Coloring a claw-free graph with Δ-1 colors

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Joint with Landon Rabern
Slides available on my webpage

George Mason CAGS
15 February 2013

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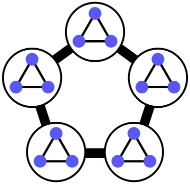

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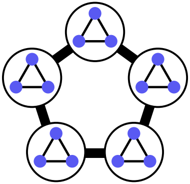

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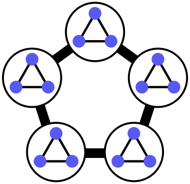

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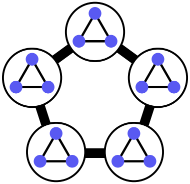

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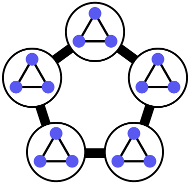

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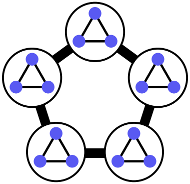

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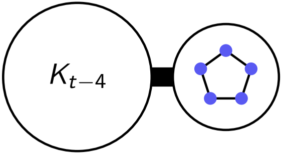

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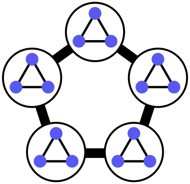

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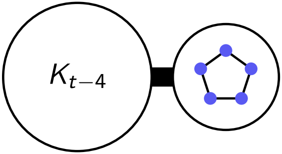

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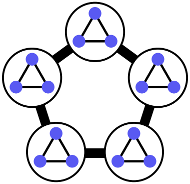

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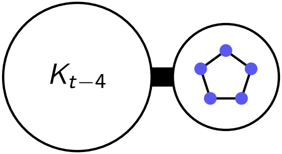

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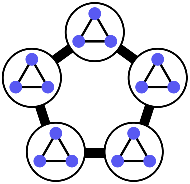

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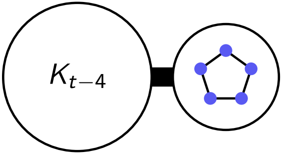

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Thm [C.-Rabern '13+, today]**:** B-K is true for claw-free graphs.

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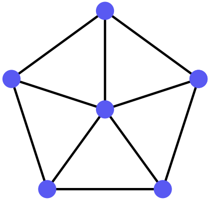

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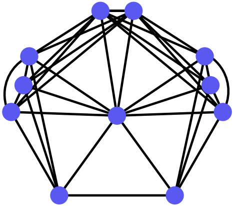

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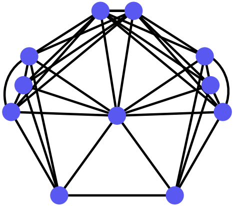

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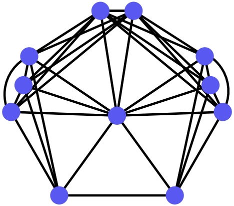

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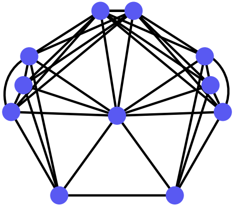

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Key Idea: No d_1-choosable graph can appear as an induced subgraph in a minimal counterexample to B-K Conj.

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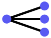
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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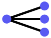
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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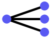
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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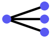
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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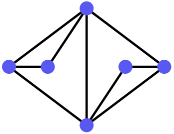

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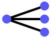
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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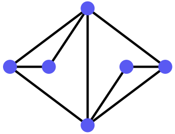

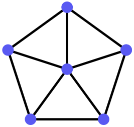

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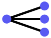
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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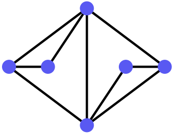

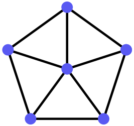

First Step: B-K Conj. is true for quasi-line graphs.

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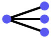
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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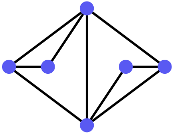
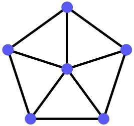
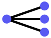




First Step: B-K Conj. is true for quasi-line graphs.

Key Lemma (Second Step): If G is claw-free, but not quasi-line, and G is a minimal counterexample to the B-K Conjecture, then G contains a vertex v such that $N(v)$ is a thickening of C_5.

Main Result

Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

Def: A quasi-line graph is one in which for each vertex v we can cover $N(v)$ with two cliques.

