)\left(\frac{s-1+\frac{2}{k-1}}{2s+1}\right) - \frac{k-3}{s(k-1)} - \left(k - \frac{2}{k-1}\right)\right](2s+1)(k-1) \\ &= \left[-\left(\frac{k-3}{k-1}\right)\left(1+\frac{1}{s}\right) + (k-1-s)\left(\frac{s-1+\frac{2}{k-1}}{2s+1}\right)\right](2s+1)(k-1) \\ &= -(k-3)\left(1+\frac{1}{s}\right)(2s+1) + (k-1-s)((s-1)(k-1)+2). \end{aligned}$$

So it suffices to show that $ch^{**}(s) \geqslant 0$ for all possible $s$. By hypothesis $s \geqslant 2$, so $ch^{**}(s) \geqslant ch^{***}(s)$, where $ch^{***}(s) := -(k-3)(\frac{3}{2})(2s+1) + (k-1-s)((s-1)(k-1)+2)$. Lemma 12.63 implies that $s \leqslant \frac{k-1}{2}$. Since $ch^{***}(s)$ is quadratic in $s$ with leading coefficient negative, we just check $ch^{***}$ at these boundaries; that is, we show that $ch^{***}(2) \geqslant 0$ and that $ch^{***}(\frac{k-1}{2}) \geqslant 0$. It is easy to check that $ch^{***}(2) = (k-3)(k - \frac{13}{2})$, which is positive, since $k \geqslant 7$. Similarly,

$$\begin{aligned} ch^{***}\left(\frac{k-1}{2}\right) &= \frac{k-1}{2}\left(\left(\frac{k-3}{2}\right)(k-1)+2\right) - k(k-3)\left(\frac{3}{2}\right) \\ &= \frac{1}{4}\left[(k-1)(k^2-4k+7) - (6k^2-18k)\right] \\ &= \frac{1}{4}\left[k^3 - 11k^2 + 29k - 7\right] \\ &= \frac{1}{4}(k-7)(k^2-4k+1). \end{aligned}$$

As above, this is nonnegative, since $k \geqslant 7$. □

## Notes

The content of Sections 12.1 and 12.5 is due to Kostochka and Yancey. In [272] they proved the much more general (and harder) result, Theorem 12.49, which gives a lower bound on $\|G\|$ for every $n$-vertex $k$-critical graph, for every integer $k \geqslant 4$. For the case $k = 4$, they later extracted the short proof [271] in Section 12.1. In subsequent work [273], they characterized all graphs for which the bound in Theorem 12.49 holds with equality. These are precisely the $k$-Ore graphs, from Section A.11.1. These results confirmed a conjecture of Gallai from 1963, and some cases of a more general conjecture of Ore from 1967. We discuss this at the start of Section 12.5.1; for more details, see the introduction of [272].

It is enlightening to note that the proof of Theorem 12.16 has some slack. Specifically, when $G$ has no 4-face, we conclude that $\|G\| \leqslant \frac{5\|G\|-10}{3}$. To get a contradiction, we only need $\|G\| < \frac{5|G|-2}{3}$. This slack suggests that if we slightly modify $G$, then the proof should still go through. This idea is due to Borodin, Kostochka, Lidický, and Yancey [68]. We investigate their approach in Exercises 1–4, where we prove various strengthenings of Theorem 12.16 (for further extensions, see [279]). Perhaps the nicest of their proofs using this idea is a short proof of Aksenov's strengthening of Grötzsch's Theorem, that every planar graph with at most three 3-cycles is 3-colorable. This result is sharp, due to $K_4$. In fact, there are infinitely many planar 4-critical graphs with exactly 4 triangles. These were characterized by Borodin, Dvořák, Kostochka, Lidický, and Yancey [55].

Theorem 12.18 is due to Borodin, Hartke, Ivanova, Kostochka, and West [73]. But the proof that we present in Section 12.2 is due to Dvořák and Postle.[17] They strengthened this result by showing [138] that if G is $C_5$-critical, then $\|G\| \geqslant (5|G|-2)/4$. Postle and Smith–Roberge [335] used a similar approach to show that if G is $C_7$-critical (and neither $C_3$ nor $C_5$), then $\|G\| \geqslant (17|G|-2)/15$. They also asked whether every $C_{2t+1}$-critical graph G has $\|G\| \geqslant (t(2t+3)|G|-(t+1)(2t-1))/(2t^2+2t-1)$, and gave constructions[18] showing that, if true, this bound is sharp.

Conjecture 4.44 (resp. Conjecture 4.45) posits that every planar graph of girth at least 4t (resp. odd-girth at least $4t+1$) has a map to $C_{2t+1}$. When G is planar, a map to $C_{2t+1}$ is equivalent to an orientation of the planar dual $G^*$ such that $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{2t+1}$. If a planar graph G has girth at least 6t, then its planar dual $G^*$ is 6t-edge-connected. Thus, $G^*$ has the desired orientation, by Theorem 6.28; so G has a map to $C_{2t+1}$. (Hence, Corollary 12.24 is implied by a very special case of Theorem 6.28.) For general t, this is the best known partial result toward these conjectures[19].

When t is small, however, the results in the previous paragraph give corollaries that are stronger. In particular, [138] shows that every planar graph with girth at least 10 (or odd-girth at least 11) maps to $C_5$. Similarly, [335] shows that every planar graph with girth at least 16 (or odd-girth at least 17) maps to $C_7$. Cranston and Li [95] proved more general results on flows in planar graphs that also yield these same corollaries. Again their approach uses the potential method, but the increased generality facilitates shorter proofs. Most recently, Cranston, Li, Wang, and Wei [96] extended this approach of flows in planar graphs to find maps to $C_9$. Specifically, they showed that every planar graph with odd-girth at least 23 maps to $C_9$.

Theorem 12.26 is due to Borodin and Kostochka [66], and it is the simplest among many related theorems. For each of the coloring problems below, the authors listed there defined an appropriate potential function and found a threshold t such that if G is critical, then its potential is at most t. Further, they constructed an infinite family of critical graphs with potential equal to t. Cranston and Yancey [107] studied $(I, F_k)$-coloring[20], in which color I induces an independent set and $F_k$ induces a forest, with each tree of order at most k. (The

[17] Luke Postle sketched this proof in a lecture at the Banff International Research Station, on 20 October 2016.

[18] These constructions begin with a $(2t+2)$-Ore graph and subdivide each edge $2t-2$ times. It is straightforward to check that the resulting graphs are $C_{2t+1}$-critical. Exercise 8 asks the reader to supply the details.

[19] The Notes in Chapter 4 discuss earlier work in this direction. And the Notes in Chapter 6 discuss work in the more general setting of flows.

[20] Their proof requires a Weak Gap Lemma and a Strong Gap Lemma, and the proof of their Strong Gap Lemma bears a striking resemblance to that of Lemma 12.61. So we recommend [107] to the reader interested in seeing another such example.

case $k = 2$ is Theorem 12.26.) Similarly, they studied $(I, F)$-coloring [106], where the order of each tree is not restricted. Borodin, Kostochka, and Yancey [64] studied 1-defective 2-coloring; here we have 2 colors and each vertex has at most one neighbor with the same color.

Finally, Borodin and Kostochka studied $(j, k)$-defective coloring. Here we have colors $j$ and $k$ and the subgraph induced by color $j$ (resp. $k$) has maximum degree at most $j$ (resp. $k$). They essentially solved the problem [67] when $k \geqslant 2j + 2$. Yancey [420] later strengthened their result by showing that, under the same hypotheses, we can also guarantee that the subgraph induced by each color class is acyclic.

Although in a slightly different vein, Theorem 12.38, on $(I^*, F)$-coloring fits best into this context of precoloring.[21] It is due to Brandt, Ferrara, Kumbhat, Loeb, Stolee, and Yancey [73]. The results in [73], [106], [107], and [67] all rely heavily on precoloring (although the last of these does not use that language). Extensive work has been done on defective coloring, although we are not aware of any other results using the potential method. For more on this topic, we recommend the excellent Dynamic Survey [416] by David Wood.

Example 12.42 is due to Thomas and Walls [374]. Theorem 12.43 is due to Moore and Smith-Roberge [310]. It extends work of Liu and Postle [285], who proved the same lower bound for the smaller class of 4-critical graphs of girth at least 5. Liu and Postle also conjectured that if $G$ is 4-critical and triangle-free, then $\|G\| \geqslant (5|G|+5)/3$. However, this conjecture was disproved by Davies (see Theorem A.40) who constructed infinitely many triangle-free 4-critical graphs $G_i$ with $\|G_i\| = (5|G_i|+4)/3$; this shows that Corollary 12.44 is nearly sharp. Postle [333] and Gao and Postle [171] strengthened the bound of Theorem 12.49 for 5-critical and 6-critical graphs (respectively) with certain subgraphs forbidden. In a similar direction, Postle characterized all 4-critical graphs with maximum degree 4 in which the 4-vertices form an independent set [334]. All of these proofs prove a more general statement (similar to Theorem 12.43) based on a modified potential function that incorporates some additional features of the graph as well, e.g., the maximum number of disjoint copies of $K_{k-1}$ or the size of a largest independent set comprised entirely of 3-vertices.

Behind nearly every proof using the potential method is a typical graph coloring proof using discharging. As we mentioned at the end of Section 1.1.3, such proofs generally translate naturally to polynomial-time algorithms. The proofs in this chapter are no exception. The main added wrinkle is that now we also need to efficiently find a (proper) subset of vertices of minimum potential. This can be viewed as a submodular minimization problem or (more concretely) as a network flow problem. So it can indeed be done in polynomial time. For more details on this topic, we recommend [106, Sections 2.3 and 5] and [272, Section 7].

The potential method is relatively new, but has already been used in numerous papers; a few early examples include [68, 275, 64, 67]. In particular, we recommend [73] and [107]. Postle gives a nice overview of the approach in [332], even though his full proof is much longer. The potential method was introduced by Xuding Zhu in work on the 9 Dragon Tree Conjecture.[22] It

---

[21] A *star* coloring is a proper coloring where each pair of color classes induces a star forest. The notion of $(I^*, F)$-coloring was introduced because each $(I^*, F)$-coloring naturally gives rise to star 4-coloring; see [8].

[22] Personal communication from André Raspaud on 24 July 2020.

first appeared, in a basic form, in [309] and later appeared more fully developed in [257].

## Exercises

12.1. Let $G$ be a triangle-free planar graph, and form $G'$ from $G$ by adding a vertex with four neighbors in $G$. Modify the proof of Theorem 12.16 to show that $G$ must be 3-colorable.

12.2. Let $G$ be a triangle-free planar graph, and form $G'$ from $G$ by adding an edge joining two vertices of $G$. Show that $G'$ must be 3-colorable.

12.3. Use the two previous exercises to show that if $G$ is a triangle-free planar graph and $f$ is $5^-$-face of $G$, then every 3-coloring of $f$ extends to a 3-coloring of $G$.

12.4. Show that if $G$ is a triangle-free planar graph, and $v$ and $w$ are non-adjacent vertices of $G$, then every coloring of $v$ and $w$ extends to a 3-coloring of $G$.

12.5. Construct an infinite family of $(1,0)$-critical graphs $G_t$ with $6|G_t| - 5\|G_t\| = -3$. Prove that they are all $(1,0)$-critical. [64].

12.6. Extend Theorem 12.26 to allow precolorings. Reprove the Gap Lemma in this more general framework.

12.7. An $(I, F_4)$-coloring of a graph $G$ partitions $V(G)$ into sets $I$ and $F_4$ such that $I$ is independent and $F_4$ induces a forest where each component has order at most 4. Use precoloring (which can be simulated by gadgets, as in Section 12.4.1) and the potential method to show that a graph $G$ has an $(I, F_4)$-coloring whenenver $\text{mad}(G) \leqslant 30/11$. [107] [This exercise is longer than most in this book, and in the Hints we give a detailed outline of the proof.]

12.8. Prove that if we begin with a $(2t+2)$-Ore graph and subdivide every edge $2t-2$ times, then the resulting graph is $C_{2t+1}$-critical. [335]

12.9. Let $G_1$ and $G_2$ be graphs with $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. A Hajós join of $G_1$ and $G_2$ is formed by deleting $u_1v_1$ and $u_2v_2$, identifying $u_1$ and $u_2$, and adding the edge $v_1v_2$. Show that if $G_1$ and $G_2$ are $k$-critical, then so is this Hajós join. Find 4-critical graphs on 6 vertices and on 8 vertices (with 10 and 13 edges) to complete the details of the remark at the start of Section 12.5.1.

12.10. Provide the details for the proofs of Lemma 12.53(b) and Corollary 12.54(b).

12.11. When $k = 5$, show that $G$ in the proof of Theorem 12.49 has the following two properties. (i) Each cluster has only one vertex. (ii) Each copy of $K_4$ in $G$ contains at most one 4-vertex. If $u$ and $v$ are adjacent 4-vertices, then each of $u$ and $v$ is in a copy of $K_4$.

12.12. Using the previous two exercises, provide the details of the discharging argument in the proof of Theorem 12.49 when $k = 5$.

# Appendix A

# The Rest of the Story

What I cannot create, I do not understand.

—Richard Feynman

In this appendix we collect various results that are both important and closely related to material that we discuss elsewhere, but that do not fit well into any of our previous chapters. Some sections are not strictly about graph coloring, but provide tools that are useful in proving coloring results. The sections are ordered roughly by increasing difficulty.

## A.1 A Brief Introduction to Complexity Theory

The *running time* of an algorithm to solve a graph problem is an infinite sequence $a_1, a_2, \ldots$, where $a_i$ denotes the maximum number of steps used by the algorithm on any input of order $i$. An algorithm runs in *polynomial time* if there exists a polynomial $f$ such that $a_i \leqslant f(i)$ for every positive integer $i$. The study of running times forms a core part of algorithm analysis. We recommend to the interested reader the treatment by Erickson [154].

polynomial time

A graph problem is *polynomial-time solvable* (or simply *polynomial-time*) if there exists some algorithm to solve it that runs in polynomial time. For short, we say the problem is *in P*. Examples of such problems include determining: the degeneracy of $G$ (see Lemma 1.23), $\text{mad}(G)$ (see Exercise 1.7), the maximum $k$ such that $G$ is $k$-edge-connected (this can be done using a maximum-flow algorithm), the maximum size of a matching in $G$, and many other maxima and minima.

polynomial-time solvable

in P

A problem is *non-deterministic polynomial-time solvable* if there exists an algorithm, called a *verifier*, such that the following holds. For every input graph $G$ for which the answer to the problem is yes, there exists a certificate $c(G)$—think of a bit string—such that the verifier can use $c(G)$ to confirm that the answer is yes in polynomial time. (If the certificate exists, but we do not know what it is, then we can think of guessing the certificate *non-deterministically*, which motivates the name above.)

non-deterministic polynomial-time solvable

All problems in P are also trivially in NP. Examples of problems in NP, that are not known to be in P, include determining whether a graph is $k$-colorable for some fixed integer $k$ (the certificate is the coloring), determining whether $\omega(G) \geqslant k$, for some input graph $G$ and integer $k$ (the certificate is a list of vertices forming a $k$-clique), and determining whether an input graph $G$ contains a path that includes every vertex in the graph, a so-called Hamiltonian path, (the certificate is the order in which the vertices appear in the path).

NP-hard

Informally, a problem is *NP-hard* if it is at least as hard to answer correctly as every other problem in NP. Formally, a problem $A$ is NP-hard if for every problem $B$ in NP there exists a polynomial $f$ such that for every instance $b$ of $B$ we can construct an instance $a$ of $A$ such that $|a| \leqslant f(|b|)$ and the answer to question $A$ on instance $a$ agrees with the answer to question $B$ on instance $b$. A problem is *NP-complete* if it is both NP-hard and also in NP.

NP-complete

Most problems that we consider in this book are in NP, so we do not emphasize the distinction between NP-complete and the more general NP-hard. However, many problems are known to be NP-hard, but conjectured not to be in NP. In other words, they are conjectured to be harder than NP-complete problems. (Problems in one such class are known as PSPACE-complete.) These include generalized versions of many well-known games, such as Checkers, Hex, Rush Hour, Sokoban, and Super Mario Bros., to name a few. To read more about the complexity of various games, we recommend Hearn and Demaine [211]. For a more general introduction to complexity theory, see Wigderson [413].

## A.2 The Petersen Graph is Not 3-Edge-Colorable

The Petersen graph was introduced in 1898 by Petersen [328] as a counterexample to Tait's claim that every bridgeless 3-regular graph is 3-edge-colorable.[1] For completeness, we mention that the Petersen graph can be represented with its vertices as the 2-element subsets of $\{1, \ldots, 5\}$, where two vertices are adjacent when their corresponding subsets are disjoint; see Figure A.1. It is well-known to be vertex-transitive and edge-transitive; further, the Petersen graph has exactly 6 perfect matchings. (Proving these facts is an easy exercise, so we omit the details.)

As mentioned above, the Petersen graph is not 3-edge-colorable. (So it also has no NZ 4-flow, by Exercise 6.4.) However, most proofs of this result require tedious case analysis. The following short proof is due to Naserasr and Škrekovski [317].

**Theorem A.1.** *If $P$ denotes the Petersen graph, then $\chi'(P) = 4$.*

*Proof.* By Vizing's Theorem, $\chi'(P) \leqslant 4$. So it suffices to prove $\chi'(P) > 3$.

Suppose, to the contrary, that $\chi'(P) = 3$ and fix a 3-edge-coloring of $P$. We view $P$ as an outer 5-cycle, an inner 5-cycle (drawn as a star), and a matching between them; see Figure A.1. We call these $C$, $C'$, and $M$, respectively. Suppose color 1 is used on an edge $vw$ of $C$. Let $vv'$ and $ww'$ be the edges of $M$ that are incident to $v$ and $w$. Since $P$ is 3-regular, every color

[1] In fact, the Petersen graph was presented in 1886 by Kempe for a different purpose.

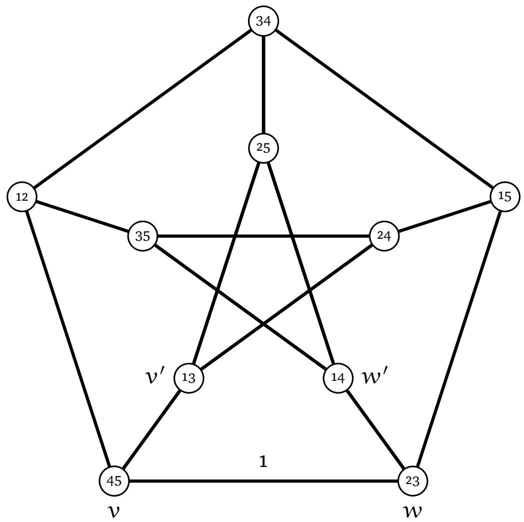



Figure A.1: The Petersen graph has no 3-edge-coloring.

appears incident to each vertex. Since color 1 is not used on $vv'$ or $ww'$ (and $v' \not\leftrightarrow w'$), color 1 must be used on two edges of $C'$. Since $\chi'(C) = 3$, each of colors 1, 2, and 3 must be used on $C$. So each of colors 1, 2, and 3 must be used on two edges of $C'$. But this is impossible, since $3(2) > 5 = \|C'\|$. This contradiction shows $\chi'(P) > 3$. □

## A.3 Hall's Theorem

Hall's Theorem, proved by Philip Hall in 1935 [199], gives necessary and sufficient conditions for a matching in a bipartite graph that saturates one part. It is a surprisingly versatile result, and has been proved and generalized in many ways. The standard formulation is only for finite graphs. However Marshall Hall Jr., no relation to Philip, proved an extension [198] to infinite graphs in which each degree is finite. (This hypothesis is necessary as witnessed by the following example, which has $|N(S)| \geqslant |S|$ for all $S \subseteq X$, but has no matching saturating $X$: $X = \{x_i : i \geqslant 1\}$, $Y = \{y_i : i \geqslant 1\}$, $N(x_1) = Y$, and $N(x_i) = \{y_{i-1}\}$ for all $i \geqslant 2$.) The proof we present below is essentially due to Halmos and Vaughan [200].

**Theorem A.2 (Hall's Theorem).** *Let $G$ be a bipartite graph with parts $X$ and $Y$. Now $G$ has a matching saturating $X$ if and only if $|N(S)| \geqslant |S|$ for every $S \subseteq X$.*

*Proof.* Clearly this hypothesis is necessary, since if $|N(S)| < |S|$, then no matching saturates $S$. So we assume $G$ satisfies the hypothesis. Our proof is by induction on $|X|$. For each $S \subseteq X$, let $f(S) := |N(S)| - |S|$. We restate the hypothesis as $f(S) \geqslant 0$ for all $S \subseteq X$. f

**Case 1: $f(S) > 0$ for all nonempty $S$ with $S \subsetneq X$.** Let $xy$ be an arbitrary edge of $G$, let $G' := G - \{x, y\}$, and let $f'(S) := |N_{G'}(S)| - |S|$ for each $S \subseteq X \setminus \{x\}$. Note that $N_{G'}(S) \supseteq (N_G(S) \setminus \{y\})$, so $f'(S) \geqslant f(S) - 1 \geqslant 0$. By induction, $G'$ has a matching $M'$ that saturates $X \setminus \{x\}$. Thus, $M' \cup \{xy\}$ saturates $X$, as desired.

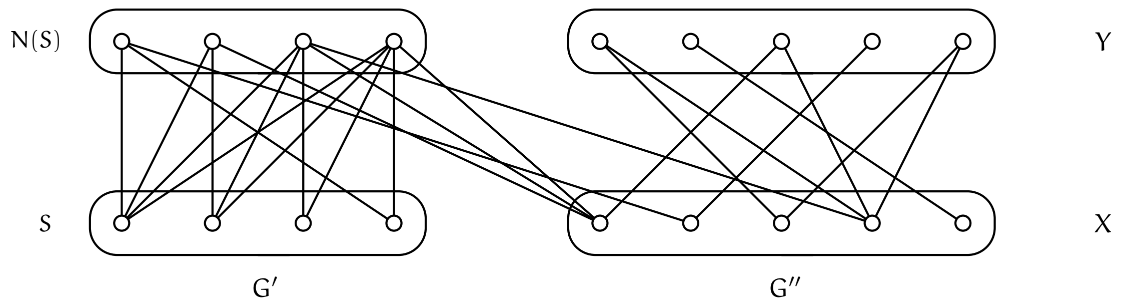



Figure A.2: An example of the case $f(S) = 0$ in the proof of Hall's Theorem.

**Case 2: $f(S) = 0$ for some nonempty $S$ with $S \subsetneq X$.** (See Figure A.2.) Among all such $S$, choose one that is smallest but nonempty. Let $G' := G[S \cup N(S)]$ and $G'' := G \setminus (S \cup N(S))$. For all $T \subseteq S$, we have $N_G(T) \subseteq N_G(S)$, so $N_{G'}(T) = N_G(T)$. Thus, $f'(T) = f(T) \geqslant 0$. So, by induction, subgraph $G'$ has a matching $M'$ saturating $S$. Let $X'' := X \cap V(G'')$. For each $T \subseteq X''$, let $f''(T) := |N_{G''}(T)| - |T|$. By hypothesis, we must have $f''(T) \geqslant 0$ for all $T \subseteq X''$. If not, then $f(S \cup T) = |N(S \cup T)| - |S \cup T| = (|N(S)| - |S|) + (|N_{G''}(T)| - |T|) = f''(T) < 0$, contradicting the hypothesis. Now, by induction, subgraph $G''$ has a matching $M''$ saturating $X''$. Thus $M' \cup M''$ saturates $X$. $\square$

We mention one beautiful application of Hall's Theorem. A *permutation matrix* is a 0/1 square matrix with each row and each column containing exactly one 1. A *doubly-stochastic matrix* is a square matrix with all entries being nonnegative real numbers, such that each row and each column sums to 1. (So each permutation matrix is doubly-stochastic, but not vice versa.) A *convex combination* is a linear combination where all coefficients are nonnegative and sum to 1. Exercise: Prove that every doubly-stochastic matrix is a convex combination of permutation matrices. [40, 405, 414]

## A.4 The Strong Chromatic Number is Well-defined

In this section we consider the strong chromatic number of a graph, which we discuss in Section 10.3. We need the following definitions.

**Definition A.3.** A $k$-*partition* of a graph $G$ is a partition of $V(G)$ into parts of size $k$, first adding $\lceil |G|/k \rceil k - |G|$ isolated vertices, so the total number of vertices is a multiple of $k$. We often call these added isolated vertices *fake* vertices. A proper $k$-coloring $\varphi$ of a graph $G$ *respects* *a* $k$*-partition* of $G$ if each color class of $\varphi$ is an independent transversal of the partition; that is, each color class contains exactly one vertex from each part of the partition. A graph $G$ is *strongly* $k$*-colorable* if, given any $k$-partition of $G$, there exists a $k$-coloring of $G$ that respects the partition. The *strong chromatic number*, $\chi_s(G)$, of a graph $G$ is the minimum $k$ such that $G$ is strongly $k$-colorable.

It is true, but not obvious, that every strongly $k$-colorable graph is also strongly $(k+1)$-colorable. Fellows [160] proved this for infinite graphs; here we adapt his proof for finite graphs.

**Theorem A.4.** *If* $G$ *is strongly* $k$*-colorable, then* $G$ *is also strongly* $(k+1)$*-colorable.*

Our basic idea is to turn a $(k+1)$-partition of $G$ into a $k$-partition of $G$, get a $k$-coloring $\varphi'$ respecting the $k$-partition (by hypothesis), and turn $\varphi'$ into a $(k+1)$-coloring $\varphi$ that respects the original $(k+1)$-partition. But the proof is subtle: we actually turn our $(k+1)$-partition of $G$ into a $k$-partition of some subgraph of $G$; that $G$ is strongly $k$-colorable is also used twice, rather than just once, as we might naively expect.

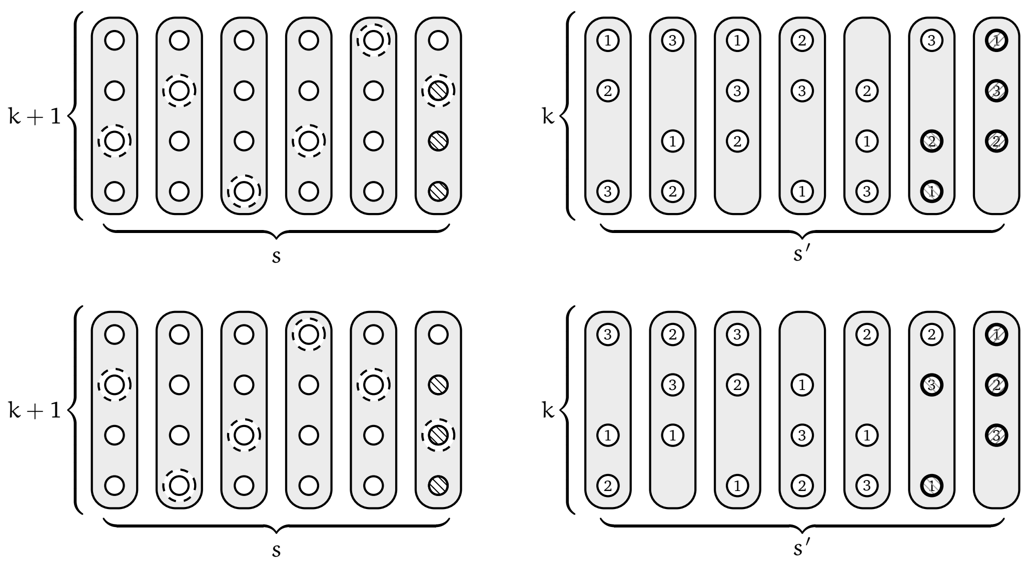



Figure A.3: An example of the proof of Theorem A.4, with $k = 3$, $s = 6$, and $s' = 7$. For clarity, edges are omitted throughout, and parts are denoted by shaded regions. Top Left: A $(k+1)$-partition $\mathcal{P}$ is given. An arbitrary transversal $I$ of $\mathcal{P}$ is chosen (possibly not independent). Three "fake" vertices are added to reach an order that is a multiple of $k+1$. Top Right: The $k$-partition $\mathcal{P} - I$ of $G - I$ is extended to a $k$-partition $\mathcal{P}'$ of $G$. And a strong $k$-coloring $\varphi'$ (w.r.t. $\mathcal{P}'$) of $G$ is found by hypothesis. Bottom Left: $\varphi^{-1}(2)$ is an independent transversal of $G$ (w.r.t. $\mathcal{P}$), so $\mathcal{P} - \varphi^{-1}(2)$ is a $k$-partition of $G - \varphi^{-1}(2)$. Bottom Right: The $k$-partition $\mathcal{P} - \varphi^{-1}(2)$ of $G - \varphi^{-1}(2)$ is extended to a $k$-partition $\mathcal{P}''$ of $G$. By assumption, $G$ has a strong $k$-coloring $\varphi'$ w.r.t. $\mathcal{P}''$. Now $\varphi''$ extends $\varphi'$ to $\varphi^{-1}(2)$, using color $k+1$ there, to get a strong $(k+1)$-coloring w.r.t. the original $(k+1)$-partition $\mathcal{P}$.

$s, s'$ *Proof.* Let $G$ be a graph that is strongly $k$-colorable. Let $s := \lceil |G|/(k+1) \rceil$ and $s' := \lceil |G|/k \rceil$.
$V_i$ Suppose we are given $V_1 \uplus \cdots \uplus V_s$, a $(k+1)$-partition of $G$. Let $I$ be a transversal of this
$W_i$ $(k+1)$-partition (not necessarily independent). Form $W_1 \uplus \cdots \uplus W_{s'}$ from $V_1 \uplus \cdots \uplus V_s$ by deleting the vertex in $I$ from each part and adding $s' - s$ new parts, each consisting entirely of
$G'$ $k$ fake vertices. Note that $W_1 \uplus \cdots \uplus W_{s'}$ is a $k$-partition of some subgraph $G'$ of $G$. (It has the right total number of vertices, and every vertex either came from $G$ or is a fake vertex added when forming either $V_1 \uplus \cdots \uplus V_s$ or $W_1 \uplus \cdots \uplus W_{s'}$. Any extra fake vertices can be viewed as arising by deleting all edges incident to some real vertices of $G$.)

Since $G' \subseteq G$, by hypothesis, $G'$ has a $k$-coloring $\varphi$ respecting $W_1 \uplus \cdots \uplus W_{s'}$. Let $I'$ be an arbitrary color class (independent transversal) of $\varphi$. Form $X_1 \uplus \cdots \uplus X_{s'}$ from $V_1 \uplus \cdots \uplus V_s$ by deleting the vertex of $I'$ from each part and adding $s' - s$ new parts, each consisting entirely of $k$ fake vertices. Similar to above, this is a $k$-partition of some subgraph $G''$ of $G$. (The key difference is that $I'$ is an independent transversal of $V_1 \uplus \cdots \uplus V_s$, whereas $I$ is a transversal, but not necessarily independent.) By hypothesis, $G''$ has a $k$-coloring $\varphi''$ respecting $X_1 \uplus \cdots \uplus X_{s'}$. When we restrict $\varphi''$ to $V_1 \uplus \cdots \uplus V_s$, this gives a $k$-coloring $\varphi$ of $G \setminus I'$. Starting from $\varphi$, and using color $k+1$ on $I'$, gives a $(k+1)$-coloring of $G$ that respects $V_1 \uplus \cdots \uplus V_s$, as desired. □

## A.5 The Equivalence of a face-k-coloring and an NZ k-flow

Recall that we typically shorten *nowhere-zero flow* to *NZ flow*. We stated the result below as Theorem 6.6, but have deferred the proof until now; it was first proved by Tutte [390].

**Theorem A.5.** *A plane graph has a proper face-$k$-coloring if and only if it has an NZ $k$-flow.*

*Proof.* Let $\varphi$ be a proper face-coloring of a plane graph $G$ with colors $\{1, \ldots, k\}$. We define a nowhere-zero $k$ flow $\varphi'$ as follows. For an arbitrary edge $e$, say the faces bounding $e$ are $f_1$ and $f_2$, with $\varphi(f_1) > \varphi(f_2)$. Orient $e$ so $f_1$ is on its right and let $\varphi'(e) = \varphi(f_1) - \varphi(f_2)$. Clearly $0 < \varphi'(e) < k$ for each edge $e$. Now we must check that the net flow into each vertex is 0. Consider a vertex $v$ with incident faces $f_1, \ldots, f_s$ (where we let $s := d(v)$) in clockwise order and let $e_i$ be the edge between $f_i$ and $f_{i+1}$ (subscripts modulo $s$). Observe that the net flow into $v$ is $(\varphi(f_1) - \varphi(f_2)) + (\varphi(f_2) - \varphi(f_3)) + \cdots + (\varphi(f_s) - \varphi(f_1)) = 0$. This is easy to check by orienting all edges into $v$, and negating the flow on each edge we reverse, since now each $f_i$ is on the right of $e_i$ and the left of $e_{i-1}$.

Now we reverse the process above. Given an NZ $k$-flow $\varphi$ on $G$, we find a proper face-$k$-coloring of $G$. We initially color the faces with integers (not necessarily nonnegative), and at the end we take residues modulo $k$. Color the outer face 0. To color the remaining faces, we repeat the following. See Figure A.4. Let $e$ be an edge bounded by faces $f_1$ and $f_2$, with $f_1$ colored using $\alpha$ and $f_2$ uncolored. If $f_1$ lies to the left (resp. right) of $e_1$, then we assign $f_2$ color $\alpha - \varphi(e)$ (resp. $\alpha + \varphi(e)$). We only need to check this this coloring is consistent, that is, the color assigned to each face $f$ does not depend on the sequence of faces we took from the outer face to $f$.

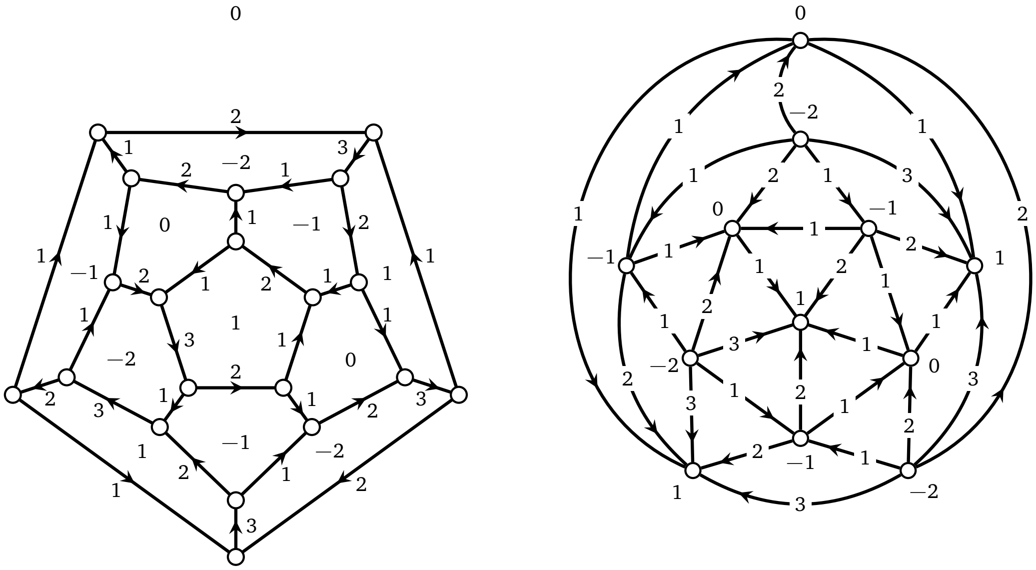


Figure A.4: The proof of Theorem A.5. Left: An NZ 3-flow and the face-4-coloring that it induces (ultimately, all face colors are taken modulo 4). Right: A corresponding flow in the dual graph such that the net flow along every closed cycle is 0.

For the planar dual $G^*$, we show the following: Fix $v, w \in V(G^*)$, where $v$ is colored and $w$ is not. For any two paths, $P_1$ and $P_2$, from $v$ to $w$, vertex $w$ is assigned the same color whether we color along $P_1$ or $P_2$. We assume that $P_1$ and $P_2$ are internally disjoint, since if they have a common internal vertex $x$, we can split into paths from $v$ to $x$ and paths from $x$ to $w$ (formally, this uses induction on $|V(P_1) \cap V(P_2)|$).

We claim that $w$ gets assigned the same color along $P_1$ as it does along $P_2$. This is equivalent to saying that if we color along $P_1$, and continue along the reverse of $P_2$, then $v$ is assigned the same color at the end as it began with. We rephrase this latter statement as follows: If we view $P_1 \cup P_2$ as a closed curve in the plane, then the net flow into the vertex subset of G that lies inside $P_1 \cup P_2$ is 0. And this final version holds because the net flow into each vertex of G is 0; alternately, from Observation 6.3. (Formally, we could use induction on the number of vertices in G inside the region bounded by $P_1 \cup P_2$.) $\square$

## A.6 Menger's Theorem

In Section 3.1 we stated part of Lemma A.6(c) below as Theorem 3.17 and used it to prove Theorem 3.16. Here we state the theorem in a more general form and also provide a proof. This result is a cornerstone in connectivity and plays a central role in that arena akin to the role played by Hall's Theorem or Tutte's 1-Factor Theorem in the area of matching.

Part (a) is a direct consequence of the Max-flow/Min-cut Theorem. Each part of Menger's Theorem has various proofs, but most are essentially by induction. For the version with vertex-disjoint paths in an undirected graph, Diestel [117, Section 3.3] gives three different proofs.

**Theorem A.6 (Menger's Theorem).** *Fix a digraph $D$, a positive integer $k$, and $v_1, v_2 \in V(D)$. (a) Now $D$ has $k$ edge-disjoint $v_1, v_2$-paths if and only if the smallest $v_1, v_2$-separating edge set has size at least $k$. (b) Assume $\overrightarrow{v_1v_2} \notin E(D)$. Now $D$ has $k$ vertex-disjoint $v_1, v_2$-paths if and only if the smallest $v_1, v_2$-separating set has size at least $k$. (c) The statements analogous to (a) and (b) for an undirected graph $G$ are also true.*

*Proof.* We first prove (a), use (a) to prove (b), and use (a) and (b) to prove (c).

(a) If $D$ has $k$ edge-disjoint $v_1, v_2$-paths, then every $v_1, v_2$-separating edge set intersects each path, so has size at least $k$. Now we prove the other direction by induction on $|E(D)|$.

k-edge-connected between $v_1$ and $v_2$

If the smallest $v_1, v_2$-separating set in $D$ has size at least $k$, then we say that $D$ is $k$-*edge-connected between* $v_1$ *and* $v_2$. If there exists $e \in E(D)$ such that $D - e$ is $k$-edge-connected between $v_1$ and $v_2$, then $D - e$ has $k$ edge-disjoint $v_1, v_2$-paths by induction; thus, so does $D$. So assume instead that no such $e$ exists; in particular $d^-(v_1) = 0$ and $d^+(v_2) = 0$.

Suppose there exists $e_1 \in E(D)$ that is incident with neither $v_1$ nor $v_2$. Since $D - e_1$ is not $k$-edge-connected between $v_1$ and $v_2$, there exist edges $e_2, \ldots, e_k$ that form a $v_1, v_2$-separating edge set in $D - e_1$. So $\{e_1, \ldots, e_k\}$ separates $v_1$ and $v_2$ in $D$. In $D - \{e_1, \ldots, e_k\}$, let $V_1$ denote the set of vertices reachable from $v_1$ and let $V_2$ denote the vertices from which $v_2$ is reachable. Note that $V(D) = V_1 \cup V_2$. Form digraphs $D_1$ and $D_2$ from $D$ by contracting (respectively) sets $V_2$ and $V_1$; call these new vertices $v_2'$ and $v_1'$. Since $D_1$ is $k$-edge-connected between $v_1$ and $v_2'$ (and $|E(D_1)| < |E(D)|$), by induction $D_1$ has $k$ edge-disjoint $v_1, v_2'$-paths, $P_1, \ldots, P_k$; since $d^-_{D_1}(v_2') = k$, we assume each $P_i$ ends with $e_i$. Similarly, $D_2$ has $k$ edge-disjoint $v_1', v_2$-paths, $Q_1, \ldots, Q_k$, and we assume each $Q_i$ starts with edge $e_i$. Thus, $P_1 \cup Q_1, \ldots, P_k \cup Q_k$ are edge-disjoint $v_1, v_2$-paths in $D$.

So assume instead that each edge in $D$ is incident with $v_1$ or $v_2$ (or both). If $v_1v_2 \in E(D)$, then we remove it and use induction on $k$. Now the maximum number of edge-disjoint $v_1, v_2$-paths through each vertex $w \in V(D) \setminus \{v_1, v_2\}$ is simply $\min\{\mu(v_1w), \mu(wv_2)\}$. So the maximum number of edge-disjoint $v_1, v_2$-paths is $\sum_{w \in V(D)\setminus\{v_1,v_2\}} \min\{\mu(v_1w), \mu(wv_2)\}$. To get a $v_1, v_2$-separating edge set of the same size, for each $w$ we take either the edges from $v_1$ to $w$ or those from $w$ to $v_2$, whichever are fewer. This proves (a).

(b) We form a new digraph $D'$ from $D$ and apply part (a) to $D'$, as follows. For each vertex $x \in V(D)$, we create $x_{\text{in}}, x_{\text{out}} \in V(D')$ and $\overrightarrow{x_{\text{in}}x_{\text{out}}} \in E(D')$. Further, for each $\overrightarrow{vw} \in E(D)$, we have $\overrightarrow{v_{\text{out}}w_{\text{in}}} \in E(D')$. Finally, we delete $v_{1\text{in}}$ and $v_{2\text{out}}$; see Figure A.5. We show that (i) $D$ has $k$ vertex-disjoint $v_1, v_2$-paths if and only if $D'$ has $k$ edge-disjoint $v_{1\text{out}}v_{2\text{in}}$-paths and (ii) $D$ is $k$-connected between $v_1$ and $v_2$ if and only if $D'$ is $k$-edge-connected between $v_{1\text{out}}$ and $v_{2\text{in}}$. We start with (i). Given paths in $D'$, we contract each edge $\overrightarrow{x_{\text{in}}x_{\text{out}}}$ to get paths in $D$. Reversing this process takes vertex-disjoint paths in $D$ to edge-disjoint paths in $D'$. So (i) holds.

Now we prove (ii). Let $S'$ be a $v_1, v_2$-separating edge set in $D'$. Let $S := \emptyset$. Now for each edge $e$ in $S'$, add to $S$ an endpoint of $e$ that is outside $\{v_1, v_2\}$. It is easy to check that $S$ is a

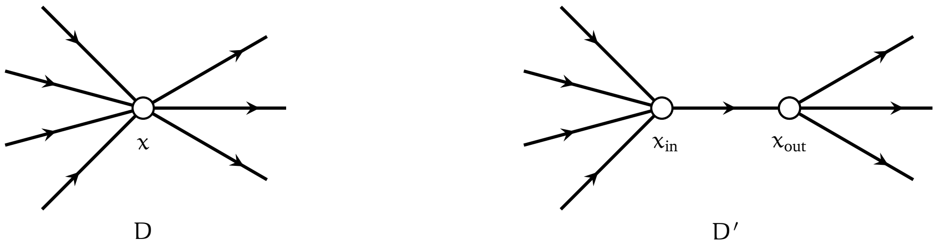



Figure A.5: Constructing a digraph $D'$ from $D$ in the proof of Theorem A.6(b).

$v_1, v_2$-separating set in $D$, and $|S| \leqslant |S'|$. Again this process can be reversed. Let $S$ be a $v_1, v_2$ separating set in $D$, and let $S' := \emptyset$. For each vertex $x \in S$, add to $S'$ the edge $x_{in}x_{out}$. Now $S'$ is a $v_1, v_2$-separating edge set in $D'$ and $|S'| = |S|$. This proves (ii), finishing the proof of (b).

Finally, we prove (c). Suppose we are given an undirected graph $G$, and specified vertices $v_1$ and $v_2$, and we want to find $k$ undirected vertex-disjoint $v_1, v_2$-paths in $G$. To form $D$ from $G$, we simply direct each edge $e$ of $G$ both ways. If the desired paths exist in $G$, then they clearly also do in $D$. Similarly, if the desired paths exist in $D$, then they also do in $G$. (Note, for each edge $xy \in E(G)$, that the paths cannot use both edge $\overrightarrow{xy}$ and edge $\overrightarrow{yx}$, since they are vertex-disjoint.)

Suppose instead that we are given an undirected graph $G$, and specified vertices $v_1$ and $v_2$, and we want to find $k$ undirected edge-disjoint $v_1, v_2$-paths in $G$. We construct $D$ from $G$ as above. Again, the correspondence is clear, except that possibly we have a path $P_1$ using directed edge $\overrightarrow{xy}$ and another path $P_2$ using directed edge $\overrightarrow{yx}$. We can write $P_1$ as $P_1'\overrightarrow{xy}P_1''$ and write $P_2$ as $P_2'\overrightarrow{yx}P_2''$. But now we can replace $P_1$ and $P_2$ with $P_1'P_2''$ and $P_2'P_1''$. (Thus, to ensure that our set of directed $v_1, v_2$-paths in $D$ corresponds to a set of undirected *edge-disjoint* $v_1, v_2$-paths in $G$, it suffices to choose our set of $v_1, v_2$-paths in $D$ to minimize its total length.) □

It is enlightening to notice the similarity between the proof of Theorem A.6(a) and that of Theorem 3.16, which proves Hadwiger's Conjecture for all line graphs of multigraphs.

## A.7 More Results on Coloring Games

In this section we include a few interesting results on coloring games that we had to omit from Chapter 9, since they do not relate directly to the activation strategy.

For convenience, we defined the marking game, as well as the chromatic game and the defective coloring game, as starting with Alice's turn. But we could just as easily consider a version of each game that starts with Bob's turn. For a graph $G$, let $\chi_g^A(G)$ and $\chi_g^B(G)$ denote, respectively, the usual game chromatic number, where Alice plays first, and the alternate version, where Bob plays first. For individual graphs $G$, the values of $\chi_g^A(G)$ and $\chi_g^B(G)$ can differ greatly. For example, consider $K_{n,n} - nK_2$, as in Exercise 3. But for many graph classes $\mathcal{G}$, it is straightforward to check that $\chi_g^A(\mathcal{G}) = \chi_g^B(\mathcal{G})$.

**Lemma A.7.** *If a graph class $\mathcal{G}$ is closed under (i) taking disjoint unions and (ii) adding an isolated vertex, then $\chi_g^A(\mathcal{G}) = \chi_g^B(\mathcal{G})$.*

*Proof.* Let $\mathcal{G}$ satisfy the hypotheses, and let $k := \chi_g^A(\mathcal{G})$. So $\mathcal{G}$ contains a graph $G$ such that $\chi_g^A(G) > k-1$. That is, Bob has a strategy to win the chromatic game on $G$, played with $k-1$ colors, when Alice plays first. Form $G^+$ from $G$ by adding an isolated vertex $v$. To show that $\chi_g^B(G^+) > k-1$, Bob first colors $v$ with an arbitrary color, and then plays on $G^+ - v$ as he does to show that $\chi_g^A(G) > k-1$. Thus, $\chi_g^B(\mathcal{G}) \geqslant \chi_g^A(\mathcal{G})$.

Now let $\ell := \chi_g^B(\mathcal{G})$. So $\mathcal{G}$ contains a graph $G$ such that $\chi_g^B(G) > \ell - 1$. First suppose that $|G|$ is odd. Let $2G$ consist of two vertex disjoint copies of $G$, say $G'$ and $G''$. Bob shows that $\chi_g^A(2G) > \ell - 1$, as follows. By symmetry, assume that Alice plays first in $G'$. Now Bob plays in $G''$ using his strategy to show that $\chi_g^B(G) > \ell - 1$. Whenever Alice plays in $G''$, Bob responds in $G''$ using this strategy. And whenever Alice plays in $G'$, Bob colors an arbitrary vertex in $G'$; this is possible because $|G'| = |G| \equiv 1 \pmod 2$. Thus $\chi_g^A(2G) \geqslant \chi_g^B(G) > \ell - 1$. Suppose instead that $|G|$ is even. Form $(2G)^+$ from $2G$ by adding an isolated vertex. Now the same strategy Bob used above shows that $\chi_G^A((2G)^+) \geqslant \chi_g^B(G) > \ell - 1$. Hence, $\chi_g^A(\mathcal{G}) \geqslant \chi_g^B(\mathcal{G})$. $\square$

It is straightforward to check that the proof of Lemma A.7 also proves analogous statements for the marking game as well as the defective coloring game.

**Lemma A.8.** *If $H$ is a subgraph of $G$, then $col_g(H) \leqslant col_g(G)$.*

*Proof.* Let $k := |G| - |H|$. We use strong induction on $k$. The base case, $k = 0$ is trivial. First suppose that $k \geqslant 2$. Form $G'$ from $G$ by deleting $k-1$ of the vertices in $V(G) \setminus V(H)$. The induction hypothesis, applied to $G$ and $G'$, shows that $\mathrm{col}_g(G') \leqslant \mathrm{col}_g(G)$. Similarly, $\mathrm{col}_g(H) \leqslant \mathrm{col}_g(G') \leqslant \mathrm{col}_g(G)$, and we are done. Thus, it suffices to consider the case $k = 1$.

Denote $V(G) \setminus V(H)$ by $\{v\}$. To determine how to play on $H$, Alice plays a game on $G$, mirroring Bob's moves on $H$ and responding as prescribed by her strategy to achieve $\mathrm{col}_g(G)$. At some point Alice's strategy to play on $G$ may tell her to mark $v$; of course this is impossible on $H$, since $v \notin V(H)$. Now she marks $v$ on $G$ and pretends that Bob responds on $G$ by marking an arbitrary vertex of minimum degree among all unmarked vertices in $G$; call this vertex $w_1$. Now Alice responds to $w_1$ on $G$, according to her strategy, and plays that move on $H$. Alice continues playing on $H$ mirroring her moves in the game on $G$.

At some point Bob may actually play $w_1$ on $H$. Now Alice needs to play on $H$, but she has no move for Bob on $G$. So again Alice chooses for Bob an arbitrary vertex of minimum degree among all unmarked vertices in $G$; call this vertex $w_2$. She plays on $H$ with her prescribed response in $G$ to $w_2$. Alice continues in this way, mirroring her moves from $G$ on $H$; whenever Bob plays on $H$ the most recent vertex $w_i$, Alice generates a new $w_{i+1}$, as described above, and uses it on $G$ to determine her next move in $H$. Now we must show that this strategy witnesses $\mathrm{col}_g(H) \leqslant \mathrm{col}_g(G)$.

For all $v \in V(G)$, let $s(v)$ denote the number of neighbors of $v$ marked before $v$ in the real game on $H$, and let $s'(v)$ denote the number of neighbors of $v$ marked before $v$ in the imaginary game on G. Let $z'$ denote the final vertex marked on G. If $v \neq w_i$ for all $i$, then $s(v) \leqslant s'(v) \leqslant \mathrm{col}_g(G) - 1$. So instead consider some vertex $w_i$. Now $s(w_i) \leqslant d(w_i) \leqslant d(z') = s(z') \leqslant \mathrm{col}_g(G) - 1$, as desired. □

## A.8 Line-Perfect Graphs

A multigraph G is *line-perfect* if its line graph $L(G)$ is perfect; that is, if $\chi(H) = \omega(H)$ for each induced subgraph H of $L(G)$. A cycle is *odd* if its length is odd. line-perfect

The goal of this section is to prove Theorem A.10, which gives two equivalent characterizations of line-perfect graphs. To make the proof more easily digestible, we handle half of it in the following lemma.

**Lemma A.9.** *A multigraph* G *is line-perfect if and only if* G *has no odd* $5^+$*-cycle.*

This lemma is best viewed as a generalization of König's Theorem. To prove König's Theorem, we use induction on $\|G\|$. Each time we add an edge $vw$, we can use a single Kempe swap to get a common color $\alpha$ unused at both $v$ and $w$, then color $vw$ with $\alpha$. To prove the present lemma, the main idea is that, with a bit more effort, we can sidestep any problems created by triangles. This is because the triangles only interact in very simple ways; otherwise, G would have an odd $5^+$-cycle.

*Proof.* Suppose G has an odd $5^+$-cycle C. Note that the line graph of C is isomorphic to C and $\chi(C) = 3 > 2 = \omega(C)$. Thus, G is not line-perfect.

Now we assume G has no odd $5^+$-cycle and show G is line-perfect. For distinct $v, w, x \in V(G)$, let $t(vwx) := \mu(vw) + \mu(vx) + \mu(wx)$, and let $t(G) := \max_{v,w,x \in V(G)} t(vwx)$. Let $\ell := \max\{\Delta(G), t(G)\}$. Clearly, $\chi'(G) \geqslant t(G)$. We will show that $\chi'(G) = t(G)$. $t(vwx)$, $t(G)$ $\ell$

We start with an arbitrary $\ell$-edge-coloring $\varphi$ of G (not necessarily proper) and repeatedly modify it, using Kempe swaps, until it is proper. For each $v \in V(G)$, let $f(v)$ be the number of colors used by $\varphi$ incident to $v$. If $\varphi$ is not proper, then we modify it to get a new coloring $\varphi'$. We show $f'(v) \geqslant f(v)$ for all vertices $v$ and $f'(w) \geqslant f(w) + 1$ for some vertex $w$. We repeat this process until $f(v) = d(v)$ for all $v$; that is, the final coloring is proper. (More formally, we use induction on $\sum_{v \in V(G)}(d(v) - f(v))$. The base case is trivial, since if $\sum_{v \in V(G)}(d(v) - f(v)) = 0$, then $f(v) = d(v)$ for all $v$, so the coloring is proper.) $f(v)$

Fix a vertex $v$ such that $f(v) < d(v)$. Let $\alpha$ be a color used at least twice incident to $v$, say on $vw$ and $vx$, and let $\beta$ be a color not used incident to $v$. Starting from $v$, we form a walk $W$ that begins with edge $vw$ and alternates colors $\alpha$ and $\beta$ until either (i) we have no edge to further extend it or (ii) we revisit a vertex. Let $\varphi_{\alpha,\beta}(v)$ be the subgraph induced by $E(W)$. If $W$ does not end at $v$, then we recolor $\varphi_{\alpha,\beta}(v)$ (all edges of $\varphi_{\alpha,\beta}(v)$ previously colored $\alpha$ are now colored $\beta$, and those previously colored $\beta$ are now colored $\alpha$); call the new coloring $\varphi'$. Now $f'(v) \geqslant f(v) + 1$ and $f'(y) \geqslant f(y)$ for all $y \in V(G)$, so we are done.

Instead assume that $W$ ends at $v$. Since no edge incident to $v$ uses $\beta$, walk $W$ has odd length, which means that it has length 3 (by hypothesis); that is, $E(\varphi_{\alpha,\beta}(v)) = \{vw, vx, wx\}$.

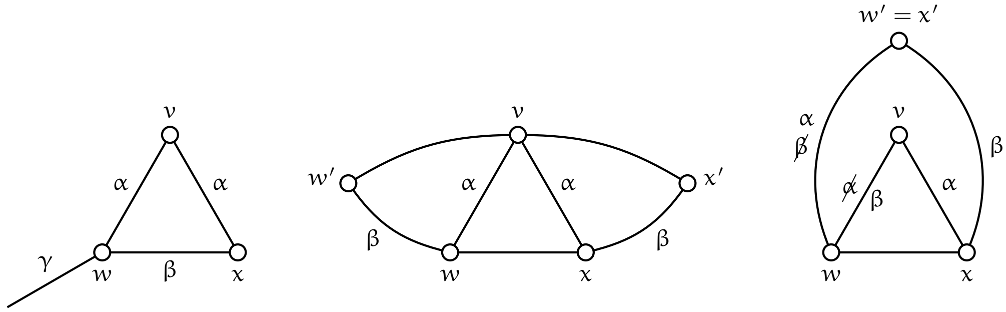



Figure A.6: We modify an arbitrary edge-coloring of G, essentially by Kempe swaps, to get a proper edge-coloring.