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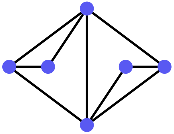

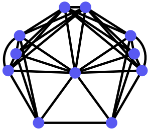

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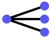
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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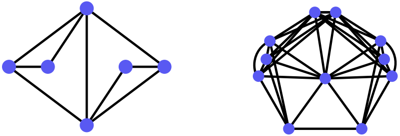

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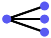
Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

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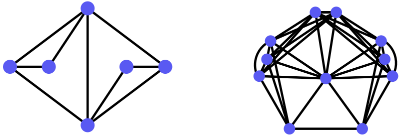

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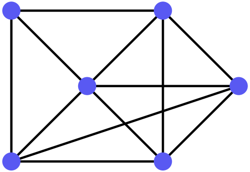
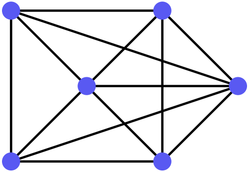
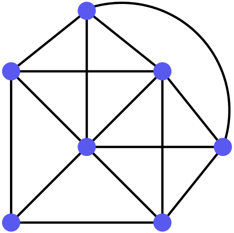
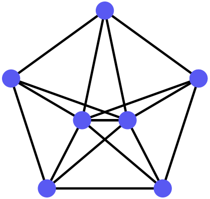
Gallery of d_1-choosble graphs









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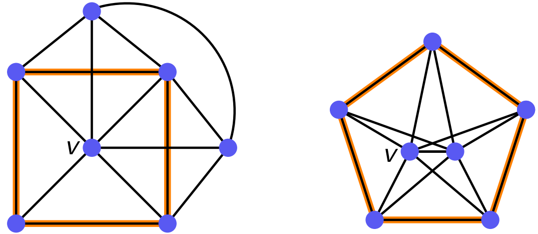

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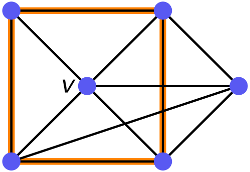
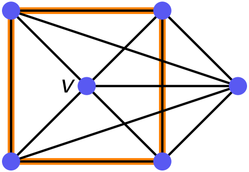
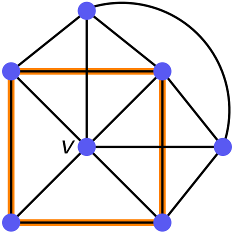


D_6



$C_4 \vee K_2$



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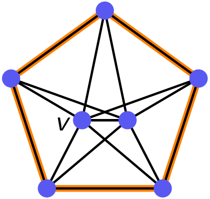

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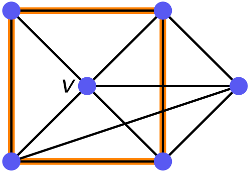

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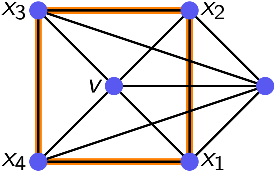

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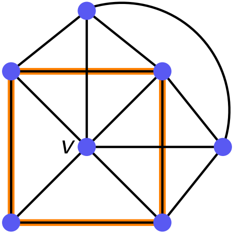

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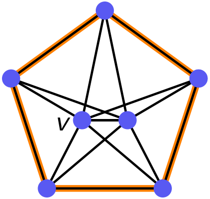

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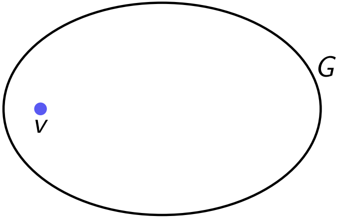

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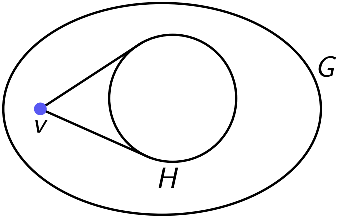

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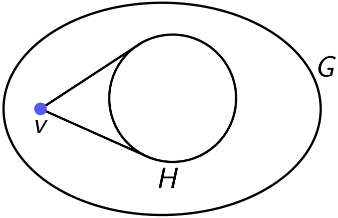

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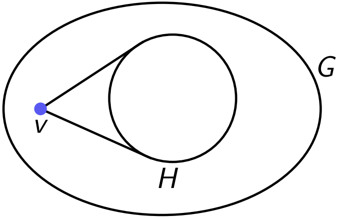

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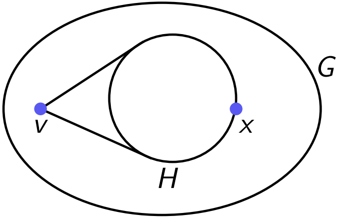

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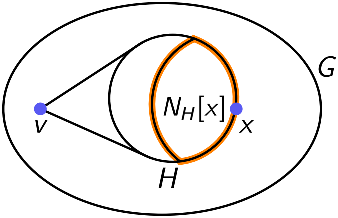