See the left of Figure A.6. Since $\ell \geqslant t(vwx)$ and color $\alpha$ is used on both $vw$ and $wx$, some color $\gamma$ is not used on any edge of $G[\{v, w, x\}]$. Now we will grow a maximal walk containing $wx$ and alternating colors $\beta$ and $\gamma$. Similar to above, starting from $w$ on an edge colored $\gamma$, we alternate edges colored $\beta$ and $\gamma$ as long as we can (or until we revisit a vertex); call this walk $\varphi_{\gamma,\beta}(w)$. Analogously, we form $\varphi_{\gamma,\beta}(x)$. Note that $\varphi_{\gamma,\beta}(w) + wx + \varphi_{\gamma,\beta}(x)$ does not contain $v$, since no edge incident to $v$ uses $\beta$, and if either walk visits $v$ on an edge colored $\gamma$, then G contains an odd $5^+$-cycle, a contradiction. Thus, we recolor $\varphi_{\gamma,\beta}(w) + wx + \varphi_{\gamma,\beta}(x)$, and doing so does not recolor any edge incident to $v$.

After recoloring as above, let $\varphi^1_{\alpha,\beta}(v)$ and $\varphi^2_{\alpha,\beta}(v)$ denote maximal walks alternating colors $\alpha$ and $\beta$ starting from $v$ along edges $vw$ and $vx$, respectively. If either $\varphi^i_{\alpha,\beta}(v)$ does not return to $v$, then we recolor it and call the new coloring $\varphi'$. Now $f'(y) \geqslant f(y)$ for all vertices $y$ and $f'(v) \geqslant f(v) + 1$, so we are done. Assume instead that each $\varphi^i_{\alpha,\beta}(v)$ revisits $v$. Since $\beta$ is not used incident to $v$ and G has no odd $5^+$-cycle, each $\varphi^i_{\alpha,\beta}(v)$ has length 3; denote their vertices by $v, w, w'$ and $v, x, x'$. Since $\beta$ is not used on $G[\{v, w, x\}]$, we must have $w' \neq x$ and $x' \neq w$. See the center and right of Figure A.6. If also $w' \neq x'$, then $v, w', w, x, x'$ is a 5-cycle, a contradiction. So we must have $w' = x'$. Now we recolor $vw$ with $\beta$ and $ww'$ with $\alpha$, and call this coloring $\varphi''$. Note that $f''(v) = f(v) + 1$, $f''(w') \geqslant f(w')$, and $f''(y) = f(y)$ for all other $y$. So again we are done. □

**Theorem A.10.** *For a multigraph* G*, the following three properties are equivalent.*

*(a)* G *is line-perfect.*

*(b)* G *has no odd* $5^+$*-cycle.*

*(c) Every block of* G *has as its underlying simple graph either (i) a bipartite graph, (ii)* $K_4$*, or (iii) the complete tripartite graph* $K_{1,1,t}$*, for some integer* $t \geqslant 1$.

*Proof.* Lemma A.9 proves the equivalence between (a) and (b). So here we prove the equivalence between (b) and (c). If (b) is false, then let C denote an odd $5^+$-cycle in G. The block of G containing C does not satisfy any of (i), (ii), or (iii), so (c) is also false.

Now we assume (b) is true and show (c) is also true. We fix a block $B$ and consider the clique number $\omega(B)$ of $B$. If $\omega(B) \geqslant 5$, then $B$ contains a 5-cycle, a contradiction. If $\omega(B) \leqslant 2$, then $B$ contains no 3-cycle. By hypothesis, $B$ contains no odd $5^+$-cycle, so $B$ is bipartite and satisfies (i). Thus, we only need to consider the cases $\omega(B) \in \{3, 4\}$.

Suppose $\omega(B) = 4$, and let $v, w, x, y$ be the vertices of some clique in $B$. See the left of Figure A.7. If $B$ contains no other vertex, then $B$ satisfies (ii). So assume $B$ contains another vertex $z$. By symmetry, assume $z \leftrightarrow y$. Since $B$ is 2-connected, $B$ also contains a path $P$ (disjoint from $y$) to $v$, $w$, or $x$; by symmetry, say it is $v$. Now $G$ contains an odd $5^+$-cycle, either $P + vw + wy + yz$ or $P + vw + wx + xy + yz$, which is a contradiction.

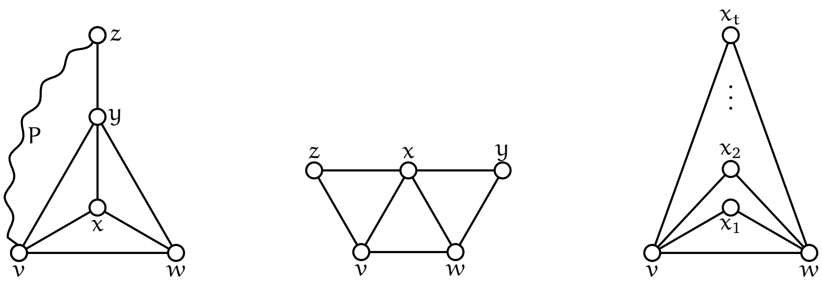



Figure A.7: The proof of Theorem A.10. Left: $\omega(B) = 4$. Center: The proof of Claim 1. Right: $\omega(B) = 3$.

Finally, assume $\omega(B) = 3$. To show $B$ satisfies (iii) above, we prove the following claim.

**Claim 1.** *Each triangle (3-cycle) of $B$ has at most one edge in other triangles.*

Proof. Let $vwx$ be a triangle contradicting the claim; see the center of Figure A.7. Suppose both $vx$ and $wx$ lie on other triangles; say on triangles $vxz$ and $wxy$. If $y \neq z$, then $B$ contains the 5-cycle $vwyxz$, which contradicts (b). But if $y = z$, then $v, w, x, y$ induce $K_4$, which contradicts our assumption that $\omega(B) = 3$. $\diamondsuit$

Now let $vwx_1$ be a triangle in $B$, and assume that only edge $vw$ appears in other triangles. Say that $vwx_i$ is a triangle for each $i \in \{1, \dots, t\}$, for some $t$. Note that $x_i \not\leftrightarrow x_j$ for all distinct $i, j \in \{1, \dots, t\}$, since otherwise $v, w, x_i, x_j$ induces $K_4$, which contradicts that $\omega(B) = 3$. If $B$ contains no other vertices, then $B$ satisfies (iii), so we are done. Assume instead that $B$ contains another vertex $y$. Since $B$ is 2-connected, $B$ contains two vertex-disjoint paths from $y$ to distinct vertices in $\{v, w, x_1, \dots, x_t\}$; call the paths $P_1$ and $P_2$ and let $Q := P_1 \cup P_2$. In each case (depending on the endpoints of $Q$ and the parity of $|E(Q)|$) we will show that $B$ contains an odd $5^+$-cycle, contradicting (b).

If $Q$ has endpoints $x_i$ and $x_j$, then our odd $5^+$-cycle is either $Q + x_i v + v x_j$ or else $Q + x_i v + vw + w x_j$. If $Q$ has endpoints $x_i$ and $v$ (or $w$, by symmetry), then Claim 1 implies $Q + v x_i$ is not a triangle, so $Q$ has length at least 3. Thus, our odd $5^+$-cycle is either $Q + x_i w + wv$ or else $Q + x_i v$. Finally, if $Q$ has endpoints $v$ and $w$, then note that $Q$ has length at least 3, since $y \notin \{x_1, \dots, x_t\}$. Now our odd $5^+$-cycle is either $Q + vw$ or $Q + v x_1 + x_1 w$. $\square$

## A.9 The Tree-Packing Theorem

cross-edge

For a graph $G$ and a partition $\mathcal{P}$ of $V(G)$ a *cross-edge* $e \in E(G)$ is one with its endpoints in distinct parts of $\mathcal{P}$.

**Theorem A.11 (Tree-Packing Theorem).** *A multigraph $G$ contains contains $k$ edge-disjoint spanning trees if and only if for each partition $\mathcal{P}$ of $V(G)$ the graph $G$ contains at least $k(|\mathcal{P}|-1)$ cross-edges.*

Since all spanning trees are connected, each spanning tree must contain at least $|\mathcal{P}|-1$ cross-edges, for each partition $\mathcal{P}$. Thus, the hypothesis in Theorem A.11 is necessary. Now we prove the other direction. Our proof is by induction $|G|$, but needs the following Key Lemma. We first prove the theorem assuming the lemma. Afterwards, we prove the lemma.

**Key Lemma.** If $G$ satisfies the hypothesis of Theorem A.11, then it contains edge-disjoint forests $F_1, \dots, F_k$ and $U \subseteq V(G)$ with $|U| \geqslant 2$ such that subgraph $F_i[U]$ is connected for all $i \in [k]$.

*Proof of Theorem A.11.* Our proof is by induction on $|G|$. The base case, $|G| = 1$, is trivial. So assume that $|G| \geqslant 2$. Let $F_1, \dots, F_k$ and $U$ be as guaranteed by the Key Lemma. Let $G' := G/U$, and denote the new vertex in $G/U$ by $U$. Note that each partition $\mathcal{P}'$ of $G'$ induces a partition $\mathcal{P}$ of $G$ with the same number of parts and the exact same cross-edges as $\mathcal{P}'$. By assumption, $G$ has at least $k(|\mathcal{P}|-1)$ cross-edges for $\mathcal{P}$. Thus, $G'$ has at least $k(|\mathcal{P}|-1) = k(|\mathcal{P}'|-1)$ cross-edges for $\mathcal{P}'$. By the induction hypothesis, $G'$ contains $k$ edge-disjoint spanning trees $F'_1, \dots, F'_k$. By replacing vertex $U$ in each $F'_i$ with the spanning tree $F_i[U]$ of $U$ in $G$, we get the desired $k$ edge-disjoint spanning trees of $G$. □

A crucial step in proving the Key Lemma is explicitly describing such a set $U$. We use the following definitions.

$\mathcal{F}$

edge replacement

$F^0, \mathcal{F}^0$

**Definition A.12.** A $k$-forest packing $F$ consists of $k$ edge-disjoint forests $F_1, \dots, F_k$. Let $E(F) := \cup_{i=1}^{k} E(F_i)$. A $k$-forest packing $F$ is *maximum* if it maximizes $\|F\|$. Let $\mathcal{F}$ denote the set of all maximum $k$-forest packings of $G$. Consider $F \in \mathcal{F}$ and $e \in E(G) \setminus E(F)$. Fix $i \in [k]$. Since $F \in \mathcal{F}$, we know that $F_i + e$ contains a cycle $C$. Consider an edge $e' \in E(C) \setminus \{e\}$. Form $F'$ from $F$ by replacing $F_i$ with $F_i + e - e'$. We call this an *edge replacement*. Clearly, $F'_i \in \mathcal{F}$, since $\|F'\| = \|F\|$. Pick $F^0 \in \mathcal{F}$ arbitrarily. Let $\mathcal{F}^0$ denote the subset of $\mathcal{F}$ that can be formed by a sequence of edge replacements starting from $F^0$. Let

$E^0$

$$E^0 := \bigcup_{F \in \mathcal{F}^0} (E(G) \setminus E(F)).$$

$G^0$

So $E^0$ is the set of edges that are each missed by some maximum $k$-forest packing that is reachable from $F^0$ by edge replacements. Let $G^0$ be the spanning subgraph of $G$ with $E^0$ as its edge set.

Now we can state a more technical version of the Key Lemma.

**Lemma A.13.** *Fix $e^0 \in E(G) \setminus E(F^0)$, let $C^0$ denote the component of $G^0$ that contains $e^0$, and let $U := V(C^0)$. Now $F_i^0[U]$ is connected for each $i \in [k]$.* $C^0$ $U$

Before proving Lemma A.13, we show that it implies the Key Lemma.

**Observation A.14.** *Lemma A.13 implies the Key Lemma.*

*Proof.* It suffices to show $E(G) \setminus E(F^0) \neq \emptyset$, since then we fix $e^0 \in E(G) \setminus E(F^0)$ and are done by Lemma A.13. Take $\mathcal{P}$ to be the partition of $V(G)$ with each vertex in its own part. By hypothesis, the number of cross-edges in $G$ is at least $k(|G| - 1)$. If $\|F^0\| = k(|G| - 1)$, then each forest in $F^0$ is a tree, so the Key Lemma holds with $U := V(G)$. Otherwise, $E(G) \setminus E(F^0) \neq \emptyset$, as desired. □

Now we prove Lemma A.13.

*Proof of Lemma A.13.* Fix $i \in [k]$, and recall that $U := V(C^0)$. Our idea is to show, for each $xy \in E(C^0)$, that $x$ and $y$ are connected in $F_i^0$ and that the path connecting them lies in $C^0$. Since $C^0$ is connected, this will show that $F_i^0[U]$ is connected, as desired. The set $E^0$ can be viewed as being defined inductively (implicitly counting the minimum number $s$ of edge replacements starting from $F^0$ to reach $F \in \mathcal{F}^0$ such that $e \notin E[F]$, for a given $e \in E^0$). So we will also prove the present lemma by induction on $s$. As observed above, the base case, $s = 0$, holds trivially, since $F^0 \in \mathcal{F}^0$. So we focus on the inductive step. For this we will use the following claim.

**Claim 1.** *Fix $F \in \mathcal{F}^0$ and let $F'$ be formed from $F$ by an edge replacement. If two vertices $x$ and $y$ are connected in $F_i' \cap C^0$, then they are also connected in $F_i \cap C^0$.*

<u>Proof.</u> Suppose that $F'$ is formed from $F$ by adding edge $e'$ and removing edge $e$. Choose $x, y$ that are connected in $F_i' \cap C^0$, and let $P'$ denote the $x,y$-path in $F_i' \cap C^0$; see Figure A.8. If $e' \notin P'$, then $P' \subseteq F_i \cap C^0$, so we are done. Assume instead that $e' \in P'$, and let $v, w$ be the endpoints of $e'$. By the definition of edge replacement, $F_i$ contains a $v,w$-path $P$. Now $(P' - e') \cup P$ contains an $x,y$-walk. By our definition of $P'$, we have $P' - e' \subseteq E(C^0)$. Also, $P' - e' \subseteq E(F_i)$. So it suffices to show that $P \subseteq E(C^0)$. Consider an arbitrary edge $e''$ on the cycle $P + e'$ such that $e'' \neq e'$. Note that $e'' \in E^0$ since $e''$ can be replaced in $F_i$ by $e'$. Thus, $x$ and $y$ are connected in $F_i' \cap C^0$. ♢

Now we prove the lemma by induction, as suggested above.

Fix $e \in E^0$ and let $x, y$ be the endpoints of $e$. We show that $x$ and $y$ are in the same component of $F_i^0[U]$, for each $i \in [k]$. Let $s$ be the minimum integer such that $e \notin E(F^s)$ and there exists a sequence $F^0, \ldots, F^s$ where each $F^r$ is formed from $F^{r-1}$ by an edge replacement. The proof is by induction on $s$. The case $s = 0$ is easy, since $F^0 \in \mathcal{F}^0$. So assume $s \geqslant 1$. By Claim 1 and the induction hypothesis, it suffices to show that $x$ and $y$ are in the same component of $F_i^s$. But this is true because $F^s$ is maximum and $xy \notin F^s$. □

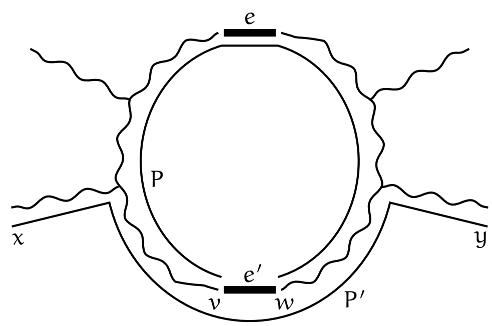



Figure A.8: Claim 1 in the proof of Lemma A.13. The wavy subgraph combines with edges $e$ and $e'$ to form trees $T$ and $T'$ (respectively). We let $P$ denote the $v,w$-path in $T$ and let $P'$ denote the $x,y$-path in $T'$.

The Tree-Packing Theorem was proved by Nash-Williams [318] and Tutte [391]. It is a striking, though not atypical, example of results where the obvious necessary conditions are sufficient (TONCAS). Other examples include Hall's Theorem and Menger's Theorem. The Tree-Packing Theorem also gives a hint of deeper and more general results in matroid theory. In particular, it is an easy corollary of the Matroid Union Theorem, due to Edmonds and Fulkerson [144] and, independently, to Nash-Williams [319].

## A.10 Mader's Splitting Off Theorem

Many theorems about edge-coloring and flows have a hypothesis that a graph $G$ be $k$-edge-connected, for some positive integer $k$. To prove these results, it is often convenient to restrict to $k$-regular graphs. This motivated Lovász to consider the idea of splitting off a pair of edges, as defined below.

$\lambda(v,w)$
split off
$G_{yz}$
$V_x$
splittable pair
$d(X)$

**Definition A.15.** For a multigraph $G$ and each $v, w \in V(G)$, let $\lambda(v,w)$ denote the maximum number of edge-disjoint $v,w$-paths in $G$. To *split off* a pair of edges $xy, xz$ from a vertex $x$ in $G$ means to delete $xy$ and $xz$ and to add edge $yz$; we denote the resulting graph by $G_{yz}$. For convenience, in this section we always assume that some vertex $x$ is fixed and we are splitting edges off of $x$. Let $V_x := V(G) \setminus \{x\}$. An edge pair $\{xy, xz\}$ is a *splittable pair* if $\lambda_{G_{yz}}(v,w) = \lambda(v,w)$ for all $v, w \in V_x$. Recall, for each $X \subseteq V(G)$, that we write $d(X)$ to denote the number of edges with one endpoint in $X$ and one endpoint in $V(G) \setminus X$.

Lovász proved (see Frank [166]) that if $d(x)$ is even and $\lambda(v,w) \geqslant k \geqslant 2$ for all $v, w \in V(G) \setminus \{x\}$, then for each $y \in N(x)$ there exists $z \in N(x)$ such that $\lambda_{G_{yz}}(v,w) \geqslant k$, for all $v, w \in V_x$. More generally, Lovász conjectured that all edges incident to $x$ could be split off, in pairs, such that the resulting graph $G'$ satisfies $\lambda_{G'}(v, w) \geqslant k$ for all $v, w \in V_x$. This was proved by Mader [293], and is stated below as Theorem A.16; also see Theorem A.22 and Lemma A.24. We will prove an alternate formulation, Theorem A.23. The original proof of Theorem A.16 was not easy, but the proof below (due to Frank [166]) is more accessible.

The aim of this section is to prove the following result.

**Theorem A.16 (Mader's Splitting Off Theorem).** *Let* $G$ *be an undirected multigraph, and fix* $x \in V(G)$. *If* $d(x) \neq 3$ *and* $x$ *is not incident to any cut-edge, then* $x$ *is incident to a splittable pair.*

By Menger's Theorem, for some $v, w \in V_x$ and some $y, z \in N(x)$, the graph $G_{yz}$ has $\lambda_{G_{yz}}(v, w) < \lambda_G(v, w)$ if and only if there exists $W \subseteq V_x$ such that $v \in W$ and $w \notin W$ and $d_{G_{yz}}(W) < \lambda_G(v, w)$. This motivates the following definitions and easy proposition.

**Definition A.17.** For $W \subseteq V_x$, let $f(W) := \max_{v \in W, w \in V_x \setminus W} \lambda(v, w)$. Always $d(W) \geqslant f(W)$. So let $s(W) := d(W) - f(W)$; we call $s(W)$ the *surplus* of $W$. A set $W$ is *tight* if $s(W) = 0$ and $W$ is *dangerous* if $s(W) \leqslant 1$. (This definition of dangerous is natural, since splitting off a pair of edges decreases the surplus of each set by at most 2, and this causes a problem only if some surplus becomes negative.)

$f(W)$
$s(W)$
surplus, tight
dangerous

Note the similarity between $s$ in Definition A.17 and $f$ in the proof of Hall's Theorem.

**Proposition A.18.** *An edge pair* $\{xy, xz\}$ *is* splittable *if and only if no dangerous set contains vertices* $y$ *and* $z$.

*Proof.* Suppose such a dangerous set $W$ exists in $G$ and form $G_{yz}$ from $G$ by splitting off $xy$ and $xz$. This decreases $d(W)$ by 2; so the new surplus of $W$ is negative, which creates a problem. See the left of Figure A.9. Formally, $s_{G_{yz}}(W) = d_{G_{yz}}(W) - f(W) = d_G(W) - 2 - f(W) \leqslant 1 - 2 < 0$. Conversely, if no such set $W$ exists, then each $W \subseteq V_x$ with $y, z \in W$ has $s_G(W) \geqslant 2$. So $s_{G_{yz}}(W) = d_{G_{yz}}(W) - f(W) = d_G(W) - 2 - f(W) \geqslant 2 - 2 = 0$. And each $W \subseteq V_x$ with $\{y, z\} \not\subseteq W$ has $s_{G_{yz}}(W) = s_G(W) \geqslant 0$. □

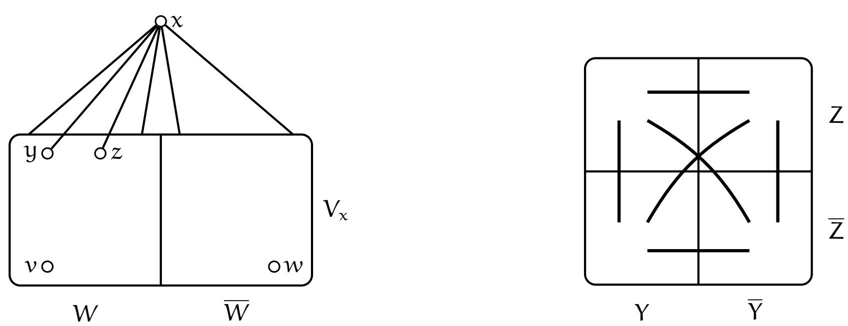



Figure A.9: Left: A dangerous set $W$, in the proof of Proposition A.18, shows that the edge pair $\{xy, xz\}$ is not a splittable pair. Right: An aid to verifying Proposition A.19.

$\overline{Y}$
$d(Y,Z)$
$\overline{d}(Y,Z)$

To prove Theorem A.16, we need a few easy results counting the number of edges between various subsets. For each $Y \subseteq V_x$, let $\overline{Y} := V_x \setminus Y$. For all $Y, Z \subseteq V(G)$, let $d(Y, Z)$ denote the number of edges with one endpoint in each of $Y \setminus Z$ and $Z \setminus Y$. Let $\overline{d}(Y, Z)$ denote $d(Y \cap Z, V(G) \setminus (X \cup Y))$. If $Y = \{x\}$, then we typically write $d(x, Z)$ rather than $d(\{x\}, Z)$.

**Proposition A.19.** *For an arbitrary multigraph* $H$ *and all* $Y, Z \subseteq V(H)$*, both of the following inequalities hold:*

$$d(Y) + d(Z) = d(Y \cap Z) + d(Y \cup Z) + 2d(Y, Z) \tag{A.1}$$

$$d(Y) + d(Z) = d(Y \setminus Z) + d(Z \setminus Y) + 2\overline{d}(Y, Z). \tag{A.2}$$

*Proof.* Each identity holds because every edge in $H$ contributes equally to both sides of the identity. This is easy to see if we consider the sets $Y \cap Z$, $Y \setminus Z$, $Z \setminus Y$, and $V(H) \setminus (Y \cup Z)$, and the number of edges between each pair of these sets. See the right of Figure A.9. □

**Proposition A.20.** *For all* $Y, Z \subseteq V_x$*, at least one of the following inequalities holds:*

$$f(Y) + f(Z) \leqslant f(Y \cap Z) + f(Y \cup Z) \tag{A.3}$$

$$f(Y) + f(Z) \leqslant f(Y \setminus Z) + f(Z \setminus Y). \tag{A.4}$$

The idea is to consider $y_1, y_2$ such that $f(Y) = \lambda(y_1, y_2)$. We have a few possibilities, depending on which of the sets $Y \cap Z$, $Y \setminus Z$, $Z \setminus Y$, and $\overline{Y} \cap \overline{Z}$ contain each of $y_1$ and $y_2$.

*Proof.* Recall that, by definition, $f(W) = f(\overline{W})$ for all $W \subseteq V_x$. As a result, replacing $Z$ with $\overline{Z}$ in (A.3) yields (A.4), and vice versa. By symmetry between $Y$ and $Z$, we assume that $f(Y) \geqslant f(Z)$. Choose $y_1, y_2$ such that $f(Y) = \lambda(y_1, y_2)$, with $y_1 \in Y$ and $y_2 \notin Y$; see Figure A.10. By possibly replacing $Z$ with $\overline{Z}$, we also assume $y_1 \in Z$. If $y_2 \notin Z$, then $f(Y) = f(Z) = f(Y \cap Z) = f(Y \cup Z) = \lambda(y_1, y_2)$. (To see this, assume, to the contrary that $f(Y \cap Z) > \lambda(y_1, y_2)$ or $f(Y \cup Z) > \lambda(y_1, y_2)$. Note that any pair $y_1', y_2'$ witnessing this also witnesses that $f(Y) > \lambda(y_1, y_2)$ or $f(Z) > \lambda(y_1, y_2)$, both of which are contradictions.) Thus, (A.3) holds with equality.

Assume instead that $y_2 \in Z$. Now $f(Y) = f(Y \cap Z) = f(Z \setminus Y)$. Thus, it suffices to show either $f(Z) \leqslant f(Y \cup Z)$ or $f(Z) \leqslant f(Y \setminus Z)$, since adding either of these inequalities to the equality in the previous sentence yields (A.3) or (A.4). Now we are done, since $f(Z) = f(\overline{Z}) \leqslant \max\{f(\overline{Z} \cap \overline{Y}), f(\overline{Z} \cap Y)\} = \max\{f(Y \cup Z), f(Y \setminus Z)\}$. So either (A.3) or (A.4) holds. □

**Proposition A.21.** *For all* $Y, Z \subseteq V_x$*, at least one of the following inequalities holds:*

$$s(Y) + s(Z) \geqslant s(Y \cap Z) + s(Y \cup Z) + 2d(Y, Z) \tag{A.5}$$

$$s(Y) + s(Z) \geqslant s(Y \setminus Z) + s(Z \setminus Y) + 2\overline{d}(Y, Z). \tag{A.6}$$

*Proof.* To reach (A.5), subtract (A.3) from (A.1). To reach (A.6), subtract (A.4) from (A.2). □

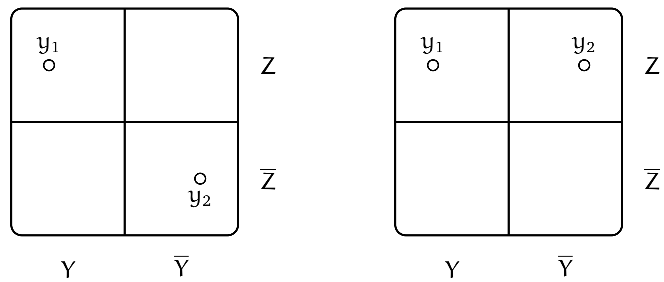



Figure A.10: The two key cases in the proof of Proposition A.20.

Now we turn to proving Mader's Theorem; for convenience, we restate it. We will actually prove an equivalent form, Theorem A.23.

**Theorem A.22 (Mader's Theorem).** *Let* $G$ *be a connected undirected multigraph, with a specified vertex* $x$*. If* $d(x) \neq 3$ *and* $x$ *has no incident cut-edge, then* $x$ *is incident to a splittable pair of edges.*

**Theorem A.23 (Mader's Theorem (Variant Formulation)).** *Let* $G$ *be a connected undirected multigraph, with a specified vertex* $x$*. If* $d(x)$ *is even and* $x$ *is not incident to any cut-edge, then the edges incident to* $x$ *can be partitioned into splittable pairs.*

First, we show that Theorems A.22 and A.23 are indeed equivalent.

**Lemma A.24.** *Theorem A.22 and Theorem A.23 are equivalent.*

*Proof.* Suppose that Theorem A.22 holds, and that $d(x)$ is even and $x$ has no incident cut-edge. By Theorem A.22, $x$ has an incident splittable pair, $\{xy, xz\}$. Since $d_G(x)$ is even, also $d_{G_{yz}}(x)$ is even. Since $\{xy, xz\}$ is a splittable pair (and $x$ is not incident to any cut-edge in $G$), we have $\lambda_{G_{yz}}(v, w) = \lambda_G(v, w) \geqslant 2$ for each pair $v, w \in N_{G_{yz}}(x)$. Thus, $x$ is also not incident to any cut-edge in $G_{yz}$. By induction on $d(x)$, this proves Theorem A.23.

Now suppose that Theorem A.23 is true, that $d(x) \neq 3$, and that $x$ is not incident to any cut-edge. If $d(x)$ is even, then we are done. So assume instead that $d(x)$ is odd and at least 5. Add three parallel edges $xx'$ to some new vertex $x'$, and call this graph $G'$. Now apply Theorem A.23 to $G'$. Since $d_G(x) \geqslant 5$, we have $d_G(x) > (d_G(x) + 3)/2 = (d_{G'}(x)/2)$. Thus, at least one of the splittable edge pairs incident to $x$ in $G'$ consists of two edges in $G$. This proves Theorem A.22. $\square$

Our next lemma will allow us to assume that every tight set consists of a single element.

**Lemma A.25.** *Let* $T$ *be a tight set in* $G$*, where* $\emptyset \subsetneq T \subseteq V_x$*. A pair of edges* $\{e_1, e_2\}$ *incident to* $x$ $T, e_1, e_2$
*is splittable in* $G$ *if the corresponding pair* $\{e_1', e_2'\}$ *is splittable in* $G'$*, where* $G' := G/T$. $G', e_1', e_2'$

The idea is to assume that the pair $\{e_1, e_2\}$ is not splittable. By Lemma A.18, the endpoints of $e_1$ and $e_2$ are contained in a dangerous set $D$. Now we use $D$ to construct a dangerous set $D'$ in $G'$ containing $e_1', e_2'$. This $D'$ witnesses that $e_1', e_2'$ are in fact *not* splittable in $G'$, proving the contrapositive.

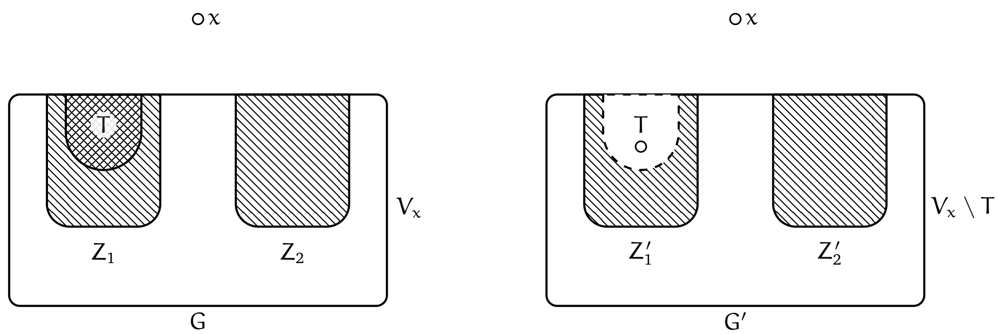



Figure A.11: Vertex subsets $Z_1$ and $Z_2$ in $G$, with $T \subseteq Z_1$ and $T \cap Z_2 = \emptyset$, and their corresponding vertex subsets $Z_1'$ and $Z_2'$ in $G/T$.

T *Proof.* Choose $y, z \in V_x$ such that $e_1 = xy$ and $e_2 = xz$. In $G'$, let $T$ be the single vertex that
Z′ arose from contracting $T$ in $G$. If either $T \subseteq Z \subseteq V_x$ or $Z \subseteq V_x \setminus T$, then let $Z'$ be the subset of $V(G')$ corresponding to $Z$; see Figure A.11. For such a $Z$, we have $d_{G'}(Z') = d_G(Z)$ and $f'(Z') \geqslant f(Z)$. Thus, $s'(Z') \leqslant s(Z)$. So, if $Z$ is dangerous in $G$, then $Z'$ is dangerous in $G'$.

Assume that $\{e_1, e_2\}$ is not splittable in $G$. So there exists a dangerous set $X$ containing $y, z$. Let $Z := X \cup T$. If $Z$ is dangerous in $G$, then also $Z'$ is dangerous in $G'$ (by the previous paragraph), so $\{e_1', e_2'\}$ is not splittable in $G'$, as desired. So we assume $Z$ is not dangerous. That is, $s(X \cup T) \geqslant 2$. Suppose inequality (A.5) holds for $X$ and $T$. Now we get

$$1 + 0 \geqslant s(X) + s(T) \geqslant s(X \cap T) + s(X \cup T) \geqslant 0 + 2,$$

a contradiction. Thus, inequality (A.6) holds for $X$ and $T$, by Proposition A.21. This gives

$$1 + 0 \geqslant s(X) + s(T) \geqslant s(X \setminus T) + s(T \setminus X) + 2\overline{d}(X, T) \geqslant 0 + 0 + 2\overline{d}(X, T).$$

So $2\overline{d}(X, T) = 0$ and $s(X \setminus T) \leqslant 1$. This inequality shows that $X \setminus T$ is dangerous in $G$. Since $\overline{d}(X, T) = 0$ and $y, z \in X$ and $x \in \overline{X} \cap \overline{T}$, we must have $y, z \in X \setminus T$. Let $D := X \setminus T$. Since $D$ is dangerous in $G$ and $y, z \in D$, also $D'$ is dangerous in $G'$ (by the first paragraph) and $y, z \in D'$. This implies that $\{e_1', e_2'\}$ is not splittable in $G'$, as desired. □

**Lemma A.26.** *If every tight set in* $G$ *consists of a single element, then we must have* $\lambda(v, w) = \min\{d(v), d(w)\}$ *for every pair* $v, w \in V_x$.

*Proof.* By Menger's Theorem, $\lambda(v, w) = \min_{\{W : |\{v,w\} \cap W| = 1\}} d(W)$. Each set $W$ attaining equality is tight. So if $W \neq \{v\}$ and $W \neq \{w\}$, then $W$ contradicts the hypothesis. □

Now we prove Theorem A.23.

*Proof of Theorem A.23.* Suppose the theorem is false, and choose a counterexample $G$ and $x$ to minimize $|G| + d(x)$. Recall that $d(x)$ is even, by hypothesis. As in the proof of Proposition A.24, it suffices to find a single splittable pair $\{xy, xz\}$ incident to $x$, since then we can proceed by induction (i.e., $G_{yz}$ is smaller than $G$, which is a minimal counterexample).

Suppose that $G$ contains a tight set $T \subseteq V_x$ that is not a single element. Since $G$ is minimal, $G/T$ contains a splittable pair $\{e_1', e_2'\}$ incident to $x$. By Lemma A.25, the pair $\{e_1, e_2\}$ is splittable in $G$, and we are done. Thus, we assume that each tight set consists of a single element.

Let $X := N(x)$. Fix $y \in X$ of minimum degree. $X, y$

**Claim 1.** *If* $W \subseteq V_x$ *with* $y \in W$ *and* $|W \cap X| \geqslant 2$*, then* $f(W \setminus \{y\}) \geqslant f(W)$.

Proof. As noted above, every tight set consists of a single element. Thus, Lemma A.26 implies that $f(W) = \lambda(v, w) = \min\{d(v), d(w)\}$ for some $v \in W$ and $w \in \overline{W}$. If $v \neq y$, then $f(W \setminus \{y\}) \geqslant \lambda(v, w) = f(W)$. Assume instead that $v = y$. Recall, from our choice of $y$, that $d(y) \leqslant d(y')$ for all $y' \in X$. So, for an arbitrary $y' \in (W \cap X) \setminus \{y\}$, we have $f(W) = \lambda(y, w) = \min(d(y), d(w)) \leqslant \min(d(y'), d(w)) = \lambda(y', w) \leqslant f(W \setminus \{y\})$. $\diamondsuit$

Recall that $f(Z) = f(\overline{Z})$, for each $Z \subseteq V_x$.

**Claim 2.** *If* $W$ *is dangerous, then* $d(x, W) \leqslant d(x, \overline{W})$.

Proof. Since $W$ is dangerous, $f(W) \geqslant d(W) - 1$. And trivially, $d(\overline{W}) \geqslant f(\overline{W})$. This gives

$$\begin{aligned} f(\overline{W}) = f(W) &\geqslant d(W) - 1 = d(\overline{W}) - d(x, \overline{W}) + d(x, W) - 1 \\ &\geqslant f(\overline{W}) - d(x, \overline{W}) + d(x, W) - 1. \end{aligned}$$

The first and last expressions imply that $1 + d(x, \overline{W}) \geqslant d(x, W)$. If the inequality holds with equality, then $d(x) = d(x, W) + d(x, \overline{W}) = 1 + 2d(x, \overline{W})$. So $d(x)$ is odd, which contradicts the hypothesis of Theorem A.23. Thus, the inequality is strict, which proves the claim. $\diamondsuit$

Since no pair $\{xy, xz\}$ is splittable, every neighbor $z$ of $x$ appears in a dangerous set containing $y$, by Proposition A.18. Recall that $X := N(x)$. Let $\mathcal{F}$ be a minimal family of dangerous sets containing $y$ such that $X \subseteq \cup_{F \in \mathcal{F}} F$.

**Claim 3.** $|\mathcal{F}| \geqslant 3$.

Proof. By the previous claim, each dangerous set contains at most half of the elements of $X$. So, $|\mathcal{F}| \geqslant 2$. But, since any two sets in $\mathcal{F}$ intersect in $y$, in fact we must have $|\mathcal{F}| \geqslant 3$. More formally, suppose instead that $\mathcal{F} = \{F_1, F_2\}$. So $F_1 \cup F_2 \supseteq X$. By Claim 2, we get $d(x, F_1) \leqslant d(x, \overline{F_1}) < d(x, F_2) \leqslant d(x, \overline{F_2}) < d(x, F_1)$, a contradiction. Thus, $|\mathcal{F}| \geqslant 3$. $\diamondsuit$

**Claim 4.** *Each pair* $F_1, F_2$ *of distinct elements of* $\mathcal{F}$ *satisfies inequality* (A.6).

Proof. Suppose, to the contrary, that inequality (A.6) does not hold for $F_1$ and $F_2$. By Proposition A.20, inequality (A.5) must hold for $F_1$ and $F_2$. Since $\mathcal{F}$ is minimal, $F_1 \cup F_2$ cannot be dangerous. That is, $s(F_1 \cup F_2) \geqslant 2$. Now inequality (A.5) gives $1 + 1 \geqslant s(F_1) + s(F_2) \geqslant s(F_1 \cap F_2) + s(F_1 \cup F_2) \geqslant 0 + 2$. So $s(F_1 \cap F_2) = 0$, i.e., $F_1 \cap F_2$ is tight. By definition,

$y \in F_1 \cap F_2$. At the start of this proof we showed that each tight set consists of a single element. Thus, $F_1 \cap F_2 = \{y\}$. So $F_1 \setminus F_2 = F_1 \setminus \{y\}$ and $F_2 \setminus F_1 = F_2 \setminus \{y\}$. By Claim 1, we get $f(F_1) \leqslant f(F_1 \setminus \{y\}) = f(F_1 \setminus F_2)$ and $f(F_2) \leqslant f(F_2 \setminus \{y\}) = f(F_2 \setminus F_1)$. Summing these inequalities and subtracting from (A.2) gives inequality (A.6), as desired. $\diamondsuit$

**Claim 5.** *Each pair* $F_i, F_j \in \mathcal{F}$ *satisfies* $|F_i \setminus F_j| = |F_j \setminus F_i| = 1$ *and* $\overline{d}(F_i, F_j) = 1$.

Proof. By symmetry, it suffices to consider the pair $F_1, F_2$. Note that $y \in F_1 \cap F_2$ and $x \in \overline{F_1} \cap \overline{F_2}$, so $\overline{d}(F_1, F_2) \geqslant 1$. By the previous claim,

$$1 + 1 \geqslant s(F_1) + s(F_2) \geqslant s(F_1 \setminus F_2) + s(F_2 \setminus F_1) + 2\overline{d}(F_1, F_2) \geqslant 0 + 0 + 2.$$

Thus, $\overline{d}(F_1, F_2) = 1$ and $s(F_1 \setminus F_2) = s(F_2 \setminus F_1) = 0$. Since $F_1 \setminus F_2$ and $F_2 \setminus F_1$ are both tight, they each consist of a single element, as noted at the start of the proof. $\diamondsuit$

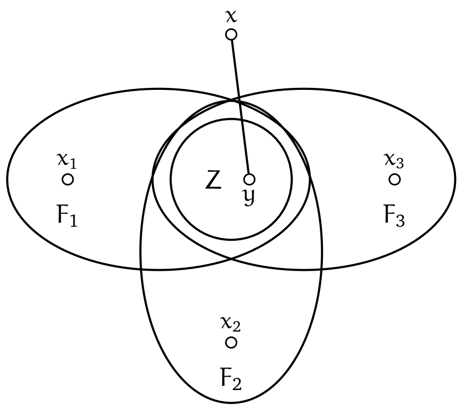



Figure A.12: The end of the proof of Theorem A.26.

By the minimality of $\mathcal{F}$, each $F_i$ contains an element $x_i \in X$ that is not contained in any other member of $\mathcal{F}$. Let $Z := F_1 \cap F_2 \cap F_3$. Claim 5 implies that $F_i = Z \cup \{x_i\}$ and $\overline{d}(F_i, F_j) = 1$ for all distinct $i, j \in [3]$. See Figure A.12. Recall that $x \in \overline{F_1} \cap \overline{F_2} \cap \overline{F_3}$. If $Z$ has an edge to any vertex other than $x$, then $\overline{d}(F_i, F_j) \geqslant 2$ for some distinct $i, j \in [3]$, a contradiction. But now the edge $xy$ is a cut-edge incident to $x$, which is a contradiction. This completes the proof. $\square$

## A.11 k-Critical Graphs

k-critical A graph $G$ is $k$-*critical* if $\chi(G) = k$ and $\chi(G - e) \leqslant k - 1$ for each edge $e \in E(G)$. Every graph $G$ contains a $\chi(G)$-critical subgraph. Thus, various conjectures (such as Hadwiger's Conjecture) are known to hold if and only if they hold for all critical graphs. But critical graphs have more structure, which may help us to prove a given conjecture.

We often want to find a sparsest infinite family of $k$-critical graphs in some graph class. We may also want to characterize these sparsest $k$-critical graphs. One natural approach is to start with a sparse $k$-critical graph and modify it to get a larger sparse $k$-critical graph. We want to define general types of modifications that always yield $k$-critical graphs. We need the following two definitions. Intuitively, they describe subgraphs that simulate either an edge or a vertex.

**Definition A.27.** A $k$-*quasi-$vw$-edge* is a graph $H_1$ with specified vertices $v$ and $w$ such that k-quasi-$vw$-edge

(a) $H_1$ is $(k-1)$-colorable,

(b) every $(k-1)$-coloring $\varphi$ of $H_1$ has $\varphi(v) \neq \varphi(w)$, and

(c) for every $e \in E(H_1)$, the subgraph $H_1 - e$ has a $(k-1)$-coloring $\varphi_e$ such that $\varphi_e(v) = \varphi_e(w)$.

A $k$-*quasi-$vw$-vertex* is a graph $H_2$ with specified vertices $v$ and $w$ such that k-quasi-$vw$-vertex

(a) $H_2$ is $(k-1)$-colorable,

(b) every $(k-1)$-coloring $\varphi$ of $H_2$ has $\varphi(v) = \varphi(w)$, and

(c) for every $e \in E(H_2)$, the subgraph $H_2 - e$ has a $(k-1)$-coloring $\varphi_e$ such that $\varphi_e(v) \neq \varphi_e(w)$.

Let $H$ be a $k$-quasi-$vw$-edge. Given a $k$-critical graph $G$ and $x, y \in V(G)$ such that $xy \in E(G)$, a $k$-*quasi-edge substitution* for $xy$ is formed from $G - xy + H$ by identifying $v$ with $x$ and identifying $w$ with $y$. k-quasi-edge substitution

The key idea, which motivates this definition, is that we can substitute any quasi-edge for any edge in a $k$-critical graph, and we get another $k$-critical graph.

**Proposition A.28.** *Every* $k$*-quasi-edge substitution in a* $k$*-critical graph yields a* $k$*-critical graph.*

This follows directly from the definitions. For completeness, we include the details.

*Proof.* Let $G^H_{xy}$ be formed by substituting a $k$-quasi-$vw$-edge $H$ for an edge $xy$ in a $k$-critical graph $G$. Since $G$ is $k$-critical and $G - xy \subseteq G^H_{xy}$, every proper $(k-1)$-coloring $\varphi_0$ of $G^H_{xy}$ has $\varphi_0(x) = \varphi_0(y)$. Since $H$ is a $k$-quasi-edge substituted for $xy$, by Definition A.27(b) every proper $(k-1)$-coloring $\varphi_0$ of $G^H_{xy}$ has $\varphi_0(x) \neq \varphi_0(y)$. Thus, $G^H_{xy}$ has no $(k-1)$-coloring; that is, $G^H_{xy}$ is not $(k-1)$-colorable.

Now consider an arbitrary edge $e \in E(G^H_{xy})$. We must show that $G^H_{xy} - e$ has a proper $(k-1)$-coloring. If $e \in H$, then we combine a $(k-1)$-coloring $\varphi_0$ of $G - xy$ such that $\varphi_0(x) = \varphi_0(y)$ with a $(k-1)$-coloring $\varphi_e$ of $H - e$ such that $\varphi_e(x) = \varphi_e(y)$, permuting colors so that $\varphi_0$ and $\varphi_e$ agree on $\{x, y\}$. Similarly, if $e \in G - xy$, then we combine a $(k-1)$-coloring $\varphi_e$ of $G - e$ (where $\varphi_e(x) \neq \varphi_e(y)$) with a $(k-1)$-coloring $\varphi_0$ of $H$, where also $\varphi_0(x) \neq \varphi_0(y)$, again permuting colors so that $\varphi_e$ and $\varphi_0$ agree on $\{x, y\}$. $\square$

Similarly, we can substitute any quasi-vertex for any vertex in a k-critical graph, and we again get another k-critical graph. And we can also iterate these substitutions.

split

**Definition A.29.** To *split* a vertex $v$ in a graph $G$, we delete $v$, add new vertices $v'$ and $v''$, and, for each vertex $w$ in $N_G(v)$, add either edge $wv'$ or edge $wv''$. Let $H$ be a k-quasi-*vw*-vertex. Given a k-critical graph $G$ and $x \in V(G)$, a k-*quasi-vertex substitution* for $x$ is formed from $G$ by splitting $x$ into vertices $x'$ and $x''$ and then identifying $v$ with $x'$ and identifying $w$ with $x''$.

k-quasi-vertex substitution

**Proposition A.30.** *A substitution of a* k*-quasi-vw-vertex* $H$ *for a vertex* $x$ *in a* k*-critical graph* $G$ *yields a* k*-critical graph whenever the graph formed by splitting* $x$ *into* $x'$ *and* $x''$ *is not* k*-critical. In particular, this is true when at least one of* $x'$ *and* $x''$ *inherits at most* $k-2$ *neighbors from* $x$.

*Proof.* Let $G_x^H$ denote a substitution as in the proposition.

Denote by $G'$ the graph formed from $G$ by splitting $x$ into $x'$ and $x''$. Since $G$ is k-critical, $G'-e$ is $(k-1)$-colorable for each $e \in E(G')$; moreover, for each $e$ there exists a $(k-1)$-coloring $\varphi_e$ of $G'-e$ with $\varphi_e(x') = \varphi_e(x'')$. If $G'$ is not k-critical, then it is because $\chi(G') \leqslant k-1$. Suppose this is true, and let $\varphi_0$ denote a $(k-1)$-coloring of $G'$. Since $\chi(G) = k$, we know that $\varphi_0(x') \neq \varphi_0(x'')$.

The definitions immediately imply that $G_x^H$ is not $(k-1)$-colorable. Now consider $e \in E(G_x^H)$. If $e \in E(H)$, then $G'$ (resp. $H-e$) has a $(k-1)$-coloring $\varphi_1$ (resp. $\varphi_2$) that differs on $x'$ and $x''$ (resp. on $v$ and $w$). By permuting colors, we can assume that $\varphi_1(x') = \varphi_2(v)$ and $\varphi_1(x'') = \varphi_2(w)$. This gives a $(k-1)$-coloring of $G_x^H - e$.

Now suppose $e \in E(G')$. As above, $G'-e$ has a $(k-1)$-coloring $\varphi_e$ such that $\varphi_e(x') = \varphi_e(x'')$. Recall that $H$ also has a $(k-1)$-coloring $\varphi_0$ such that $\varphi_0(v) = \varphi_0(w)$. Again, by permuting colors, we assume $\varphi_e(x') = \varphi_e(x'') = \varphi_0(v) = \varphi_0(w)$; so we are done as above.

For the final statement: every k-critical graph has minimum degree at least $k-1$. □

Our next two observations follow directly from the definitions.

**Observation A.31.** *If* $H$ *is formed from a* k*-quasi-vw-vertex by adding a pendent edge* $wx$*, then* $H$ *is a* k*-quasi-vx-edge.*

**Observation A.32.** *A graph* $H$ *with specified vertices* $v$ *and* $w$ *is a* k*-quasi-vw-vertex if and only if* $H + vw$ *is* k*-critical.*

Along the same lines as Observation A.32, we have the following.

**Proposition A.33.** *A* $(k-1)$*-colorable graph* $H$ *with specified vertices* $v$ *and* $w$ *is a* k*-quasi-vw-edge if and only if* $N(v) \cap N(w) = \emptyset$ *and identifying* $v$ *with* $w$ *yields a* k*-critical graph.*

*Proof.* If $N(v) \cap N(w) = \emptyset$ and identifying $v$ with $w$ gives a k-critical graph, then $H$ is a k-quasi-*vw*-edge; this simply follows from the definitions. Similarly, suppose that $H$ is a k-quasi-*vw*-edge, and form $H'$ by identifying $v$ and $w$. Again, from the definitions, $\chi(H') > k-1$ and $\chi(H'-e) \leqslant k-1$ for every edge $e$. So it suffices to show that $N(v) \cap N(w) = \emptyset$. Suppose to the contrary that there exists $x \in N(v) \cap N(w)$. By definition, $H - vx$ has a $(k-1)$-coloring $\varphi_0$ such that $\varphi_0(v) = \varphi_0(w)$. So $\varphi_0(v) = \varphi_0(w) \neq \varphi_0(x)$. Thus, $\varphi_0$ is also a $(k-1)$-coloring of $H$, a contradiction. □

**Definition A.34.** A $k$-*quasi-loop* is formed from a $k$-quasi-$vw$-edge and a $k$-quasi-$xy$-vertex by identifying $v$ with $x$ and identifying $w$ with $y$. k-quasi-loop

**Observation A.35.** *Every* $k$-*quasi-loop is* $k$-*critical.*

*Proof.* Let $H_1$ and $H_2$ denote the $k$-quasi-vertex and $k$-quasi-edge, respectively. By Observation A.32, the graph $H_1 + xy$ is $k$-critical. By Proposition A.28, substituting $H_2$ for $xy$ in $H_1 + xy$ yields a $k$-critical graph, which is the given $k$-quasi-loop. □

**Definition A.36.** An *Ore composition* of graphs $G_1$ and $G_2$ is formed by deleting some edge $vw$ in $G_1$, splitting some vertex $x$ in $G_2$ into $x'$ and $x''$, and identifying $v$ with $x'$ and identifying $w$ with $x''$. It is often denoted $O(G_1, G_2)$. Graph $G_1$ is called the *edge side* and graph $G_2$ is called the *split side*. However, such a composition is not uniquely determined, even given $G_1$, $G_2$, $v$, $w$, and $x$, since it depends on how we split $x$. Ore composition edge side split side

**Observation A.37.** *In an Ore composition of* $k$-*critical graphs* $G_1$ *and* $G_2$*, if the graph* $G_2'$ *formed by splitting* $x$ *into* $x'$ *and* $x''$ *is not* $k$-*critical, then the Ore composition is* $k$-*critical.*

*Proof.* Deleting some $vw$ from $G_1$ gives a $k$-quasi-$vw$-vertex. Splitting some $x$ in $G_2$ gives a $k$-quasi-$x'x''$-edge, as long as $G_2'$ is not $k$-critical. Now we are done by Observation A.35. □

**Observation A.38.** *A Hajós join of graphs* $G_1$ *and* $G_2$ *is formed from the disjoint union* $G_1 + G_2$ *by deleting some edge* $v_1w_1$ *from* $G_1$*, deleting some edge* $v_2w_2$ *from* $G_2$*, identifying* $v_1$ *and* $v_2$*, and adding the edge* $w_1w_2$*. If* $G_1$ *and* $G_2$ *are* $k$-*critical, then so is each Hajós join of* $G_1$ *and* $G_2$*.*

*Proof.* The Hajós join is a special instance of a $k$-quasi-loop. □

Note that a Hajós join is equivalent to the special case of an Ore composition where $x'$ or $x''$ (arising on the split side) has degree 1 before being identified with a vertex of the edge side. Thus, every Hajós join is an Ore composition, but not vice versa. Below we discuss in more detail a particularly interesting family of critical graphs, called $k$-Ore graphs.

### A.11.1 Properties of k-Ore Graphs

For each integer $k$ with $k \geqslant 3$, the set of $k$-Ore graphs is defined recursively. The graph $K_k$ is a $k$-Ore graph. And every Ore composition of two $k$-Ore graphs is also a $k$-Ore graph.

Below we list some properties of $k$-Ore graphs. These can all be proved by induction on the order of the graph (or, equivalently, induction on the number of Ore compositions used to form the graph). So we omit the proofs. Kostochka and Yancey proved (Theorem 12.49) that every k-critical graph G has $\|G\| \geqslant |G|\frac{(k+1)(k-2)}{2(k-1)} - \frac{k(k-3)}{2(k-1)}$. Further, they showed [273] that equality in this bound holds if and only if G is a k-Ore graph.

Here we are particularly interested in large cliques in a k-Ore graph in which all vertices have degree $k-1$. A subgraph D of a graph G is a *diamond* in G if $D \cong K_k - vw$ and $d_G(x) = k-1$ for all $x \in V(D) \setminus \{v, w\}$; see the left of Figure A.13. A subgraph $D'$ of a graph G is an *emerald* in G if $D' \cong K_{k-1}$ and $d_G(x) = k-1$ for all $x \in V(D')$; see the right of Figure A.13.

diamond

emerald

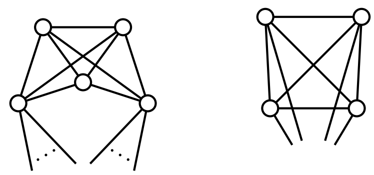


Figure A.13: Left: A diamond, when $k = 5$.
Right: An emerald, when $k = 5$.

1. Every k-Ore graph is k-critical.
2. Every k-Ore graph has order equal to 1 modulo $k-1$.
3. Every k-Ore graph G has size $\|G\| = |G|\frac{(k+1)(k-2)}{2(k-1)} - \frac{k(k-3)}{2(k-1)}$.
4. If G is k-Ore and $v \in V(G)$, then $G - v$ contains a diamond or an emerald in G.
5. If G is a k-Ore graph larger than $K_k$ and H is a copy of $K_{k-1}$ in G, then $G - H$ contains a diamond or an emerald in G.
6. For each integer $k \geqslant 3$ and each integer $i \geqslant 1$ there exists a k-Ore graph $G_{k,i}$ of order $1 + (k-1)i$ with two edges, $e_1$ and $e_2$, such that $G_{k,i} - e_1 - e_2$ has no copies of $K_{k-1}$.