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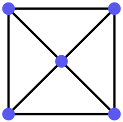

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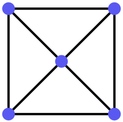

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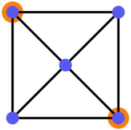

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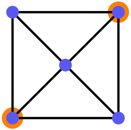

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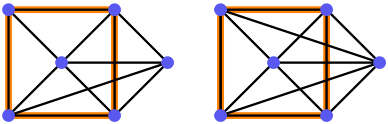

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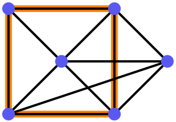

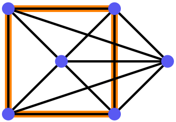

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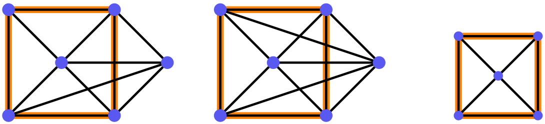

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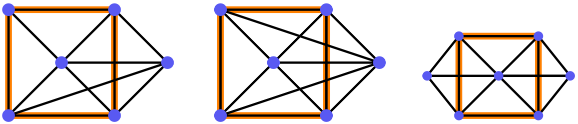

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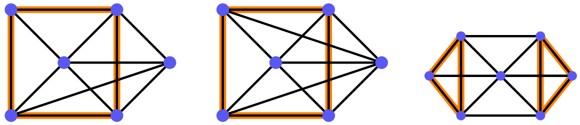

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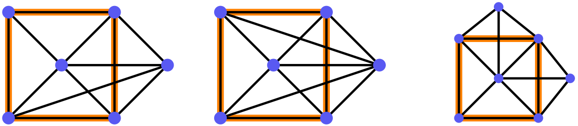

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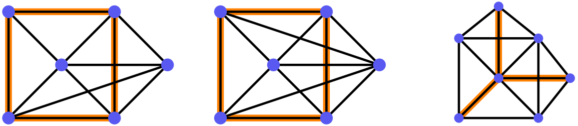

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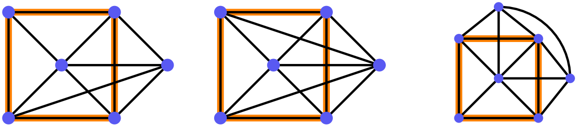

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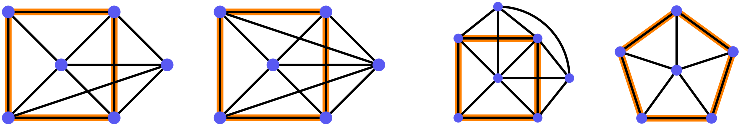

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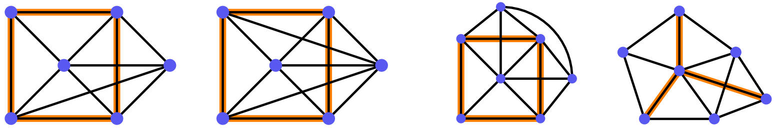

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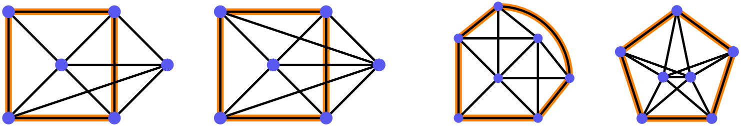

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So H contains a C_5. Each other neighbor y of v must be adj. to at least 3 succesive verts on the C_5 or we get a claw. If y is adj. to 4 or 5 cycle verts, we get a d_1-choosable subgraph.

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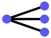
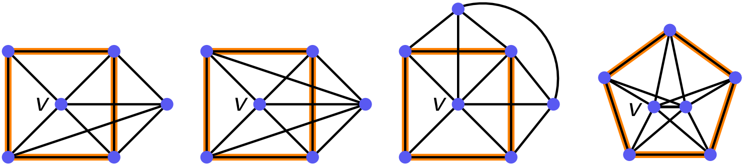
Summary

Main Thm: The B-K Conj. is true for claw-free graphs, i.e., if G has no induced , $\Delta \geq 9$, and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.

Key Idea: A minimal counterexample to B-K Conjecture cannot contain a d_1-choosable graph as an induced subgraph.

- **First Step:** B-K Conj. is true for quasi-line graphs.
- **Key Lemma:** If G is claw-free, but not quasi-line, and G is a minimal counterexample to the B-K Conjecture, then G contains a vertex v such that $N(v)$ is a thickening of C_5.



- **Final Step:** Since G is claw-free, nbrs of verts in thickening attach in a structured way, so we get a d_1-choosable subgraph.