Much work has focused on determining the sparsest k-critical graphs that satisfy some other criteria as well. It is natural to consider forbidding $K_{k-1}$. In this case, the general lower bound of Kostochka and Yancey (Theorem 12.49) can only be improved by an additive constant. The reason is that there exist arbitrarily large k-Ore graphs G with two edges, $e_1$ and $e_2$, such that $G - e_1 - e_2$ has no copy of $K_{k-1}$; see (6) in the list above. Figure 12.11 shows the case $k = 4$. Postle conjectured that if G is k-critical with no copy of $K_{k-2}$, then the lower bound on $\|G\|$ proved by Kostochka and Yancey can be improved asymptotically; this conjecture has been proved when $k = 5$ [285], $k = 6$ [171], and $k \geqslant 33$ [181]. See the Chapter 12 Notes for more details. When we forbid only $K_{k-1}$, Moore [311] proposed the following.

**Conjecture A.39.** *Fix an integer* $k \geqslant 4$. *If* G *is a* k*-critical graph with no copy of* $K_{k-1}$, *then*

$$\|G\| \geqslant |G|\frac{(k+1)(k-2)}{2(k-1)} + \frac{k(k-3)}{2(k-1)}.$$

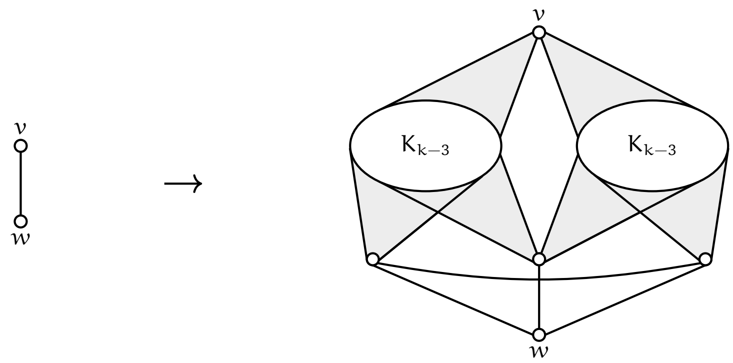



Figure A.14: An edge $vw$ and a k-quasi-$vw$-edge.

Davies (unpublished) showed that this conjecture is best possible when $k \geqslant 5$.

**Theorem A.40.** *Fix an integer $k \geqslant 5$. There exist arbitrarily large $k$-critical graphs $G$ with no copy of $K_{k-1}$ and with $\|G\| = |G|\frac{(k+1)(k-2)}{2(k-1)} + \frac{k(k-3)}{2(k-1)}$.*

*Proof.* It is easy to show, by induction on $|G|$, that there exist arbitrarily large k-Ore graphs with two edges, $e_1$ and $e_2$, such that $G - e_1 - e_2$ has no copy of $K_{k-1}$; see (6) in the list above. For each $i \in \{1, 2\}$ denote the endpoints of $e_i$ by $v_i$ and $w_i$. We simply use two k-quasi-$v_i w_i$-edge substitutions, and require that each k-quasi-edge is $K_{k-2}$-free. Our choice of k-quasi-edge is shown in Figure A.14. It is straightforward to check that it has no copy of $K_{k-1}$. So all that remains is to compute the numbers of vertices and edges in this k-quasi-edge and show that these numbers satisfy the desired equality.

Each of the two times we substitute this k-quasi-edge, we increase the number of vertices by $2(k-3)+3 = 2k-3$. At the same time, we increase the number of edges by $2(\binom{k-1}{2} - 1) + 2(k-3) + 3 = (k^2 - 3k) + (2k-6) + 3 = k^2 - k - 3$. Recall from (3) above that every k-Ore graph $G$ has $\|G\| = |G|\frac{(k+1)(k-2)}{2(k-1)} - \frac{k(k-3)}{2(k-1)}$. So it suffices to show that each substitution increases the number of edges by $(2k-3)\frac{(k+1)(k-2)}{2(k-1)} + \frac{k(k-3)}{2(k-1)}$, since the first substitution will cancel the additive term in the edge count for k-Ore graphs and the second substitution will reintroduce that additive term, but with a positive sign. Thus

$$\begin{aligned}
(2k-3)\frac{(k+1)(k-2)}{2(k-1)} + \frac{k(k-3)}{2(k-1)} &= (k+1)(k-2) - \frac{(k+1)(k-2)}{2(k-1)} + \frac{k(k-3)}{2(k-1)} \\
&= k^2 - k - 2 + \frac{-(k^2-k-2) + (k^2-3k)}{2(k-1)} \\
&= k^2 - k - 2 + \frac{(-2k+2)}{2(k-1)} \\
&= k^2 - k - 3.
\end{aligned}$$

$\square$

## A.12 Fractional Edge-Coloring: The Matching Polytope

We do not develop the theory of polyhedra, but instead give only those definitions needed to state our theorem. For a thorough introduction to this topic, see [90, Chapter 3].

polyhedron
polytope
face
defining hyperplane
facet
incidence vector
convex combination
matching polytope

**Definition A.41.** A *polyhedron* in $\mathbb{R}^d$ is the set of points satisfying some collection of linear inequalities; a *polytope* is a bounded polyhedron. A *face* of a polyhedron P is a set $P \cap \{x \in \mathbb{R}^n : ax = b\}$ where the inequality $ax \leqslant b$ holds for every point $x \in P$. The hyperplane $ax = b$ is the *defining hyperplane* for this face. A *facet* is an inclusionwise maximal face.

The *incidence vector*, $x_F$, of a set $F \subseteq E(G)$ is the vector in $\mathbb{R}^{E(G)}$ with a 1 in coordinate $e$ if $e \in F$ and a 0 otherwise. For an arbitrary vector $a \in \mathbb{R}^{E(G)}$ and a set $F \subseteq E(G)$, we write $a(F) = x_F a = \sum_{e \in F} a_e$. For a single edge $e$, we may write $x_e$ for $x_{\{e\}}$. We denote the set of edges incident to a vertex $v$ by $\nabla(v)$. A *convex combination* is a linear combination with all coefficients nonnegative and with coefficients summing to 1. The *matching polytope* of a graph G is the set of all convex combinations of incidence vectors of matchings of G.

The following theorem is due to Edmonds [143] and it inspired the development of much of Polyhedral Combinatorics (see [90, Chapter 4]). This theorem leads to a short proof determining $\chi'_f(G)$, the fractional chromatic index of any graph G. In fact, we will see that the upper bound on $\chi'$ given in the Goldberg–Seymour Conjecture (which is now proved) does indeed hold for $\chi'_f$. The proof we present is due to Lovász [289].

**Theorem A.42.** *The matching polytope is given by the following constraints.*

$$x_e \geqslant 0 \qquad \textit{for all } e \in E(G) \tag{A.1}$$
$$x(\nabla(v)) \leqslant 1 \qquad \textit{for all } v \in V(G) \tag{A.2}$$
$$x(E(G[S])) \leqslant (|S|-1)/2 \qquad \textit{for every odd set } S \subseteq V(G) \tag{A.3}$$

*Proof.* Let P denote the polytope defined by inequalities (A.1)–(A.3), and let $\tilde{P}$ denote the set of convex combinations of incidence vectors of matchings of G. Since each incidence vector of a matching of G satisfies (A.1)–(A.3), so do their convex combinations. Thus, $\tilde{P} \subseteq P$.

Now we prove that also $P \subseteq \tilde{P}$. The matching polytope $\tilde{P}$ has dimension $|E(G)|$. This follows from the fact that the matchings $\emptyset, e_1, e_2, \ldots, e_{|E(G)|}$ are affinely independent. Every facet has dimension $|E(G)| - 1$, so contains $|E(G)|$ affinely independent matchings. Let F be a facet of P with defining hyperplane H, and let $\mathcal{M}^*$ be the set of matchings that lie in H. Thus, the affine hull of $\mathcal{M}^*$ is H. So, in fact, every hyperplane that contains all of $\mathcal{M}^*$ is H. This means that if $F'$ is a facet defined by hyperplane $H'$ and $H'$ contains all of $\mathcal{M}^*$, then $H'$ is H, so $F'$ is F. (We rule out the possibility that F and $F'$ have opposite inequalities, since then P lies completely in H, which is impossible, since P has dimension $|E(G)|$.) So, to prove the theorem, it suffices to show that for every facet F of P, the matchings $\mathcal{M}^*$ lie in a hyperplane of the form (1) $x_e = 0$, for some $e \in E(G)$, or (2) $x(\nabla(v)) = 1$, for some $v \in V(G)$, or (3) $x(E(G[S])) = (|S|-1)/2$, for some odd set $S \subseteq V(G)$.

Let $F$ be a facet of $P$ with supporting hyperplane $\mathbf{a}\mathbf{x} = b$. First suppose that there exists some edge $e$ such that $\mathbf{a}_e < 0$. Suppose that there exists $M \in \mathcal{M}^*$ with $e \in M$. Let $M' = M - e$. Now $\mathbf{a}\mathbf{x}_{M'} = \mathbf{a}\mathbf{x} - \mathbf{a}_e = b - \mathbf{a}_e > b$, a contradiction. Thus, for each $M \in \mathcal{M}^*$, we have $e \notin M$. Thus, $\mathcal{M}^*$ is contained in the hyperplane $\mathbf{a}_e = 0$, which is of form (1) in the previous paragraph. Similarly, suppose there exist $v \in V(G)$ such that every $M \in \mathcal{M}^*$ saturates $v$. Now $\mathcal{M}^*$ is contained in the hyperplane $\mathbf{x}(\nabla(v)) = 1$, which is of form (2) above.

Now assume that $\mathbf{a}_e \geqslant 0$ for all $e$ and for every $v \in V(G)$, some matching $M_v \in \mathcal{M}^*$ does not saturate $v$. Let $G'$ be the subgraph of $G$ induced by edges with $\mathbf{a}_e > 0$. If $G'$ is disconnected, then we can write the inequality $\mathbf{a}\mathbf{x} \leqslant b$ as the sum of two inequalities $\mathbf{a}^1\mathbf{x} \leqslant b_1$ and $\mathbf{a}^2\mathbf{x} \leqslant b_2$. Since each $\mathbf{x} \in P$ satisfies both of these inequalities, and each $M \in \mathcal{M}^*$ lies in the defining hyperplane for $F$, we conclude that each $M \in M^*$ lies in both of the hyperplanes $\mathbf{a}^1 x = b_1$ and $a^2 x = b_2$. However, this implies that $F$ has dimension at most $|E(G)| - 2$, a contradiction. Thus, we conclude that $G'$ is connected. We will show that $\mathcal{M}^*$ lies in the hyperplane $\mathbf{x}(E(G[S])) \leqslant (|S| - 1)/2$, where $S = V(G')$.

Suppose that some $M \in M^*$ saturates every vertex of $G'$. Choose $v \in V(G')$ and $M_v \in M^*$ such that $v$ is not saturated by $M_v$. (Such an $M_v$ exists, by assumption.) Now $\mathbf{a}\mathbf{x}(M) > \mathbf{a}\mathbf{x}(M_v) = b$, a contradiction. Thus, every extremal matching has some vertex of $G'$ unsaturated. We will show that every extremal matching has exactly one vertex of $G'$ unsaturated. Suppose, to the contrary, that some extremal matching has two unsaturated vertices. Choose $M_1 \in \mathcal{M}^*$ and $u$ and $v$, both unsaturated by $M_1$, to minimize $\mathrm{dist}_{G'}(u, v)$. Clearly, $\mathrm{dist}_{G'}(u, v) > 1$, since otherwise $M + uv$ violates the constraint $\mathbf{a}\mathbf{x} \leqslant b$. Let $w$ be the neighbor of $u$ on a shortest $u, v$ path in $G'$. Choose $M_2 \in \mathcal{M}^*$ that leaves $w$ unsaturated. Let $J = M_1 \cup M_2$, and note that $\Delta(J) \leqslant 2$. Since $w$ is unsaturated in $M_2$ and saturated in $M_1$, the component of $J$ containing $w$ is a path; call it $P$. Let $M_1' = M_1 \oplus P$ and $M_2' = M_2 \oplus P$. Note that $\mathbf{a}\mathbf{x}(M_1') + a\mathbf{x}(M_2') = \mathbf{a}\mathbf{x}(M_1) + \mathbf{a}\mathbf{x}(M_2)$. Thus, $M_1', M_2' \in \mathcal{M}^*$ (otherwise one of $M_1'$ and $M_2'$ would violate the constraint $\mathbf{a}\mathbf{x} \leqslant b$.

However, at least one of $u$ and $v$ is unsaturated in $M_1'$. Further, $w$ is unsaturated in $M_1'$. Thus, $M_1' \in \mathcal{M}^*$ and $M_1'$ has two unsaturated vertices that are closer in $G'$ than $u$ and $w$. So $M_1'$ contradicts our choice of $M_1$. Thus, each extremal matching has exactly one unsaturated vertex. Hence, each extremal matching lies in the hyperplane $\mathbf{x}(E(G[S])) \leqslant (|S| - 1)/2$, where $S = V(G')$, as desired. $\square$

# Hints

. . . one cannot so well seize a thing and make it one's own, when it has been learned from another, as when one has himself discovered it.

—René Descartes

## Chapter 1

1.1. Use induction on $k$.

1.2. A *fractional orientation* assigns a nonnegative fraction of each edge to each of its endpoints, with the fractions assigned to the endpoints of each edge summing to 1. Find a fractional orientation of $H - v$, with each vertex assigned fractions of incident edges summing to at most $18/13$. (Why does this solve the problem?)

1.3. (a) Show that if $G$ is connected and chordal but not a clique, then every minimal cutset is a clique. Use this to prove that every chordal graph that is not a clique contains two simplicial vertices that are non-adjacent. (b) Consider paths.

1.4. Use induction. For some vertex $v \in V(G)$, compare $d_G(v)$ and $d_{\overline{G}}(v)$ with $\chi(G - v)$ and $\chi(\overline{G - v})$.

1.5. Compare with the proof of Lemma 1.16.

1.6. Start with plane triangulations and subdivide edges to ensure that every face of the resulting graph has length exactly $g$.

1.7. Since $\text{mad}(G)$ is a rational number with denominator in $[|G|]$, reduce this problem to one of finding an orientation with minimum outdegree for each of $|G|$ multigraphs.

1.9. Use the previous exercise. In Exercise 3.8 we extend this result to include the case $k = 7$.

1.10. When $k = 5\ell$, build your construction from the icosahedron by adding, for each edge $vw$, exactly $\ell - 1$ paths of length 2 joining $v$ and $w$.

1.11. Color greedily by non-increasing distance.

1.12. Consider a list-assignment $L$ for a graph $G$ such that (i) $wx, xy \in E(G)$ and $wy \notin E(G)$ and $|L(w)| + |L(y)| > |L(x)|$. Show how to "save a color for $x$" when you color $w$ and $y$, regardless of whether or not they share a common color.

1.13. The characterization is the same as for degree-choosable graphs, and the proof is nearly the same as that of Theorem 1.37.

1.14. Consider a graph $G$ such that $\mathrm{mad}(G) = \frac{5}{2}$ but $G$ does not contain any of the reducible configurations used in the proof of Theorem 1.41. Describe the degrees of all vertices in $G^{(2)}$ and list-color that graph directly (without appealing to reducible configurations).

1.15. Carefully account for all of the (negative) charge.

1.16. Every 3-face should take charge from adjacent long faces, and every long face should take charge from incident $4^+$-vertices.

1.18. If a $\Delta$-vertex has only a single 3-vertex, then it is natural to have that $\Delta$-vertex sponsor that 3-vertex.

1.19. Use face charging, and give each 2-vertex some charge from its incident $4^+$-face.

1.20. Color the vertices first.

1.21. Yes. Yes. See Section 1.2.3.

## Chapter 2

2.1. Construct lists so that every possible coloring of the small side exhausts the list of exactly one vertex on the big side.

2.2. (a) Consider a hypothetical $k$-coloring $\varphi$ of $G_{k+1}$ and a maximum set vertex subset $S$ such that $S \subseteq \cup_{i=1}^{k} V(G_i)$ and $|V(G_i) \cap S| \leqslant 1$ for all $i \in [k]$ and $|\varphi^{-1}(j) \cap S| \leqslant 1$ for all $j \in [k]$. (b) Assume a $k$-coloring of $G_{k+1}$ and modify it to get a $(k-1)$-coloring of $G_{k-1}$.

2.3. Induct on the number of blocks.

2.4. Copy ideas from the proof of Theorem 2.3.

2.5. (a) Start with an independent set $S$ of size $\lfloor \frac{3}{2}k \rfloor$ and add vertices so that $S$ induces a clique in $G^2$.

2.6. Consider the case that each cycle is a 4-cycle and the path has length 0 or 1.

2.7. Use induction on $a + b + c$.

2.8. Consider separately the cases $k \leqslant (3/2)^{\ell}$ and $k > (3/2)^{\ell}$. For the latter, let $a := k^{1/\ell}$ and prove bounds, in terms of $a$, on $n$ and the number of 3-colorings of $G(v, w, k, \ell)$.

2.9. Show that if the list assignment on the bottom is restricted to any of the four "crosses", then it is isomorphic to the one on top.

2.10. (a) This graph is small enough to essentially use brute force. (b) Start with an arbitrary drawing of a graph in the plane (allowing edge-crossings). For each edge with at least one crossing, between each pair of successive crossings of an edge, add a new vertex in the interior of that edge. Also add a new vertex between the initial endpoint and the first crossing (but not between the last crossing and the final endpoint; here first and last are arbitrary for each edge). In the resulting graph, each edge is in at most one crossing. Substitute a crossover gadget for each edge-crossing, identifying the external connectors with the endpoints of the two edges in the crossing. Note that the resulting graph is planar. Show that it is 3-colorable if and only if the original graph is.

2.11. (a) Orient edges so that (nearly) every vertex has outdegree at most $k-1$. (b) Avoid recursion, and just slightly modify $(d,s,t)$-graphs for appropriate values of $s$ and $t$.

## Chapter 3

3.1. Focus on the case when $i := \Delta - 1$.

3.2. What can you say about cycles in such graphs?

3.3. It suffices to consider regular graphs.

3.5. Let $G' := G - M$. Show that the $\Delta$-vertices in $G'$ form an independent set. This implies that $\chi'(G') \leqslant \Delta + \mu(G) - 1$. (This result is old, but an easy proof is in [146].)

3.6. Use Vizing's Adjacency Lemma.

3.8. Extend the ideas in Exercise 1.9. Use Euler's Formula to show that if $G$ is sufficiently large and embeddable in $S$ and $\delta(G) = 6$, then $G$ contains a 6-vertex that has six 6-neighbors and that is incident to six 3-faces. Use this configuration to show that $G$ is not 7-critical.

3.9. When seeking to extend a 5-coloring of $G - v$, for some 5-vertex $v$, use planarity to show that if all neighbors of $v$ use distinct colors, then some pair of neighbors of $v$ lie in distinct Kempe components induced by their two colors.

3.10. Show that a minimal counterexample $G$ must be $\Delta$-regular. Consider an arbitrary vertex $w$ of $G$, the neighbors $v_1, \ldots, v_\Delta$ of $G$, and a $\Delta$-coloring $\varphi$ of $G - w$. By symmetry, we assume $\varphi(v_i) = i$ for each $i \in [\Delta]$. Determine the structure of the Kempe components $\mathcal{C}_{i,j}$ containing $v_i$ and/or $v_j$ for each pair $i, j \in [\Delta]$.

3.11. What does a Kempe swap do to the color classes in one of these colorings?

3.12. Given a subgraph $H$ with $|H|$ odd, delete some vertex to get a smaller subgraph yielding a bound that is no smaller.

## Chapter 4

4.1. Start with a long odd cycle $C$, and add gadgets to form $G_k$, so that in every theoretical map $G_k \to C_{2k+1}$ some vertex of $C_{2k+1}$ is not the image of any vertex on $C$.

4.2. Modify the proof of Theorem 1.24. How do we "save a color" for a vertex $v_2$ when coloring its neighbors $v_1$ and $v_3$ that are non-adjacent?

4.3. (a) Be greedy. (b) How do we "set aside" colors for a vertex that we are deleting?

4.5. (b) Mirror the proof for choosability, by using "disjoint sets of colors" for distinct blocks that contain a common cut-vertex.

## Chapter 5

5.1. Modify the proof of Lemma 5.4.

5.3. Prove that an appropriate invariant holds through each round of the Proposal Algorithm.

5.4. (b) Let $f(n)$ be the maximum number of stable matchings admitted by fixed preference lists for $n$ men and $n$ women. Prove that $f(a+b) \geqslant f(a)f(b)$.

5.5. Shorten the lists to allow only a single stable matching (the one that the women want).

5.6. Color a single part and proceed by induction.

5.7. (a) Consider the case that some non-adjacent pair shares a common color in their lists. (b) Use a series of gadgets to force a partial $L'$-coloring and conclude that it cannot be extended to all of $G$. (This portion of the exercise belongs more to Chapter 2.)

5.8. If a color is common to all vertices in a part, then use it there. Otherwise, each color used elsewhere will deplete the lists of fewer vertices.

5.9. Refer to Section 1.2.3.

5.10. Color the vertices first.

## Chapter 6

6.1. Use induction on $|E(G) \setminus E(T)|$.

6.2. For even $n \geqslant 6$, use induction.

6.3. Consider all possible orientations for $\partial(v)$, for some $v \in V(G)$, in a modulo 3 orientation.

6.4. Given a 3-edge-coloring, find two sets of vertex-disjoint cycles such that each edge is used in at least one set. Given an NZ 4-flow $f$, show that the edges with $|f(e)| = 2$ induce a perfect matching $M$ and that each cycle in $G - M$ has even length.

6.5. Which three-element multisets in $\mathbb{Z}_2^2$ sum to 0?

6.6. The Hamiltonian cycle already has a $\mathbb{Z}_2$-flow. Find another $\mathbb{Z}_2$-flow that covers all remaining edges, as in the proof of Lemma 6.12.

6.7. For a triangle $C$, extend a 4-flow of $G/C$ to $G$.

6.8. (a) Copy ideas from Exercise 6.3. (b) Construct a $\beta$-orientation greedily, selecting the final two vertices in the vertex ordering carefully.

6.10. Use the trick in the proof of Theorem 6.10 of finding a directed path and adding a flow to it.

6.11. Given a single-cross graph $G$, delete its crossing edges, add a vertex-disjoint 4-cycle $C$, and add a matching $M$ between the four vertices of $C$ and the four endpoints of the crossing edges, so that the resulting graph is planar. Given a 3-edge-coloring $\varphi$ of this new graph, modify it to get a 3-edge-coloring of $G$. (Prove that $\varphi$ colors the edges of $M$ in one of only a few possible ways.)

## Chapter 7

7.2. Modify the proofs in Sections 7.3.2 and 7.3.6.

7.3. Consider a 2-coloring of the branch vertices, and bound the number of edges that must be subdivided to make the 2-coloring proper.

## Chapter 8

8.1. Note that $L(K_{3,3}) = C_3 \square C_3$. Up to isomorphism, there are not too many orientations of $C_3 \square C_3$ with maximum outdegree 2.

8.2. Do not directly compute EE and OE; instead, do one of the following. (a) Find a parity-reversing "near-involution" that pairs most elements of EE with elements of OE, leaving only a few to count; or (b) prove that $|EE|$ and $|OE|$ are both even, but that $|EE| + |OE|$ is $2 \pmod 4$.

8.3. (a) Use the Coefficient Formula.

8.5. Follow the proof of Theorem 8.27.

8.6. Consider the cases that $G$ is (a) a wheel (cycle joined to a dominating vertex) with an even number of vertices, (b) a wheel with an odd number of vertices, or (c) something else. The first two cases are easy, and the third case requires finding some reducible configurations. The proof is not hard, but it is longer than most exercises in this book.

8.7. Expand the permanent on column $\vec{v}$.

## Chapter 9

9.1. Consider replies for Bob on vertices "opposite" the vertices where Alice played most recently.

9.4. Generalize the construction in the remark following Theorem 9.16. Given a set of isolated vertices, partition it first into sets of size $D$ ($k$ times) and second into sets of size $k$ ($D-k$ times), so that no two vertices are in two or more common parts. One way to do this uses the Hajnal–Szemeredi Theorem (Theorem 10.28).

9.5. Consider Theorem 9.18.

9.6. (a) Copy the proof of Corollary 9.11, but strengthen the bound $|Y| \leqslant \omega(G)$ to $|Y| \leqslant \omega(G) - 1$. (b) Consider a vertex $vw$ in $H$. Fix $M$ and $N$ as in the proof of Theorem 9.7 and let $P$ be the outedges in $H$ from $vw$. For each of $M$, $N$, and $P$ consider separately those edges "due to" $v$ and those due to $w$.

9.7. For the upper bound, let $f : V(G) \to [n]$ be a bijection that minimizes the quantity $\max_{vw \in E(G)} |f(v) - f(w)|$. For the lower bound, use (and prove) the following statement: If $G$ has treewidth at most $k$, then there exists $S \subseteq V(G)$ such that each component of $G - S$ has at most $|G|/2$ vertices.

9.8. In each case add a vertex inside each of various faces of a Platonic or Archimedean solid, making each new vertex adjacent to every vertex on the boundary of its face.

9.9. Analyze more carefully the proof of Lemma 9.29, bounding the charge ending on each forbidden edge. For the sharpness of the coefficients, consider the constructions from the previous exercise.

9.10. (a) Consider stars. (b) Solve (a) and "glue together" multiple copies of the graphs $G_d$, possibly adding some edges.

## Chapter 10

10.1. Consider cliques.

10.2. Use a vertex shuffle with two parts.

10.3. Let $s = 2$.

10.4. Copy the proof of Lemma 10.4.

10.5. Each vertex $v_i$ sends charge $\frac{r}{r+2}$ to each adjacent $v_j$ later in the order.

10.6. For a $k$-coloring $\varphi$ with defect 2, partition the vertex set of each component of $G[\varphi]$ into sets of size $2\Delta$ (and possibly one smaller set) and find an appropriate IT.

10.7. The proof is essentially the same.

10.8. For the upper bound, use Hall's Theorem. For the lower bound, when $m = n - 1$, make one class with a single vertex from each clique and every other class with all vertices in the same clique.

10.9. When $k = 2$, consider odd cycles.

10.10. Mimic the proofs of Lemmas 10.7 and 10.8.

10.11. Consider the 4-Ore graphs described in Section A.11.1. The fact that $\Theta \leqslant 7$ greatly restricts the possibilities.

10.12. When you apply Lemma 10.27, your set $I$ should be empty.

## Chapter 11

11.1. Adapt ideas from Section 1.2.3.

11.2. Pick colors $\alpha, \beta \in L(w_k) \setminus L(w_1)$ and use them to color the vertices of a path along $C$, as long as possible, that begins with $w_k$ and avoids $w_1$.

11.3. The $(3, 2)$-decomposition in Figure 11.3 is one possible solution.

11.4. Reuse the gadget at the top of Figure 2.23, but combine copies of it a bit differently than in the graph at the bottom of that figure (which was chosen to minimize the order, rather than to enforce $|L(v) \cap L(w)| \leqslant 3$). In fact, the paper of Mirzakhani [300] presents both the graph in Figure 2.23 and also the graph we construct in the present exercise.

## Chapter 12

12.1. Consider the potential of subgraphs of $G'$.

12.2. Now some 4-faces may not be contractible. Handle separately the three cases (i) $G'$ has a contractible 4-face, (ii) $G'$ has at least two 4-faces, but none are contractible, and (iii) $G'$ has at most one 4-face.

12.4. How can you modify the graph to ensure that $v$ and $w$ are colored the same? Or colored differently?

12.5. Generalize the two examples shown in Figure 12.8.

12.6. A natural choice for the gadget for a vertex $v$ colored 0 would be a butterfly, identifying $v$ with the middle of the butterfly. However, such a gadget has weight 0, which is unhelpful. Instead, add a pendent edge at some 2-vertex in a butterfly and identify $v$ with the endpoint of this pendent edge. The resulting gadget has weight 1, rather than 0, which is needed to prove the Gap Lemma.

12.7. We extend the theorem to precolored graphs, in which a vertex can be colored I, colored F, or uncolored. If a vertex is uncolored, it can have $i$ "fake neighbors" that are colored F, where $i \in \{0, 1, 2, 3, 4\}$. Similarly, if a vertex is colored F, it can have $j$ "fake neighbors" that are colored F, where $j \in \{0, 1, 2, 3\}$. We denote by $U_i$ the sets of vertices that are uncolored but have $i$ fake neighbors colored F. And we denote by $F_j$ the sets of vertices that are precolored F and have $j$ fake neighbors colored F. Finally, we denote by I the set of vertices colored I (slightly abusing notation). We use the potential function

$$\rho_G^4(R) := \sum_{i=0}^{4} C_{U,j}|U_j \cap R| + \sum_{j=1}^{k} C_{F,j}|F_j \cap R| + C_I|I \cap R| - C_E|E(G[R])|,$$

for each $R \subseteq V(G)$, where $C_E = 11$, $C_I = 4$, $C_{U,i} = 15-3i$ for all $i \in \{0, 1, 2, 3, 4\}$, and $C_{F,1} = 8$, $C_{F,2} = 5$, $C_{F,3} = 3$, $C_{F,4} = 0$.

A precolored graph G is $(I, F_4)$-critical if it does not admit any $(I, F_4)$-coloring, but every proper subgraph does and so does each precolored graph $G'$ formed by "weakening the precoloring" (moving a vertex in $F_3$ to $F_2$, for example). Show that if a precolored graph G is $(I, F_4)$-critical, then $\rho_G^4(V(G)) \leqslant -3$. (Why does this imply the theorem stated above in terms of $\mathrm{mad}(G)$ for every graph G without a precoloring?)

(a) First, prove a Weak Gap Lemma: If $R \subseteq V(G)$ and $|R| \geqslant 1$, then $\rho_G^4(R) \geqslant 1$.

(b) Now use this to prove a Strong Gap Lemma: If $R \subseteq V(G)$ and $G[R]$ contains at least one edge, then $\rho_G^4(R) \geqslant 4$.

(c) Prove that in a minimal counterexample G to the theorem, $I = \emptyset$.

(d) Now use discharging to show that no counterexample G exists. Let $\mathrm{ch}(v) := 11(d(v) - 3) + 3 + 6j$ for all $v \in U_j$. Let $\mathrm{ch}(v) := 11(d(v) - 2) + 6j$ for all $v \in F_j$, where $j \leqslant 2$. Finally, let $\mathrm{ch}(v) := 11d(v) - 6$ for all $v \in F_j$ with $j \geqslant 3$. Show that the sum of these charges is $-2\rho(V(G))$, so if G is a counterexample, then the sum is at most 4. Now use discharging to show that it is greater than 4. Let each vertex $v \in U_1$ of degree 2 take charge 1 from each neighbor.

12.8. Consider $x, y$-walks of length $2t - 1$ in $C_{2t+1}$ for various vertex pairs $x, y$.

12.9. Restrict the degree sequence for each graph to two possibilities. Every 3-vertex should be in at least one triangle and every higher degree vertex should be in more.

12.12. Use the following two rules. (R1) Each 5-vertex in a $K_4$ and each $6^+$-vertex gives $\frac{1}{6}$ to each neighbor. (R2) After applying (R1) all 4-vertices and 5-vertices not in a $K_4$ average their charges. Use Lemma 12.54 to analyze the final charges affected by (R2).

# References

In addition to standard journal citations, whenever possible we also provide links to freely accessible, often preliminary, versions of the papers. Typically, these are on preprint servers such as the arXiv.


[1] H. L. Abbott and B. Zhou. On small faces in 4-critical planar graphs. *Ars Combin.*, 32:203–207, 1991. 40

[2] T. Abe, S.-J. Kim, and K. Ozeki. The Alon-Tarsi number of $K_5$-minor-free graphs. *Discrete Math.*, 345(4):Paper No. 112764, 7, 2022, `arXiv:1911.04067`. 247, 248

[3] G. Agnarsson and M. M. Halldórsson. Coloring powers of planar graphs. *SIAM J. Discrete Math.*, 16(4):651–662, 2003. 39

[4] R. Aharoni, E. Berger, and R. Ziv. Independent systems of representatives in weighted graphs. *Combinatorica*, 27(3):253–267, 2007. 304

[5] M. Aigner and G. M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, third edition, 2004. Including illustrations by Karl H. Hofmann. 70, 338

[6] V. A. Aksenov. The extension of a 3-coloring on planar graphs. *Diskret. Analiz*, 26 Grafy i Testy:3–19, 84, 1974. 140

[7] V. A. Aksenov and L. S. Mel'nikov. Essay on the theme: the three-color problem. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 23–34. North-Holland, Amsterdam-New York, 1978. 40, 69

[8] M. O. Albertson, G. G. Chappell, H. A. Kierstead, A. Kündgen, and R. Ramamurthi. Coloring with no 2-colored $P_4$'s. *Electron. J. Combin.*, 11(1):Research Paper 26, 13, 2004. 385

[9] M. O. Albertson and J. P. Hutchinson. The three excluded cases of Dirac's map-color theorem. In *Second International Conference on Combinatorial Mathematics (New York, 1978)*, volume 319 of *Ann. New York Acad. Sci.*, pages 7–17. New York Acad. Sci., New York, 1979. 38

[10] N. Alon. The strong chromatic number of a graph. *Random Structures Algorithms*, 3(1):1–7, 1992. 304

[11] N. Alon. Restricted colorings of graphs. In *Surveys in combinatorics, 1993 (Keele)*, volume 187 of *London Math. Soc. Lecture Note Ser.*, pages 1–33. Cambridge Univ. Press, Cambridge, 1993. 247

[12] N. Alon. Combinatorial Nullstellensatz. *Combinatorics Probability and Computing*, 8(1–2):7–29, 1999. 246

[13] N. Alon. High girth augmented trees are huge. *J. Combin. Theory Ser. A*, 144:7–15, 2016. 70

[14] N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. *J. Combin. Theory Ser. B*, 87(2):231–243, 2003. 306

[15] N. Alon and Z. Füredi. Covering the cube by affine hyperplanes. *European J. Combin.*, 14(2):79–83, 1993. 247

[16] N. Alon, J. Grytczuk, M. Hałuszczak, and O. Riordan. Nonrepetitive colorings of graphs. *Random Structures Algorithms*, 21(3-4):336–346, 2002. Random structures and algorithms (Poznan, 2001). 212

[17] N. Alon, A. Kostochka, B. Reiniger, D. B. West, and X. Zhu. Coloring, sparseness and girth. *Israel J. Math.*, 214(1):315–331, 2016, arXiv:1412.8002. 70, 72

[18] N. Alon, C. McDiarmid, and B. Reed. Acyclic coloring of graphs. *Random Structures Algorithms*, 2(3):277–288, 1991. 212, 213, 214

[19] N. Alon, M. B. Nathanson, and I. Ruzsa. The polynomial method and restricted sums of congruence classes. *J. Number Theory*, 56(2):404–417, 1996. 246

[20] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992. 20, 39, 69, 162, 246

[21] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977. 139

[22] K. Appel and W. Haken. The four color proof suffices. *Math. Intelligencer*, 8(1):10–20, 58, 1986. 139

[23] K. Appel and W. Haken. *Every planar map is four colorable*, volume 98 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989. With the collaboration of J. Koch. 110, 139

[24] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977. 139

[25] M. F. Aprile. Constructive aspects of lovasz local lemma and applications to graph colouring. Master's thesis, University of Oxford, 2014. 212

[26] N. R. Aravind and C. R. Subramanian. Bounds on vertex colorings with restrictions on the union of color classes. *J. Graph Theory*, 66(3):213–234, 2011. 214

[27] N. R. Aravind and C. R. Subramanian. Forbidden subgraph colorings and the oriented chromatic number. *European J. Combin.*, 34(3):620–631, 2013. 213, 214

[28] S. Arnborg, A. Proskurowski, and D. G. Corneil. Forbidden minors characterization of partial 3-trees. *Discrete Math.*, 80(1):1–19, 1990. 206

[29] A. Asadi, Z. Dvořák, L. Postle, and R. Thomas. Sub-exponentially many 3-colorings of triangle-free planar graphs. *J. Combin. Theory Ser. B*, 103(6):706–712, 2013, arXiv:1007.1430. 69

[30] A. S. Asratian. A note on transformations of edge colorings of bipartite graphs. *J. Combin. Theory Ser. B*, 99(5):814–818, 2009. 107

[31] R. Balakrishnan and X. Zhu. *The Combinatorial Nullstellensatz: With Applications to Graph Coloring*. CRC Press, Boca Raton, 2022. 246, 247, 248

[32] J. Balogh, M. Kochol, A. Pluhár, and X. Yu. Covering planar graphs with forests. *J. Combin. Theory Ser. B*, 94(1):147–158, 2005. 38

[33] T. Bartnicki, J. a. Grytczuk, H. A. Kierstead, and X. Zhu. The map-coloring game. *Amer. Math. Monthly*, 114(9):793–803, 2007. 275

[34] J. Beck. An algorithmic approach to the Lovász local lemma. I. *Random Structures Algorithms*, 2(4):343–365, 1991. 211

[35] A. Bernshteyn. The local cut lemma. *European J. Combin.*, 63:95–114, 2017. 212

[36] A. Bernshteyn. The Johansson–Molloy Theorem for DP-Coloring. *Random Structures & Algorithms*, 54(4):653–664, 2019, arXiv:1708.03843. 214

[37] A. Bernshteyn, T. Brazelton, R. Cao, and A. Kang. Counting colorings of triangle-free graphs. *J. Combin. Theory Ser. B*, 161:86–108, 2023, arXiv:2109.13376. 214

[38] A. Bernshteyn and A. Kostochka. Sharp Dirac's theorem for DP-critical graphs. *J. Graph Theory*, 88(3):521–546, 2018, arXiv:1609.09122. 142

[39] A. Bernshteyn, A. Kostochka, and S. Pron. On DP-coloring of graphs and multigraphs. *Sibirsk. Mat. Zh.*, 58(1):36–47, 2017, arXiv:1609.00763. 141

[40] G. Birkhoff. Three observations on linear algebra. *Univ. Nac. Tucumán. Revista A.*, 5:147–151, 1946. In Spanish. 390

[41] R. Bissacot, R. Fernández, A. Procacci, and B. Scoppola. An improvement of the Lovász local lemma via cluster expansion. *Combin. Probab. Comput.*, 20(5):709–719, 2011, arXiv:0910.1824. 212

[42] H. L. Bodlaender. On the complexity of some coloring games. In *Graph-theoretic concepts in computer science (Berlin, 1990)*, volume 484 of *Lecture Notes in Comput. Sci.*, pages 30–40. Springer, Berlin, 1991. 206, 275

[43] M. Bonamy. Planar graphs with $\Delta \geqslant 8$ are $(\Delta+1)$-edge-choosable. *SIAM J. Discrete Math.*, 29(3):1735–1763, 2015, arXiv:1303.4025. 40

[44] M. Bonamy, N. Bousquet, C. Feghali, and M. Johnson. On a conjecture of Mohar concerning Kempe equivalence of regular graphs. *J. Combin. Theory Ser. B*, 135:179–199, 2019, arXiv:1510.06964. 107

[45] M. Bonamy, D. W. Cranston, and L. Postle. Planar graphs of girth at least five are square $(\Delta+2)$-choosable. *J. Combin. Theory Ser. B*, 134:218–238, 2019, arXiv:1508.03663. 40

[46] M. Bonamy, O. Defrain, T. Klimošová, A. Lagoutte, and J. Narboni. On Vizing's edge colouring question. *J. Combin. Theory Ser. B*, 159:126–139, 2023, arXiv:2107.07900. 107

[47] M. Bonamy, B. Lévêque, and A. Pinlou. Graphs with maximum degree $\Delta \geqslant 17$ and maximum average degree less than 3 are list 2-distance $(\Delta+2)$-colorable. *Discrete Math.*, 317:19–32, 2014, arXiv:1301.7090. 40

[48] J. A. Bondy and U. S. R. Murty. *Graph theory with applications*. American Elsevier Publishing Co., Inc., New York, 1976. 189

[49] O. Borodin. Criterion of chromaticity of a degree prescription. In *Abstracts of IV All-Union Conf. on Th. Cybernetics*, pages 127–128, 1977. 39

[50] O. V. Borodin. A generalization of Kotzig's theorem and prescribed edge coloring of planar graphs. *Mat. Zametki*, 48(6):22–28, 160, 1990. 40, 42, 277

[51] O. V. Borodin. Precise lower bound for the number of edges of minor weight in planar maps. *Math. Slovaca*, 42(2):129–142, 1992. 277

[52] O. V. Borodin. Structural properties of plane graphs without adjacent triangles and an application to 3-colorings. *J. Graph Theory*, 21(2):183–186, 1996. 40

[53] O. V. Borodin, H. J. Broersma, A. Glebov, and J. van den Heuvel. Stars and bunches in planar graphs, Part I: Triangulations. *CDAM Research Report Series*, 2002–04, 2002. Original Russian version. *Diskretn. Anal. Issled. Oper. Ser. 1* **8** (2001), no. 2, 15–39, Available at: http://www.cdam.lse.ac.uk/Reports/Abstracts/cdam-2002-04.html. 39

[54] O. V. Borodin, H. J. Broersma, A. Glebov, and J. van den Heuvel. Stars and bunches in planar graphs, Part II: General planar graphs and colourings. *CDAM Research Report Series*, 2002–05, 2002. Original Russian version. *Diskretn. Anal. Issled. Oper. Ser. 1* **8** (2001), no. 4, 9–33, Available at: http://www.cdam.lse.ac.uk/Reports/Abstracts/cdam-2002-05.html. 39

[55] O. V. Borodin, Z. Dvořák, A. V. Kostochka, B. Lidický, and M. Yancey. Planar 4-critical graphs with four triangles. *European J. Combin.*, 41:138–151, 2014, arXiv:1306.1477. 140, 384, 445

[56] O. V. Borodin, A. N. Glebov, A. O. Ivanova, T. K. Neustroeva, and V. A. Tashkinov. Sufficient conditions for planar graphs to be 2-distance $(\Delta+1)$-colorable. *Sib. Èlektron. Mat. Izv.*, 1:129–141, 2004. 40, 69

[57] O. V. Borodin, A. N. Glebov, T. R. Jensen, and A. Raspaud. Planar graphs without triangles adjacent to cycles of lengths 3 to 9 are 3-colorable. *Sib. Elektron. Mat. Izv.*, 3:428–440, 2006. 70

[58] O. V. Borodin, A. N. Glebov, M. Montassier, and A. Raspaud. Planar graphs without 5- and 7-cycles and without adjacent triangles are 3-colorable. *J. Combin. Theory Ser. B*, 99(4):668–673, 2009. 140

[59] O. V. Borodin, A. N. Glebov, A. Raspaud, and M. R. Salavatipour. Planar graphs without cycles of length from 4 to 7 are 3-colorable. *J. Combin. Theory Ser. B*, 93(2):303–311, 2005. 40, 140

[60] O. V. Borodin, S. G. Hartke, A. O. Ivanova, A. V. Kostochka, and D. B. West. Circular $(5,2)$-coloring of sparse graphs. *Sib. Èlektron. Mat. Izv.*, 5:417–426, 2008. 141

[61] O. V. Borodin and A. O. Ivanova. List 2-distance $(\Delta+2)$-coloring of planar graphs with girth six. *European J. Combin.*, 30(5):1257–1262, 2009. 40

[62] O. V. Borodin, A. O. Ivanova, and T. K. Neustroeva. A prescribed 2-distance $(\Delta+1)$-coloring of planar graphs with a given girth. *Diskretn. Anal. Issled. Oper. Ser. 1*, 14(3):13–30, 2007. 40

[63] O. V. Borodin, S.-J. Kim, A. V. Kostochka, and D. B. West. Homomorphisms from sparse graphs with large girth. *J. Combin. Theory Ser. B*, 90(1):147–159, 2004. Dedicated to Adrian Bondy and U. S. R. Murty. 141

[64] O. V. Borodin, A. Kostochka, and M. Yancey. On 1-improper 2-coloring of sparse graphs. *Discrete Math.*, 313(22):2638–2649, 2013. 385, 386

[65] O. V. Borodin and A. V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *J. of Comb. Theory, B*, 23(2-3):247–250, 1977. 304, 306

[66] O. V. Borodin and A. V. Kostochka. Vertex decompositions of sparse graphs into an independent set and a subgraph of maximum degree at most 1. *Sibirsk. Mat. Zh.*, 52(5):1004–1010, 2011. 384, 445

[67] O. V. Borodin and A. V. Kostochka. Defective 2-colorings of sparse graphs. *J. Combin. Theory Ser. B*, 104:72–80, 2014. 385

[68] O. V. Borodin, A. V. Kostochka, B. Lidický, and M. Yancey. Short proofs of coloring theorems on planar graphs. *European J. Combin.*, 36:314–321, 2014, `arXiv:1211.3981`. 383, 385

[69] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. List edge and list total colourings of multigraphs. *J. Combin. Theory Ser. B*, 71(2):184–204, 1997. 40, 163

[70] O. V. Borodin and A. Raspaud. A sufficient condition for planar graphs to be 3-colorable. *J. Combin. Theory Ser. B*, 88(1):17–27, 2003. 70

[71] B. Bosek, J. Grytczuk, G. Gutowski, O. Serra, and M. Zając. Graph polynomials and group coloring of graphs. *European J. Combin.*, 102:Paper No. 103505, 11, 2022, `arXiv:2012.03230`. 247, 249

[72] P. Bradshaw. Graph colorings with restricted bicolored subgraphs: I. Acyclic, star, and treewidth colorings. *J. Graph Theory*, 100(2):362–370, 2022, `arXiv:2008.13274`. 214

[73] A. Brandt, M. Ferrara, M. Kumbhat, S. Loeb, D. Stolee, and M. Yancey. I,F-partitions of sparse graphs. *European J. Combin.*, 57:1–12, 2016, `arXiv:1510.03381`. 384, 385, 445

[74] R. Brooks. On colouring the nodes of a network. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 37, pages 194–197. Cambridge Univ Press, 1941. 39

[75] L. Cai, W. Wang, and X. Zhu. Choosability of toroidal graphs without short cycles. *J. Graph Theory*, 65(1):1–15, 2010. 246

[76] L. Cai and X. Zhu. Game chromatic index of k-degenerate graphs. *J. Graph Theory*, 36(3):144–155, 2001. 277

[77] S. Cambie, W. Cames van Batenburg, E. Davies, and R. J. Kang. Packing list-colourings. *Random Structures & Algorithms*, 64(1):62–93, 2024, `arXiv:2110.05230`. 162

[78] L. Cao. Total weight choosability of graphs: towards the 1-2-3-conjecture. *J. Combin. Theory Ser. B*, 149:109–146, 2021. 248

[79] P. Catlin. A bound on the chromatic number of a graph. *Discrete Mathematics*, 22(1):81–83, 1978. 304

[80] B. L. Chen, K.-W. Lih, and P.-L. Wu. Equitable coloring and the maximum degree. *European J. Combin.*, 15(5):443–447, 1994. 306

[81] G. Chen, G. Jing, and W. Zang. Proof of the Goldberg–Seymour conjecture on edge-colorings of multigraphs. 2019, `arXiv:1901.10316`. 95, 107

[82] E.-K. Cho, I. Choi, R. Kim, B. Park, T. Shan, and X. Zhu. Decomposing planar graphs into graphs with degree restrictions. *J. Graph Theory*, 101(2):165–181, 2022, `arXiv:2007.01517`. 339

[83] I. Choi, D. W. Cranston, and T. Pierron. Degeneracy and colorings of squares of planar graphs without 4-cycles. *Combinatorica*, 40(5):625–653, 2020, `arXiv:1806.07204`. 40

[84] C.-Y. Chou, W. Wang, and X. Zhu. Relaxed game chromatic number of graphs. *Discrete Math.*, 262(1-3):89–98, 2003. 276

[85] D. Christofides, K. Edwards, and A. D. King. A note on hitting maximum and maximal cliques with a stable set. *J. Graph Theory*, 73(3):354–360, 2013, `arXiv:1109.3092`. 306

[86] M. Chudnovsky, K. Edwards, K.-i. Kawarabayashi, and P. Seymour. Edge-colouring seven-regular planar graphs. *J. Combin. Theory Ser. B*, 115:276–302, 2015, `arXiv:1210.7349`. 108, 191

[87] M. Chudnovsky, K. Edwards, and P. Seymour. Edge-colouring eight-regular planar graphs. *J. Combin. Theory Ser. B*, 115:303–338, 2015, `arXiv:1209.1176`. 108, 191

[88] N. Cohen and F. Havet. Planar graphs with maximum degree $\Delta \geqslant 9$ are $(\Delta+1)$-edge-choosable—a short proof. *Discrete Math.*, 310(21):3049–3051, 2010. Available at: `https://hal.archives-ouvertes.fr/inria-00432389/en/`. 40

[89] V. Cohen-Addad, M. Hebdige, D. Král', Z. Li, and E. Salgado. Steinberg's Conjecture is false. *J. Combin. Theory Ser. B*, 122:452–456, 2017, `arXiv:1604.05108`. 40, 70, 445

[90] M. Conforti, G. Cornuéjols, and G. Zambelli. *Integer programming*, volume 271 of *Graduate Texts in Mathematics*. Springer, Cham, 2014. 414

[91] D. W. Cranston. Coloring, list coloring, and painting squares of graphs (and other related problems). *Electron. J. Combin.*, DS25(Dynamic Surveys):42, 2023, `arXiv:2210.05915`. 246

[92] D. W. Cranston. Bounding clique size in squares of planar graphs. *European J. Combin.*, 120:Paper No. 103960, 2024, `arXiv:2308.09585`. 71

[93] D. W. Cranston, S.-J. Kim, and G. Yu. Injective colorings of sparse graphs. *Discrete Math.*, 310(21):2965–2973, 2010, `arXiv:1007.0786`. 38, 42

[94] D. W. Cranston, S.-J. Kim, and G. Yu. Injective colorings of graphs with low average degree. *Algorithmica*, 60(3):553–568, 2011, `arXiv:1006.3776`. 38, 41

[95] D. W. Cranston and J. Li. Circular flows in planar graphs. *SIAM J. Discrete Math.*, 34(1):497–519, 2020, `arXiv:1812.09833`. 384

[96] D. W. Cranston, J. Li, Z. Wang, and C. Wei. Planar graphs with homomorphisms to the 9-cycle. 2024, `arXiv:2402.02689`. 384

[97] D. W. Cranston, B. Lidický, X. Liu, and A. Shantanam. Planar Turán numbers of cycles: A counterexample. *Electron. J. Combin.*, 29(3):Paper No. 3.31, 16, 2022, `arXiv:2110.02043`. 41

[98] D. W. Cranston and R. Mahmoud. Kempe equivalent list colorings. *Combinatorica*, 44(1):125–153, 2024, `arXiv:2112.07439`. 107

[99] D. W. Cranston and L. Rabern. Brooks' theorem and beyond. *Journal of Graph Theory*, 80(3):199–225, Nov 2015, `arXiv:1403.0479`. 39

[100] D. W. Cranston and L. Rabern. Graphs with $\chi = \Delta$ have big cliques. *SIAM J. Discrete Math.*, 29(4):1792–1814, 2015, `arXiv:1305.3526`. 305

[101] D. W. Cranston and L. Rabern. Painting squares in $\Delta^2 - 1$ shades. *Electron. J. Combin.*, 23(2):Paper 2.50, 30, 2016, `arXiv:1311.1251`. 163, 246

[102] D. W. Cranston and L. Rabern. List-coloring claw-free graphs with $\Delta - 1$ colors. *SIAM J. Discrete Math.*, 31(2):726–748, 2017, arXiv:1508.03574. 163, 246

[103] D. W. Cranston and L. Rabern. Planar graphs are 9/2-colorable. *J. Combin. Theory Ser. B*, 133:32–45, 2018, arXiv:1410.7233. 117, 140, 141

[104] D. W. Cranston and D. B. West. A guide to the discharging method. Extended version of [105]., June 2013, arXiv:1306.4434v1. 38

[105] D. W. Cranston and D. B. West. An introduction to the discharging method via graph coloring. *Discrete Math.*, 340(4):766–793, 2017, arXiv:1306.4434. 38, 41, 430

[106] D. W. Cranston and M. P. Yancey. Sparse graphs are near-bipartite. *SIAM J. Discrete Math.*, 34(3):1725–1768, 2020, arXiv:1903.12570. 385

[107] D. W. Cranston and M. P. Yancey. Vertex partitions into an independent set and a forest with each component small. *SIAM J. Discrete Math.*, 35(3):1769–1791, 2021, arXiv:2006.11445. 384, 385, 386

[108] D. W. Cranston and G. Yu. Cliques in squares of graphs with maximum average degree less than 4. *J. Graph Theory*, to appear, 2023, arXiv:2305.11763. 277

[109] W. Cushing and H. A. Kierstead. Planar graphs are 1-relaxed, 4-choosable. *European J. Combin.*, 31(5):1385–1397, 2010. 247

[110] S. L. Dahlberg, H. Kaul, and J. A. Mudrock. A polynomial method for counting colorings of $s$-labeled graphs. 2023, arXiv:2312.11744. 247

[111] S. L. Dahlberg, H. Kaul, and J. A. Mudrock. An algebraic approach for counting DP-3-colorings of sparse graphs. *European Journal of Combinatorics*, 118:103890, 2024, arXiv:2212.12576. 247

[112] M. DeVos, K.-i. Kawarabayashi, and B. Mohar. Locally planar graphs are 5-choosable. *J. Combin. Theory Ser. B*, 98(6):1215–1232, 2008. 338

[113] M. DeVos, R. Langhede, B. Mohar, and R. Šámal. Many flows in the group connectivity setting. May 2020, arXiv:2005.09767. 191

[114] M. DeVos, J. McDonald, and K. Nurse. Another proof of Seymour's 6-flow theorem. 2023, arXiv:2302.08625. 191

[115] M. DeVos and K. Nurse. A short proof of Seymour's 6-flow theorem, 2023, arXiv:2307.04768. 191

[116] M. DeVos, E. Rollová, and R. Šámal. A new proof of Seymour's 6-flow theorem. *J. Combin. Theory Ser. B*, 122:187–195, 2017, arXiv:1512.06214. 191

[117] R. Diestel. *Graph Theory*. Springer-Verlag, Heidelberg, 4 edition, 2010. 394

[118] G. A. Dirac. Map-colour theorems. *Canadian J. Math.*, 4:480–490, 1952. 38

[119] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc. (3)*, 2:69–81, 1952. 305

[120] M. Dębski, S. Felsner, P. Micek, and F. Schröder. Improved bounds for centered colorings. *Adv. Comb.*, pages Paper No. 8, 28, 2021, arXiv:1907.04586. 213

[121] A. Doyon, G. Hahn, and A. Raspaud. Some bounds on the injective chromatic number of graphs. *Discrete Math.*, 310(3):585–590, 2010. 38

[122] L. Dubois, G. Joret, G. Perarnau, M. Pilipczuk, and F. Pitois. Two lower bounds for p-centered colorings. *Discrete Math. Theor. Comput. Sci.*, 22(4):Paper No. 9, 7, 2020, arXiv:2006.04113. 213

[123] V. Dujmović, G. Joret, J. Kozik, and D. R. Wood. Nonrepetitive colouring via entropy compression. *Combinatorica*, 36(6):661–686, 2016, arXiv:1112.5524. 211, 212

[124] C. Dunn and H. A. Kierstead. A simple competitive graph coloring algorithm. II. *J. Combin. Theory Ser. B*, 90(1):93–106, 2004. Dedicated to Adrian Bondy and U. S. R. Murty. 276

[125] C. Dunn and H. A. Kierstead. A simple competitive graph coloring algorithm. III. *J. Combin. Theory Ser. B*, 92(1):137–150, 2004. 276, 277

[126] L. Duraj, G. Gutowski, and J. Kozik. Chip games and paintability. *Electron. J. Combin.*, 23(3):Paper 3.3, 12, 2016, arXiv:1506.01148. 20

[127] Z. Dvořák. *New Techniques in Coloring Embedded Graphs*. PhD thesis, Univerzita Karlova, 2012. Habilitation thesis, Available at: https://iuuk.mff.cuni.cz/~rakdver/habil.pdf. 140, 340

[128] Z. Dvořák. Clustered coloring of (path + $2k_1$)-free graphs on surfaces. 2023, arXiv:2306.09834. 71

[129] Z. Dvořák. A strengthening and an efficient implementation of Alon-Tarsi list coloring method. *Electron. J. Combin.*, 31(1):Paper No. 1.35, 26, 2024, arXiv:2301.06571. 248

[130] Z. Dvořák, X. Hu, and J.-S. Sereni. A 4-choosable graph that is not $(8:2)$-choosable. *Adv. Comb.*, pages Paper No. 5, 9, 2019, arXiv:1806.03880. 71

[131] Z. Dvořák, K.-i. Kawarabayashi, and D. Kráľ. Packing six T-joins in plane graphs. *J. Combin. Theory Ser. B*, 116:287–305, 2016, arXiv:1009.5912. 108, 191

[132] Z. Dvořák, D. Král, P. Nejedlý, and R. Škrekovski. Coloring squares of planar graphs with girth six. *European J. Combin.*, 29(4):838–849, 2008. Available at: kam.mff.cuni.cz/~kamserie/serie/clanky/2005/s727.ps. 40, 69

[133] Z. Dvořák, D. Král, and R. Thomas. Coloring triangle-free graphs on surfaces. Extended abstract. In *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 120–129. SIAM, Philadelphia, PA, 2009. 140

[134] Z. Dvořák, B. Lidický, and B. Mohar. 5-choosability of graphs with crossings far apart. *J. Combin. Theory Ser. B*, 123:54–96, 2017. 338

[135] Z. Dvořák, B. Lidický, B. Mohar, and L. Postle. 5-list-coloring planar graphs with distant precolored vertices. *J. Combin. Theory Ser. B*, 122:311–352, 2017. 339

[136] Z. Dvořák, B. Lidický, and R. Škrekovski. Graphs with two crossings are 5-choosable. *SIAM J. Discrete Math.*, 25(4):1746–1753, 2011. 338

[137] Z. Dvořák, B. Mohar, and R. Šámal. Exponentially many nowhere-zero $\mathbb{Z}_3$-, $\mathbb{Z}_4$-, and $\mathbb{Z}_6$-flows. *Combinatorica*, 39(6):1237–1253, 2019, arXiv:1708.09579. 191

[138] Z. Dvořák and L. Postle. Density of $5/2$-critical graphs. *Combinatorica*, 37(5):863–886, 2017, arXiv:1411.6668. 384

[139] Z. Dvořák and L. Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory Ser. B*, 129:38–54, 2018, arXiv:1508.03437. 141, 445

[140] Z. Dvořák and L. Postle. Triangle-free planar graphs with at most $64^{n^{0.731}}$ 3-colorings. *J. Combin. Theory Ser. B*, 156:294–298, 2022, arXiv:2108.12669. 69, 72, 445

[141] Z. Dvořák, R. Škrekovski, and T. Valla. Planar graphs of odd-girth at least 9 are homomorphic to the Petersen graph. *SIAM J. Discrete Math.*, 22(2):568–591, 2008. 141

[142] N. Eaton and T. Hull. Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, 25:79–87, 1999. 247

[143] J. Edmonds. Maximum matching and a polyhedron with $0, 1$-vertices. *J. Res. Nat. Bur. Standards Sect. B*, 69B:125–130, 1965. 414

[144] J. Edmonds and D. R. Fulkerson. Transversals and matroid partition. *J. Res. Nat. Bur. Standards Sect. B*, 69B:147–153, 1965. 402

[145] K. Edwards, D. P. Sanders, P. Seymour, and R. Thomas. Three-edge-colouring doublecross cubic graphs. *J. Combin. Theory Ser. B*, 119:66–95, 2016, arXiv:1411.4352. 106, 190

[146] A. Ehrenfeucht, V. Faber, and H. A. Kierstead. A new method of proving theorems on chromatic index. *Discrete Math.*, 52(2-3):159–164, 1984. 106, 108, 419

[147] M. N. Ellingham and L. Goddyn. List edge colourings of some 1-factorable multigraphs. *Combinatorica*, 16(3):343–352, 1996. 247

[148] J. A. Ellis-Monaghan and I. Moffatt, editors. *Handbook of the Tutte Polynomial and Related Topics*. Chapman and Hall/CRC, 2022. 189

[149] R. Entringer. A Short Proof of Rubin's Block Theorem. In B. Alspach and C. Godsil, editors, *Annals of Discrete Mathematics 27 - Cycles in Graphs*, volume 115 of *North-Holland Mathematics Studies*, pages 367–368. North-Holland, 1985. 39

[150] P. Erdős. Some applications of probability of graph theory and combinatorial problems. In *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, pages 133–136. Publ. House Czechoslovak Acad. Sci., Prague, 1964. 305

[151] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. Elsevier, North-Holland, Amsterdam, 1975. 213

[152] P. Erdős, A. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979. 19, 39, 45, 70, 71, 72, 163, 338

[153] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959. 70

[154] J. Erickson. *Algorithms*. 2019. Available at `http://algorithms.wtf`. 387

[155] L. Esperet and A. Parreau. Acyclic edge-coloring using entropy compression. *European J. Combin.*, 34(6):1019–1027, 2013, `arXiv:1206.1535`. 212

[156] I. Fabrici. Light graphs in families of outerplanar graphs. *Discrete Math.*, 307(7-8):866–872, 2007. 277

[157] U. Faigle, U. Kern, H. Kierstead, and W. T. Trotter. On the game chromatic number of some classes of graphs. *Ars Combin.*, 35:143–150, 1993. 275, 276, 277

[158] G. Fan and H. A. Kierstead. Hamiltonian square-paths. *J. Combin. Theory Ser. B*, 67(2):167–182, 1996. 305

[159] C. Feghali, M. Johnson, and D. Paulusma. Kempe equivalence of colourings of cubic graphs. *European J. Combin.*, 59:1–10, 2017, `arXiv:1503.03430`. 107

[160] M. R. Fellows. Transversals of vertex partitions in graphs. *SIAM J. Discrete Math.*, 3(2):206–215, 1990. 304, 306, 391

[161] G. Fertin, A. Raspaud, and B. Reed. Star coloring of graphs. *J. Graph Theory*, 47(3):163–182, 2004. 212

[162] P. M. S. Fialho, B. N. B. de Lima, and A. Procacci. A new bound on the acyclic edge chromatic number. *Discrete Math.*, 343(11):112037, 11, 2020, `arXiv:1912.04436`. 212

[163] S. Fisk. Geometric coloring theory. *Advances in Math.*, 24(3):298–340, 1977. 107

[164] H. Fleischner and M. Stiebitz. A solution to a colouring problem of P. Erdös. *Discrete Math.*, 101(1-3):39–48, 1992. Special volume to mark the centennial of Julius Petersen's "Die Theorie der regulären Graphs", Part II. 247

[165] H. Fleischner and M. Stiebitz. Some remarks on the cycle plus triangles problem. In *The mathematics of Paul Erdős, II*, volume 14 of *Algorithms Combin.*, pages 136–142. Springer, Berlin, 1997. 304

[166] A. Frank. On a theorem of Mader. *Discrete Math.*, 101(1-3):49–57, 1992. Special volume to mark the centennial of Julius Petersen's "Die Theorie der regulären Graphs", Part II. 402, 403

[167] R. Fritsch and G. Fritsch. *The four-color theorem*. Springer-Verlag, New York, 1998. History, topological foundations, and idea of proof, Translated from the 1994 German original by Julie Peschke. 110

[168] D. Gale and L. S. Shapley. College Admissions and the Stability of Marriage. *Amer. Math. Monthly*, 69(1):9–15, 1962. 162, 163

[169] T. Gallai. Kritische Graphen. I. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 8:165–192, 1963. 39

[170] F. Galvin. The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B*, 63(1):153–158, 1995. 162

[171] W. Gao and L. Postle. On the minimal edge density of $K_4$-free 6-critical graphs. 2018, `arXiv:1811.02940`. 385, 412

[172] M. Gardner. Mathematical games. *Scientific American*, 23, April 1981. 275

[173] M. R. Garey and D. S. Johnson. *Computers and intractability*. A Series of Books in the Mathematical Sciences. W. H. Freeman and Co., San Francisco, Calif., 1979. A guide to the theory of NP-completeness. 70

[174] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoret. Comput. Sci.*, 1(3):237–267, 1976. 70, 72, 445

[175] I. Gessel. Tournaments and Vandermonde's determinant. *J. Graph Theory*, 3(3):305–307, 1979. 246

[176] I. Giotis, L. Kirousis, K. I. Psaromiligkos, and D. M. Thilikos. Acyclic edge coloring through the Lovász local lemma. *Theoret. Comput. Sci.*, 665:40–50, 2017, `arXiv:1407.5374`. 212

[177] A. N. Glebov, A. V. Kostochka, and V. A. Tashkinov. Smaller planar triangle-free graphs that are not 3-list-colorable. *Discrete Math.*, 290(2-3):269–274, 2005. 69

[178] M. K. Goldberg. Multigraphs with a chromatic index that is nearly maximal. *Diskret. Analiz*, 23:3–7, 72, 1973. A collection of articles dedicated to the memory of Vitaliĭ Konstantinovič Korobkov. 94

[179] D. Gonçalves, M. Montassier, and A. Pinlou. Acyclic coloring of graphs and entropy compression method. *Discrete Math.*, 343(4):111772, 13, 2020. `https://hal.archives-ouvertes.fr/lirmm-02938618/`. 211, 213

[180] G. Gonthier. Formal proof—the four-color theorem. *Notices of the American Mathematical Society*, 55(11):1382–1393, 2008. 139

[181] R. J. Gould, V. Larsen, and L. Postle. Structure in sparse k-critical graphs. *J. Combin. Theory Ser. B*, 156:194–222, 2022. 412

[182] A. Graf. *Finding Independent Transversals Efficiently*. PhD thesis, University of Waterloo, 2019. 304

[183] A. Graf and P. Haxell. Finding independent transversals efficiently. *Combinatorics, Probability and Computing*, pages 1–27, May 2020, `arXiv:1811.02687`. 304

[184] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, second edition, 1993. 140

[185] H. Grötzsch. Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe*, 8:109–120, 1958/1959. 40, 140

[186] B. Grünbaum. Grötzsch's theorem on 3-colorings. *Michigan Math. J.*, 10:303–310, 1963. 140

[187] J. Grytczuk. Nonrepetitive colorings of graphs—a survey. *Int. J. Math. Math. Sci.*, pages Art. ID 74639, 10, 2007. 212

[188] J. Grytczuk. Nonrepetitive graph coloring. In *Graph theory in Paris*, Trends Math., pages 209–218. Birkhäuser, Basel, 2007. 212

[189] J. Grytczuk, J. Kozik, and P. Micek. New approach to nonrepetitive sequences. *Random Structures Algorithms*, 42(2):214–225, 2013, `arXiv:1103.3809`. 211

[190] J. Grytczuk, J. Przybyło, and X. Zhu. Nonrepetitive list colourings of paths. *Random Structures Algorithms*, 38(1-2):162–173, 2011. 211

[191] J. Grytczuk and X. Zhu. The Alon-Tarsi number of a planar graph minus a matching. *J. Combin. Theory Ser. B*, 145:511–520, 2020, `arXiv:1811.12012`. 248

[192] Y. Gu and X. Zhu. The Alon-Tarsi number of planar graphs revisited. *J. Graph Theory*, 105(3):398–402, 2024, arXiv:2203.16308. 247, 248

[193] D. J. Guan and X. Zhu. Game chromatic number of outerplanar graphs. *J. Graph Theory*, 30(1):67–70, 1999. 276, 339

[194] L. Guth. *Polynomial methods in combinatorics*, volume 64 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2016. 246

[195] A. Hackmann and A. Kemnitz. List edge colorings of outerplanar graphs. *Ars Combin.*, 60:181–185, 2001. 277

[196] A. Hajnal. A theorem on k-saturated graphs. *Canadian Journal of Mathematics*, 17(5):720, 1965. 304

[197] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdös. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 601–623, 1970. 305

[198] M. Hall, Jr. *Combinatorial theory*. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Inc., New York, second edition, 1986. A Wiley-Interscience Publication. 389

[199] P. Hall. On Representatives of Subsets. *J. London Math. Soc.*, 10(1):26–30, 1935. 389

[200] P. R. Halmos and H. E. Vaughan. The marriage problem. *Amer. J. Math.*, 72:214–215, 1950. 389

[201] M. Han, J. Li, Y. Wu, and C.-Q. Zhang. Counterexamples to Jaeger's circular flow conjecture. *J. Combin. Theory Ser. B*, 131:1–11, 2018. 71, 191

[202] J. Harant and S. Jendroľ. On the existence of specific stars in planar graphs. *Graphs Combin.*, 23(5):529–543, 2007. 38

[203] J. Harant and S. Jendroľ. Nonrepetitive vertex colorings of graphs. *Discrete Math.*, 312(2):374–380, 2012. 212

[204] S. G. Hartke, S. Jahanbekam, and B. Thomas. The chromatic number of the square of subcubic planar graphs. 2016, arXiv:1604.06504. 39

[205] F. Havet, J. van den Heuvel, C. McDiarmid, and B. Reed. List colouring squares of planar graphs. In *Proceedings of the 4th European conference on combinatorics, graph theory and applications, EuroComb'07, Seville, Spain, September 11–15, 2007*, pages 515–519. Amsterdam: Elsevier, 2007, arXiv:0807.3233. 39

[206] P. Haxell. A condition for matchability in hypergraphs. *Graphs Combin.*, 11(3):245–248, 1995. 304

[207] P. Haxell. A note on vertex list colouring. *Combinatorics, Probability and Computing*, 10(04):345–347, 2001. 304

[208] P. Haxell. On the strong chromatic number. *Combinatorics, Probability and Computing*, 13(06):857–865, 2004. 304

[209] P. Haxell. An improved bound for the strong chromatic number. *J. Graph Theory*, 58(2):148–158, 2008. 304

[210] P. Haxell. On forming committees. *The American Mathematical Monthly*, 118(9):777–788, 2011. 304

[211] R. A. Hearn and E. D. Demaine. *Games, puzzles, and computation*. A K Peters, Ltd., Wellesley, MA, 2009. 388

[212] P. Heawood. Map-colour theorem. *Quarterly Journal of Mathematics*, 24:332–339, 1890. 38, 106, 139

[213] D. Hefetz. On two generalizations of the Alon-Tarsi polynomial method. *J. Combin. Theory Ser. B*, 101(6):403–414, 2011, arXiv:0911.2099. 230, 247

[214] P. Hell and J. Nešetřil. *Graphs and homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004. 141

[215] P. Hell and K. Seyffarth. Largest planar graphs of diameter two and fixed maximum degree. *Discrete Math.*, 111(1-3):313–322, 1993. Graph theory and combinatorics (Marseille-Luminy, 1990). 71

[216] K. Hendrey and D. R. Wood. Defective and clustered choosability of sparse graphs. *Combin. Probab. Comput.*, 28(5):791–810, 2019, arXiv:1806.07040. 304, 305, 306, 307

[217] A. J. W. Hilton, R. Rado, and S. H. Scott. A ($< 5$)-colour theorem for planar graphs. *Bull. London Math. Soc.*, 5:302–306, 1973. 116

[218] H. Hind, M. Molloy, and B. Reed. Colouring a graph frugally. *Combinatorica*, 17(4):469–482, 1997. 212

[219] J. Hladký, D. Král, and U. Schauz. Brooks' theorem via the Alon-Tarsi theorem. *Discrete Math.*, 310(23):3426–3428, 2010, arXiv:0905.3475. 39, 41, 246

[220] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981. 70, 106, 445

[221] P.-Y. Huang, T.-L. Wong, and X. Zhu. Application of polynomial method to on-line list colouring of graphs. *European J. Combin.*, 33(5):872–883, 2012. 247

[222] E. Hurley and F. Pirot. Colouring locally sparse graphs with the first moment method. 2021, arXiv:2109.15215. 214

[223] R. W. Irving and P. Leather. The complexity of counting stable marriages. *SIAM J. Comput.*, 15(3):655–667, 1986. 162

[224] F. Jaeger. Flows and generalized coloring theorems in graphs. *J. Combin. Theory Ser. B*, 26(2):205–216, 1979. 190, 191

[225] F. Jaeger. Tait's theorem for graphs with crossing number at most one. *Ars Combin.*, 9:283–287, 1980. 190

[226] F. Jaeger. On circular flows in graphs. In *Finite and infinite sets, Vol. I, II (Eger, 1981)*, volume 37 of *Colloq. Math. Soc. János Bolyai*, pages 391–402. North-Holland, Amsterdam, 1984. 141, 190

[227] F. Jaeger. Nowhere-zero flow problems. In *Selected topics in graph theory, 3*, pages 71–95. Academic Press, San Diego, CA, 1988. 192

[228] T. Jensen and B. Toft. *Graph coloring problems*. John Wiley & Sons, 1995. 338

[229] T. R. Jensen and B. Toft. Choosability versus chromaticity—the plane unit distance graph has a 2-chromatic subgraph of infinite list-chromatic number. *Geombinatorics*, 5(2):45–64, 1995. Appendix 1 by Leonid S. Mel′nikov and Vadim G. Vizing and Appendix 2 by Noga Alon. 41

[230] G. Jing. On edge coloring of multigraphs. 2023, arXiv:2308.15588. 107

[231] A. Johansson. The choice number of sparse graphs. Preliminary version, April 1996. 214

[232] T. K. Jonas. *Graph coloring analogues with a condition at distance two: L(2,1)-labellings and list lambda-labellings*. ProQuest LLC, Ann Arbor, MI, 1993. Thesis (Ph.D.)–University of South Carolina. 38

[233] J. Kahn. Asymptotics of the chromatic index for multigraphs. *J. Combin. Theory Ser. B*, 68(2):233–254, 1996. 107

[234] J. Kahn. Asymptotics of the list-chromatic index for multigraphs. *Random Structures Algorithms*, 17(2):117–156, 2000. 108

[235] P. C. Kainen. A generalization of the 5-color theorem. *Proc. Amer. Math. Soc.*, 45:450–453, 1974. 106, 139

[236] M. Kalkowski, M. Karoński, and F. Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. *J. Combin. Theory Ser. B*, 100(3):347–349, 2010. 248

[237] R. N. Karasev and F. V. Petrov. Partitions of nonzero elements of a finite field into pairs. *Israel J. Math.*, 192(1):143–156, 2012, arXiv:1005.1177. 230, 247

[238] A. R. Karlin, S. Oveis Gharan, and R. Weber. A simply exponential upper bound on the maximum number of stable matchings. In *STOC'18—Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 920–925. ACM, New York, 2018, arXiv:1711.01032. 162

[239] H. Kaul and J. A. Mudrock. On the Alon-Tarsi number and chromatic-choosability of Cartesian products of graphs. *Electron. J. Combin.*, 26(1):Paper 1.3, 13, 2019, arXiv:1803.07455. 246

[240] H. Kaul and J. A. Mudrock. Counting packings of list-colorings of graphs. 2024, arXiv:2401.11025. 247

[241] K.-i. Kawarabayashi and C. Thomassen. Decomposing a planar graph of girth 5 into an independent set and a forest. *J. Combin. Theory Ser. B*, 99(4):674–684, 2009. 339

[242] J. B. Kelly and L. M. Kelly. Paths and circuits in critical graphs. *Amer. J. Math.*, 76:786–792, 1954. 69

[243] A. B. Kempe. On the Geographical Problem of the Four Colours. *Amer. J. Math.*, 2(3):193–200, 1879. 106, 139

[244] H. A. Kierstead. On the chromatic index of multigraphs without large triangles. *J. Combin. Theory Ser. B*, 36(2):156–160, 1984. 106, 108

[245] H. A. Kierstead. On the choosability of complete multipartite graphs with part size three. *Discrete Mathematics*, 211(1-3):255–259, 2000. 163

[246] H. A. Kierstead. A simple competitive graph coloring algorithm. *J. Combin. Theory Ser. B*, 78(1):57–68, 2000. 276

[247] H. A. Kierstead and A. V. Kostochka. An Ore-type theorem on equitable coloring. *J. Combin. Theory Ser. B*, 98(1):226–234, 2008. 306

[248] H. A. Kierstead and A. V. Kostochka. A short proof of the Hajnal-Szemerédi theorem on equitable colouring. *Combin. Probab. Comput.*, 17(2):265–270, 2008. 306

[249] H. A. Kierstead and A. V. Kostochka. Ore-type versions of Brooks' theorem. *Journal of Combinatorial Theory, Series B*, 99(2):298–305, 2009. 305, 306

[250] H. A. Kierstead and A. V. Kostochka. Every 4-colorable graph with maximum degree 4 has an equitable 4-coloring. *J. Graph Theory*, 71(1):31–48, 2012. 306

[251] H. A. Kierstead and A. V. Kostochka. Equitable list coloring of graphs with bounded degree. *J. Graph Theory*, 74(3):309–334, 2013. 306

[252] H. A. Kierstead, A. V. Kostochka, M. Mydlarz, and E. Szemerédi. A fast algorithm for equitable coloring. *Combinatorica*, 30(2):217–224, 2010. 306

[253] H. A. Kierstead and J. H. Schmerl. Some applications of Vizing's theorem to vertex colorings of graphs. *Discrete Math.*, 45(2-3):277–285, 1983. 106

[254] H. A. Kierstead and D. Yang. Very asymmetric marking games. *Order*, 22(2):93–107 (2006), 2005. 276

[255] H. A. Kierstead, D. Yang, and J. Yi. On coloring numbers of graph powers. *Discrete Math.*, 343(6):111712, 7, 2020. 276

[256] R. Kim, S.-J. Kim, and X. Zhu. The Alon-Tarsi number of subgraphs of a planar graph. 2019, `arXiv:1906.01506`. 248

[257] S.-J. Kim, A. V. Kostochka, D. B. West, H. Wu, and X. Zhu. Decomposition of sparse graphs into forests and a graph with bounded degree. *J. Graph Theory*, 74(4):369–391, 2013. 386

[258] A. D. King. Hitting all maximum cliques with a stable set using lopsided independent transversals. *J. Graph Theory*, 67(4):300–305, 2011, `arXiv:0911.1741`. 304, 306

[259] A. D. King, L. Lu, and X. Peng. A fractional analogue of Brooks' theorem. *SIAM J. Discrete Math.*, 26(2):452–471, 2012, `arXiv:1103.3524`. 307

[260] L. Kirousis and J. Livieratos. Improved bounds for acyclic coloring parameters. 2022, `arXiv:2202.13846`. 212, 213

[261] S. Klavžar and B. Mohar. The chromatic numbers of graph bundles over cycles. *Discrete Math.*, 138(1-3):301–314, 1995. 14th British Combinatorial Conference (Keele, 1993). 127

[262] W. Klostermeyer and C.-Q. Zhang. $(2+\varepsilon)$-coloring of planar graphs with large odd-girth. *J. Graph Theory*, 33(2):109–119, 2000. 122, 141, 445

[263] D. E. Knuth. *Mariages stables et leurs relations avec d'autres problèmes combinatoires*. Collection "Chaire Aisenstadt". Les Presses de l'Université de Montréal, Montreal, Que., 1976. Introduction à l'analyse mathématique des algorithmes. 162

[264] M. Kochol. Hypothetical complexity of the nowhere-zero 5-flow problem. *J. Graph Theory*, 28(1):1–11, 1998. 190

[265] M. Kochol. An equivalent version of the 3-flow conjecture. *J. Combin. Theory Ser. B*, 83(2):258–261, 2001. 191

[266] K. Kolipaka, M. Szegedy, and Y. Xu. A sharper local lemma with improved applications. In *Approximation, randomization, and combinatorial optimization*, volume 7408 of *Lecture Notes in Comput. Sci.*, pages 603–614. Springer, Heidelberg, 2012. 212

[267] J. Komlós, G. N. Sárközy, and E. Szemerédi. Proof of the Seymour conjecture for large graphs. *Ann. Comb.*, 2(1):43–60, 1998. 305

[268] D. König. über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann.*, 77(4):453–465, 1916. 106

[269] A. Kostochka. Degree, density, and chromatic number. *Metody Diskret. Anal.*, 35:45–70 (in Russian), 1980. 304

[270] A. Kostochka, L. Rabern, and M. Stiebitz. Graphs with chromatic number close to maximum degree. *Discrete Mathematics*, 312(6):1273–1281, 2012. 305, 307

[271] A. Kostochka and M. Yancey. Ore's conjecture for $k = 4$ and Grötzsch's theorem. *Combinatorica*, 34(3):323–329, 2014, `arXiv:1209.1173`. 383

[272] A. Kostochka and M. Yancey. Ore's conjecture on color-critical graphs is almost true. *J. Combin. Theory Ser. B*, 109:73–101, 2014, `arXiv:1209.1050`. 383, 385

[273] A. Kostochka and M. Yancey. A Brooks-type result for sparse critical graphs. *Combinatorica*, 38(4):887–934, 2018, `arXiv:1408.0846`. 383, 412

[274] A. V. Kostochka and J. Nešetřil. Properties of Descartes' construction of triangle-free graphs with high chromatic number. *Combin. Probab. Comput.*, 8(5):467–472, 1999. 69, 70

[275] A. V. Kostochka and B. M. Reiniger. The minimum number of edges in a 4-critical graph that is bipartite plus 3 edges. *European J. Combin.*, 46:89–94, 2015. 385

[276] J. Kozik and B. Podkanowicz. Schnyder Woods and Alon-Tarsi Number of Planar Graphs. *Electron. J. Combin.*, 31(1):Paper No. 1.59, 2024, `arXiv:2303.02683`. 247, 248

[277] J. Kratochvíl, Z. Tuza, and M. Voigt. Brooks-type theorems for choosability with separation. *J. Graph Theory*, 27(1):43–49, 1998. 339

[278] I. Kříž. A hypergraph-free construction of highly chromatic graphs without short cycles. *Combinatorica*, 9(2):227–229, 1989. 70

[279] H. La, B. Lužar, and K. Štorgel. Further extensions of the Grötzsch theorem. *Discrete Math.*, 345(6):Paper No. 112849, 12, 2022, `arXiv:2110.01862`. 384

[280] M. Las Vergnas and H. Meyniel. Kempe classes and the Hadwiger conjecture. *J. Combin. Theory Ser. B*, 31(1):95–104, 1981. 107

[281] M. Lasoń. A generalization of combinatorial Nullstellensatz. *Electron. J. Combin.*, 17(1):Note 32, 6, 2010, `arXiv:1302.4647`. 230, 247

[282] D. Leven and Z. Galil. NP completeness of finding the chromatic index of regular graphs. *J. Algorithms*, 4(1):35–44, 1983. 70, 106, 445

[283] Z. Li, Z. Shao, F. Petrov, and A. Gordeev. The Alon-Tarsi number of a toroidal grid. *European J. Combin.*, 111:Paper No. 103697, 7, 2023, `arXiv:1912.12466`. With a preface by Xuding Zhu. 246

[284] Z. Li, Q. Ye, and Z. Shao. The Alon-Tarsi number of Halin graphs. 2021, `arXiv:2110.11617`. 249

[285] C.-H. Liu and L. Postle. On the minimum edge-density of 4-critical graphs of girth five. *J. Graph Theory*, 86(4):387–405, 2017, `arXiv:1409.5295`. 385, 412

[286] L. Lovász. On decomposition of graphs. *Studia Sci. Math. Hungar.*, 1:237–238, 1966. 304

[287] L. Lovász. On chromatic number of finite set-systems. *Acta Math. Acad. Sci. Hungar.*, 19:59–67, 1968. 70

[288] L. Lovász. Three short proofs in graph theory. *J. Combin. Theory, Ser. B*, 19(3):269–271, 1975. 39

[289] L. Lovász. Graph theory and integer programming. In *Discrete optimization (Proc. Adv. Res. Inst. Discrete Optimization and Systems Appl., Banff, Alta., 1977), I*, volume 4, pages 141–158. Elsevier, 1979. 414

[290] L. M. Lovász. Tutte's flow conjectures. Unpublished essay. Available at: `https://pdfs.semanticscholar.org/8c94/64aaed5a8a8a988dc99989697aa1cddc062f.pdf`, 2012. 191

[291] L. M. Lovász, C. Thomassen, Y. Wu, and C.-Q. Zhang. Nowhere-zero 3-flows and modulo $k$-orientations. *J. Combin. Theory Ser. B*, 103(5):587–598, 2013. 141, 191

[292] H. Lu and X. Zhu. Dense Eulerian graphs are $(1, 3)$-choosable. *Electron. J. Combin.*, 29(2):Paper No. 2.54, 8, 2022, `arXiv:2109.00792`. 248

[293] W. Mader. Existenz $n$-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 37(1), pages 86–97. Springer, 1972. 403

[294] A. Martinsson. A simplified proof of the Johansson-Molloy Theorem using the Rosenfeld counting method. 2021, `arXiv:2111.06214`. 214

[295] J. McDonald. Edge-colourings. In *Topics in chromatic graph theory*, volume 156 of *Encyclopedia Math. Appl.*, pages 94–113. Cambridge Univ. Press, Cambridge, 2015. 107

[296] B. D. McKay. A note on the history of the four-colour conjecture. *J. Graph Theory*, 72(3):361–363, 2013, `arXiv:1201.2852`. 139

[297] L. S. Mel′nikov and V. G. Vizing. New proof of Brooks' theorem. *J. Combinatorial Theory*, 7:289–290, 1969. 108

[298] H. Meyniel. Les 5-colorations d'un graphe planaire forment une classe de commutation unique. *J. Combin. Theory Ser. B*, 24(3):251–257, 1978. 107

[299] M. Michałek. A short proof of Combinatorial Nullstellensatz. *The American Mathematical Monthly*, 117(9):821–823, 2010, `arXiv:0904.4573`. 246, 247

[300] M. Mirzakhani. A small non-4-choosable planar graph. *Bull. Inst. Combin. Appl.*, 17:15–18, 1996. 69, 72, 340, 423, 445

[301] B. Mohar. Akempic triangulations with 4 odd vertices. *Discrete Math.*, 54(1):23–29, 1985. 107

[302] B. Mohar. 7-critical graphs of bounded genus. *Discrete Math.*, 112(1-3):279–281, 1993. 108

[303] B. Mohar. Kempe equivalence of colorings. In *Graph theory in Paris*, Trends Math., pages 287–297. Birkhäuser, Basel, 2007. 107

[304] B. Mohar and C. Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001. 7

[305] M. Molloy. The list chromatic number of graphs with small clique number. *J. Combin. Theory Ser. B*, 134:264–284, 2019, `arXiv:1701.09133`. 214

[306] M. Molloy and B. Reed. Further algorithmic aspects of the local lemma. In *STOC '98 (Dallas, TX)*, pages 524–529. ACM, New York, 1999. 212

[307] M. Molloy and B. Reed. *Graph colouring and the probabilistic method*, volume 23 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2002. 108, 210, 211, 212, 214, 307

[308] M. Molloy and M. R. Salavatipour. A bound on the chromatic number of the square of a planar graph. *J. Combin. Theory Ser. B*, 94(2):189–213, 2005. 39

[309] M. Montassier, P. Ossona de Mendez, A. Raspaud, and X. Zhu. Decomposing a graph into forests. *J. Combin. Theory Ser. B*, 102(1):38–52, 2012. `https://hal.archives-ouvertes.fr/hal-00656800/en/`. 386

[310] B. Moore and E. Smith-Roberge. A density bound for triangle-free 4-critical graphs. *J. Graph Theory*, 103(1):66–111, 2023, `arXiv:2012.01503`. 385

[311] Moore, Benjamin. *Fractional refinements of integral theorems*. PhD thesis, University of Waterloo, 2021. `http://hdl.handle.net/10012/17138`. 412

[312] R. A. Moser. A constructive proof of the Lovász local lemma. In *STOC'09—Proceedings of the 2009 ACM International Symposium on Theory of Computing*, pages 343–350. ACM, New York, 2009. 211

[313] R. A. Moser and G. Tardos. A constructive proof of the general Lovász local lemma. *J. ACM*, 57(2):Art. 11, 15, 2010, `arXiv:0903.0544`. 211

[314] J. A. Mudrock. A Short Proof that the List Packing Number of any Graph is Well Defined. *Discrete Math. To appear.*, 2023, `arXiv:2207.11868`. 162

[315] J. Mycielski. Sur le coloriage des graphs. *Colloq. Math.*, 3:161–162, 1955. 69, 71

[316] J. Narboni. Vizing's edge-recoloring conjecture holds. 2023, `arXiv:2302.12914`. 107

[317] R. Naserasr and R. Škrekovski. The Petersen graph is not 3-edge-colorable—a new proof. *Discrete Math.*, 268(1-3):325–326, 2003. 388

[318] C. S. J. A. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *J. London Math. Soc.*, 36:445–450, 1961. 190, 402

[319] C. S. J. A. Nash-Williams. An application of matroids to graph theory. In *Theory of Graphs*, Intl. Sympos., Rome, pages 263–265. Dunod, 1966. 402

[320] S. Ndreca, A. Procacci, and B. Scoppola. Improved bounds on coloring of graphs. *European J. Combin.*, 33(4):592–609, 2012, `arXiv:1005.1875`. 212, 213

[321] J. Nešetřil and P. Ossona de Mendez. Grad and classes with bounded expansion. I. Decompositions. *European J. Combin.*, 29(3):760–776, 2008, `arXiv:math/0508323`. 213

[322] J. Nešetřil and P. Ossona de Mendez. *Sparsity*, volume 28 of *Algorithms and Combinatorics*. Springer, Heidelberg, 2012. Graphs, structures, and algorithms. 213

[323] J. Nešetřil and V. Rödl. A short proof of the existence of highly chromatic hypergraphs without short cycles. *J. Combin. Theory Ser. B*, 27(2):225–227, 1979. 70

[324] T. Nishizeki and K. Kashiwagi. On the 1.1 edge-coloring of multigraphs. *SIAM J. Discrete Math.*, 3(3):391–410, 1990. 107

[325] E. A. Nordhaus and J. W. Gaddum. On complementary graphs. *Amer. Math. Monthly*, 63:175–177, 1956. 41

[326] U. of Waterloo. Department of Combinatorics, Optimization, B. Guenin, and U. of Waterloo. Faculty of Mathematics. *Packing T-joins and Edge Colouring in Planar Graphs*. Research report (University of Waterloo. Faculty of Mathematics). Faculty of Mathematics, University of Waterloo, 2003. 108, 191

[327] C. Palmer and D. Pálvölgyi. At most $3.55^n$ stable matchings. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science—FOCS 2021*, pages 217–227. IEEE Computer Soc., Los Alamitos, CA, 2022, `arXiv:2011.00915`. 162

[328] J. Petersen. Sur le théoréme de tait. *L'Intermédiare des Mathématiciens*, 5:225–227, 1898. 388

[329] D. Peterson and D. R. Woodall. Edge-choosability in line-perfect multigraphs. *Discrete Math.*, 202(1-3):191–199, 1999. 150, 162

[330] D. Peterson and D. R. Woodall. Erratum: "Edge-choosability in line-perfect multigraphs" [Discrete Math. **202** (1999) no. 1-3, 191–199; MR1694489 (2000a:05086)]. *Discrete Math.*, 260(1-3):323–326, 2003. 150, 162

[331] F. Petrov. General parity result and cycle-plus-triangles graphs. *J. Graph Theory*, 85(4):803–807, 2017, arXiv:1512.06205. 247, 249

[332] L. Postle. On the minimum edge density of 5-critical triangle-free graphs. *Elec. Notes in Disc. Math.*, 49:667–673, 2015. 385

[333] L. Postle. On the minimum number of edges in triangle-free 5-critical graphs. *European J. Combin.*, 66:264–280, 2017, arXiv:1602.03098. 385

[334] L. Postle. Characterizing 4-critical graphs with Ore-degree at most seven. *J. Combin. Theory Ser. B*, 129:107–147, 2018, arXiv:1409.5116. 385

[335] L. Postle and E. Smith-Roberge. On the density of $C_7$-critical graphs. *Combinatorica*, 42(2):253–300, 2022, arXiv:1903.04453. 384, 386

[336] L. Postle and R. Thomas. Five-list-coloring graphs on surfaces I. Two lists of size two in planar graphs. *J. Combin. Theory Ser. B*, 111:234–241, 2015. 339

[337] L. Postle and R. Thomas. Five-list-coloring graphs on surfaces II. A linear bound for critical graphs in a disk. *J. Combin. Theory Ser. B*, 119:42–65, 2016. 339

[338] L. Rabern. On hitting all maximum cliques with an independent set. *Journal of Graph Theory*, 66(1):32–37, 2011, arXiv:0907.3705. 304

[339] L. Rabern. $\Delta$-critical graphs with small high vertex cliques. *J. Combin. Theory Ser. B*, 102(1):126–130, 2012, arXiv:1102.1023. 290, 305

[340] L. Rabern. *Coloring graphs from almost maximum degree sized palettes*. PhD thesis, Arizona State University, 2013. 305

[341] B. Reed and B. Sudakov. List colouring when the chromatic number is close to the order of the graph. *Combinatorica*, 25(1):117–123, 2004. 163

[342] M. Richardson. Solutions of irreflexive relations. *Ann. of Math. (2)*, 58:573–590; errata 60 (1954), 595, 1953. 162

[343] G. Ringel and J. W. T. Youngs. Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. U.S.A.*, 60:438–445, 1968. 8, 38

[344] N. Robertson, D. Sanders, P. Seymour, and R. Thomas. The four-colour theorem. *J. Combin. Theory Ser. B*, 70(1):2–44, 1997. 86, 139

[345] N. Robertson, P. Seymour, and R. Thomas. Excluded minors in cubic graphs. *J. Combin. Theory Ser. B*, 138:219–285, 2019, arXiv:1403.2118. 106, 190

[346] N. Robertson, P. D. Seymour, and R. Thomas. Tutte's edge-colouring conjecture. *J. Combin. Theory Ser. B*, 70(1):166–183, 1997. 106, 190

[347] N. Robertson, P. D. Seymour, and R. Thomas. Cyclically five-connected cubic graphs. *J. Combin. Theory Ser. B*, 125:132–167, 2017. 106, 190

[348] M. Rosenfeld. Another approach to non-repetitive colorings of graphs of bounded degree. *Electron. J. Combin.*, 27(3):Paper No. 3.43, 16, 2020, arXiv:2006.09094. 211, 212

[349] M. Rosenfeld. Ann wins the nonrepetitive game over four letters and the erase-repetition game over six letters. *European Journal of Combinatorics*, 118:103924, 2024, arXiv:2107.14022. 211

[350] D. Sanders and R. Thomas. Edge three-coloring cubic apex graphs. in preparation. 106

[351] D. P. Sanders and Y. Zhao. A note on the three color problem. *Graphs Combin.*, 11(1):91–94, 1995. 40

[352] D. P. Sanders and Y. Zhao. Planar graphs of maximum degree seven are class I. *J. Combin. Theory Ser. B*, 83(2):201–212, 2001. 106

[353] U. Schauz. Algebraically solvable problems: describing polynomials as equivalent to explicit solutions. *Electron. J. Combin.*, 15(1):Research Paper 10, 35, 2008. 230, 247

[354] U. Schauz. Mr. Paint and Mrs. Correct. *Electron. J. Combin.*, 16(1):Research Paper 77, 18, 2009. 39, 340

[355] U. Schauz. A paintability version of the Combinatorial Nullstellensatz, and list colorings of $k$-partite $k$-uniform hypergraphs. *Electron. J. Combin.*, 17:R176, 2010. 40, 247

[356] E. R. Scheinerman and D. H. Ullman. *Fractional graph theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1997. A rational approach to the theory of graphs, With a foreword by Claude Berge, A Wiley-Interscience Publication. 108, 115, 140

[357] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Paths, flows, matchings, Chapters 1–38. 106, 190

[358] J.-S. Sereni and J. Volec. A note on acyclic vertex-colorings. *J. Comb.*, 7(4):725–737, 2016, `arXiv:1312.5600`. 213

[359] P. D. Seymour. Problem section in combinatorics. In T. P. McDonough and V. C. Mavron, editors, *The Proceedings of the British Combinatorial Conference held in the University College of Wales, Aberystwyth, 2–6 July 1973*, London Mathematical Society Lecture Note Series, No. 13, pages v+204. Cambridge University Press, London-New York, 1974. 305

[360] P. D. Seymour. On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc. (3)*, 38(3):423–460, 1979. 94, 108

[361] P. D. Seymour. Nowhere-zero 6-flows. *J. Combin. Theory Ser. B*, 30(2):130–135, 1981. 191

[362] P. D. Seymour. On Tutte's extension of the four-colour problem. *J. Combin. Theory Ser. B*, 31(1):82–94, 1981. 108

[363] T. Slivnik. Short proof of Galvin's theorem on the list-chromatic index of a bipartite multigraph. *Combin. Probab. Comput.*, 5(1):91–94, 1996. 163

[364] C. A. B. Smith and S. Abbott. The story of Blanches Descartes. *The Mathematical Gazette*, 87(508):23–33, 2003. 69

[365] J. Spencer. Asymptotic lower bounds for Ramsey functions. *Discrete Math.*, 20(1):69–76, 1977/78. 213

[366] J. P. Steinberger. An unavoidable set of D-reducible configurations. *Trans. Amer. Math. Soc.*, 362(12):6633–6661, 2010, `arXiv:0905.0043`. 107, 140

[367] M. Stiebitz, D. Scheide, B. Toft, and L. M. Favrholdt. *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*, volume 75. Wiley, 2012. 106, 107

[368] T. Szabó and G. Tardos. Extremal problems for transversals in graphs with bounded degree. *Combinatorica*, 26(3):333–351, 2006. 304

[369] G. Szekeres and H. S. Wilf. An inequality for the chromatic number of a graph. *J. Combinatorial Theory*, 4:1–3, 1968. 38

[370] P. Tait. On the colouring of maps. *Proceedings of the Royal Society of Edinburgh, Section A*, 2(10):501–503, 1878. 139

[371] T. Tao. Moser's entropy compression argument. `https://terrytao.wordpress.com/2009/08/05/mosers-entropy-compression-argument/`. Accessed: 2022-06-21. 211

[372] V. A. Tashkinov. On an algorithm for the edge coloring of multigraphs. *Diskretn. Anal. Issled. Oper. Ser. 1*, 7(3):72–85, 100, 2000. 107

[373] R. Thomas. An update on the four-color theorem. *Notices Amer. Math. Soc.*, 45(7):848–859, 1998. 139, 140

[374] R. Thomas and B. Walls. Three-coloring Klein bottle graphs of girth five. *J. Combin. Theory Ser. B*, 92(1):115–135, 2004. 385

[375] C. Thomassen. The Jordan-Schönflies theorem and the classification of surfaces. *Amer. Math. Monthly*, 99(2):116–130, 1992. 7

[376] C. Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62(1):180–181, 1994. 39, 338, 339

[377] C. Thomassen. Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory Ser. B*, 62(2):268–279, 1994. 140

[378] C. Thomassen. 3-list-coloring planar graphs of girth 5. *J. Combin. Theory Ser. B*, 64(1):101–107, 1995. 69, 339

[379] C. Thomassen. Color-critical graphs on a fixed surface. *J. Combin. Theory Ser. B*, 70(1):67–100, 1997. 339

[380] C. Thomassen. A short list color proof of Grötzsch's theorem. *J. Combin. Theory Ser. B*, 88(1):189–192, 2003. 69, 247, 249, 339

[381] C. Thomassen. Exponentially many 5-list-colorings of planar graphs. *J. Combin. Theory Ser. B*, 97(4):571–583, 2007. 339

[382] C. Thomassen. Many 3-colorings of triangle-free planar graphs. *J. Combin. Theory Ser. B*, 97(3):334–349, 2007. 69, 140, 247

[383] C. Thomassen. The weak 3-flow conjecture and the weak circular flow conjecture. *J. Combin. Theory Ser. B*, 102(2):521–529, 2012. 191

[384] C. Thomassen. The square of a planar cubic graph is 7-colorable. *J. Combin. Theory Ser. B*, 128:192–218, 2018. 39

[385] C. Thomassen. Exponentially many 3-colorings of planar triangle-free graphs with no short separating cycles. *J. Combin. Theory Ser. B*, 158(part 1):301–312, 2023. 69

[386] A. Thue. Über unendliche zeichenreihen. *Norske vid. Selsk. Skr. Mat. Nat. Kl.*, 7:1–22, 1906. 214

[387] E. G. Thurber. Concerning the maximum number of stable matchings in the stable marriage problem. *Discrete Math.*, 248(1-3):195–219, 2002. 162

[388] L. E. Trotter, Jr. Line perfect graphs. *Math. Programming*, 12(2):255–259, 1977. 162

[389] W. T. Tutte. On the imbedding of linear graphs in surfaces. *Proc. London Math. Soc. (2)*, 51:474–483, 1949. 189, 192

[390] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, 6:80–91, 1954. 189, 392

[391] W. T. Tutte. On the problem of decomposing a graph into $n$ connected factors. *J. London Math. Soc.*, 36:221–230, 1961. 190, 402

[392] W. T. Tutte. On the algebraic theory of graph colorings. *J. Combinatorial Theory*, 1:15–50, 1966. 106, 189

[393] W. T. Tutte. A geometrical version of the four color problem. (With discussion). In *Combinatorial Mathematics and its Applications (Proc. Conf., Univ. North Carolina, Chapel Hill, N.C., 1967)*, pages 553–560. Univ. North Carolina Press, Chapel Hill, N.C., 1969. 190

[394] P. Ungar and B. Descartes. Advanced Problems and Solutions: Solutions: 4526. *Amer. Math. Monthly*, 61(5):352–353, 1954. 69

[395] J. van den Heuvel. The complexity of change. In *Surveys in combinatorics 2013*, volume 409 of *London Math. Soc. Lecture Note Ser.*, pages 127–160. Cambridge Univ. Press, Cambridge, 2013. 108

[396] J. van den Heuvel and H. A. Kierstead. Uniform orderings for generalized coloring numbers. *European J. Combin.*, 91:Paper No. 103214, 13, 2021. 276

[397] J. van den Heuvel and S. McGuinness. Coloring the square of a planar graph. *J. Graph Theory*, 42(2):110–124, 2003. Available at: `http://www.cdam.lse.ac.uk/Reports/Abstracts/cdam-99-06.html`. 38

[398] V. G. Vizing. On an estimate of the chromatic class of a $p$-graph. *Diskret. Analiz No.*, 3:25–30, 1964. 40, 106

[399] V. G. Vizing. The chromatic class of a multigraph. *Kibernetika (Kiev)*, 1965(3):29–39, 1965. 40, 89, 107, 108

[400] V. G. Vizing. Critical graphs with given chromatic class. *Diskret. Analiz No.*, 5:9–17, 1965. 106, 108

[401] V. G. Vizing. Some unsolved problems in graph theory. *Uspehi Mat. Nauk*, 23(6 (144)):117–134, 1968. 89

[402] V. G. Vizing. Vertex coloring with given colors. *Metody Diskretn. Anal.*, 29:3–10 (in Russian), 1976. 39, 338

[403] M. Voigt. List colourings of planar graphs. *Discrete Math.*, 120(1-3):215–219, 1993. 39, 69, 338

[404] M. Voigt. A not 3-choosable planar graph without 3-cycles. *Discrete Math.*, 146(1-3):325–328, 1995. 69, 445

[405] J. von Neumann. A certain zero-sum two-person game equivalent to the optimal assignment problem. In *Contributions to the theory of games, vol. 2*, Ann. of Math. Stud., no. 28, pages 5–12. Princeton Univ. Press, Princeton, NJ, 1953. 390

[406] R. Škrekovski. A Grötzsch-type theorem for list colourings with impropriety one. *Combin. Probab. Comput.*, 8(5):493–507, 1999. 247

[407] R. Škrekovski. List improper colourings of planar graphs. *Combin. Probab. Comput.*, 8(3):293–299, 1999. 247

[408] K. Wagner. über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114(1):570–590, 1937. 140

[409] W.-F. Wang and K.-W. Lih. Labeling planar graphs with conditions on girth and distance two. *SIAM J. Discrete Math.*, 17(2):264–275, 2003. 40

[410] I. M. Wanless and D. R. Wood. A general framework for hypergraph colouring. *SIAM J. Disc. Math*, 36(3), 2022, `arXiv:2008.00775`. 211, 212, 213

[411] G. Wegner. Graphs with given diameter and a coloring problem. *Technical Report, University of Dortmund*, 1977. 38

[412] D. B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996. 190

[413] A. Wigderson. *Mathematics and computation*. Princeton University Press, Princeton, NJ, [2019] ©2019. A theory revolutionizing technology and science. 388

[414] Wikipedia. Birkhoff algorithm — Wikipedia, the free encyclopedia, 2024. Online; accessed 11-May-2024, `https://en.wikipedia.org/wiki/Birkhoff_algorithm`. 390

[415] R. Wilson. *Four colors suffice*. Princeton Science Library. Princeton University Press, Princeton, NJ, 2014. How the map problem was solved, Revised color edition of the 2002 original, with a new foreword by Ian Stewart. 110, 140

[416] D. R. Wood. Defective and clustered graph colouring. *Electron. J. Combin.*, 1:Dynamic Survey 23, 2018, `arXiv:1803.07694`. 276, 305, 385

[417] J. Wu and X. Zhu. Lower bounds for the game colouring number of partial k-trees and planar graphs. *Discrete Math.*, 308(12):2637–2642, 2008. 258, 276

[418] B. Xu. On 3-colorable plane graphs without 5- and 7-cycles. *J. Combin. Theory Ser. B*, 96(6):958–963, 2006. 140

[419] R. Xu and X. Zhu. Decomposition of triangle-free planar graphs. 2022, `arXiv:2207.09659`. 248

[420] M. Yancey. Partition of sparse graphs into two forests with bounded degree. 2024, `arXiv:2403.05387`. 385

[421] D. Yang. Coloring games on squares of graphs. *Discrete Math.*, 312(8):1400–1406, 2012. 276

[422] D. Yang and X. Zhu. Game colouring directed graphs. *Electron. J. Combin.*, 17(1):Research Paper 11, 19, 2010. 276

[423] D. Yang and X. Zhu. Strong chromatic index of sparse graphs. *J. Graph Theory*, 83(4):334–339, 2016. 276

[424] D. H. Younger. Integer flows. *J. Graph Theory*, 7(3):349–357, 1983. 189

[425] D. A. Youngs. 4-chromatic projective graphs. *J. Graph Theory*, 21(2):219–227, 1996. 127

[426] M. Zając. A short proof of Brooks' theorem. May 2018, arXiv:1805.11176. 39, 41

[427] C.-Q. Zhang. *Integer flows and cycle covers of graphs*, volume 205 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1997. 191

[428] C.-Q. Zhang. Circular flows of nearly Eulerian graphs and vertex-splitting. *J. Graph Theory*, 40(3):147–161, 2002. 141

[429] L. Zhang. Every planar graph with maximum degree 7 is of class 1. *Graphs Combin.*, 16(4):467–495, 2000. 106

[430] X. Zhu. The game coloring number of planar graphs. *J. Combin. Theory Ser. B*, 75(2):245–258, 1999. 276

[431] X. Zhu. The game coloring number of pseudo partial $k$-trees. *Discrete Math.*, 215(1-3):245–262, 2000. 276, 277

[432] X. Zhu. Circular chromatic number: a survey. *Discrete Math.*, 229(1-3):371–410, 2001. 141

[433] X. Zhu. Circular chromatic number of planar graphs of large odd girth. *Electron. J. Combin.*, 8(1):Research Paper 25, 11, 2001. 141

[434] X. Zhu. Refined activation strategy for the marking game. *J. Combin. Theory Ser. B*, 98(1):1–18, 2008. 276

[435] X. Zhu. On-line list colouring of graphs. *Electron. J. Combin.*, 16(1):R127, 2009. 39

[436] X. Zhu. The Alon-Tarsi number of planar graphs. *J. Combin. Theory Ser. B*, 134:354–358, 2019, arXiv:1711.10817. 247

[437] X. Zhu. Every nice graph is $(1,5)$-choosable. *J. Combin. Theory Ser. B*, 157:524–551, 2022, arXiv:2104.05410. 248

[438] X. Zhu. List 4-colouring of planar graphs. *J. Combin. Theory Ser. B*, 162:1–12, 2023. 339

[439] A. A. Zykov. On some properties of linear complexes. *Mat. Sbornik N.S.*, 24(66):163–188, 1949. 69, 71

# Image Credits

All of the images in this book were drawn in Ti*k*Z by the author. Most of them have not appeared previously in any publication (except possibly one by the author). However, some are redrawn versions of images that have appeared elsewhere. These are listed below.

- Figure 2.4 is based on a figure that appeared in [404].
- Figures 2.5–2.7 are based on figures that appeared in [89].
- Figure 2.8 is based on figures that appeared in [140].
- Figures 2.9–2.11 and Figures 2.13–2.18 are based on figures that appeared in [282], some of which were in turn based on figures that appeared in [220].
- Figures 2.19–2.22 are based on figures that appear in the forthcoming volume "The Art of Combinatorics", by Douglas B. West.
- Figure 2.23 is based on a figure that appeared in [300].
- Figure 2.24 is based on a figure that appeared in [174].
- Figure 4.10 is based on a figure that appeared in [262].
- Figures 4.11 and 4.17 are based on figures that appeared in [139].
- Figure 12.8 is based on a figure that appeared in [66].
- Figure 12.11 is based on a figure that appeared in [55].
- Figure 12.40 is based on a figure that appeared in [73].

# Acknowledgments

Each of the following people provided feedback on various parts of the book, or made a key contribution at the right time: Peter Bradshaw, Beth Cranston, Stephen Hartke, Erica King, Jiaao Li, Ben Moore, Kacey Nelson, Shubhanshu Prasad, Greg Puleo, Ed Scheinerman, Evelyne Smith-Roberge, Benny Sudakov, and Matt Yancey.

Reem Mahmoud read the whole book, and offered extensive and invaluable feedback. Richard Hammack gave constructive critique on early versions of figures, which shaped the visual style that I ultimately adopted for the numerous images throughout the book. The expository styles in the books of Douglas West, Jeff Erickson, and Mike Molloy and Bruce Reed have all heavily influenced mine here. Landon Rabern taught me more of the math in this book than anyone else. Finally, thanks to God for putting it on my heart to write this book. I was initially reluctant, but I'm grateful that I eventually did.

I greatly benefited from the hospitality of Johns Hopkins University and Howard Community College, where I wrote the first chapters during a sabbatical, as well as support from Virginia Commonwealth University, both the Department of Mathematics and the Department of Computer Science.

The `arXiv` simplified the process of tracking down many of the articles on which these chapters are based, and the community at `latex.stackexchange.com` provided numerous insights, solutions, and hacks (typically without me even asking). Don Knuth, Leslie Lamport, and Till Tantau, creators of TeX, LaTeX, and Ti*k*Z continue to impress me with their software's elegance, versatility, and power. Thanks to the developers and maintainers of Vim, without which I would be lost.

Finally, I offer my heartfelt thanks to the graph coloring community, without whose research efforts I would have nothing to write. I feel deeply blessed to work in such a flourishing area, and to continuously partake of the creative fruits of so many other graph theorists. Thank you!

# Index