Conjectures equivalent to the Borodin-Kostochka Conjecture: Coloring a graph with Δ-1 colors

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Joint with Landon Rabern

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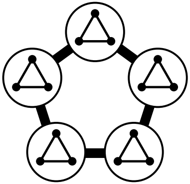

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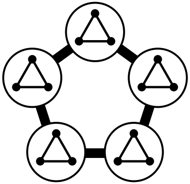

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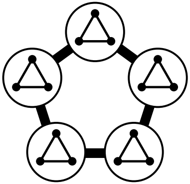

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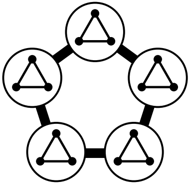

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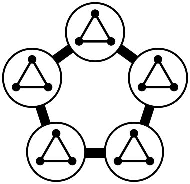

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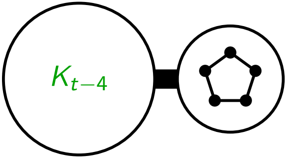

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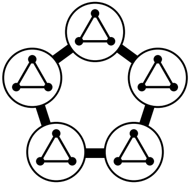

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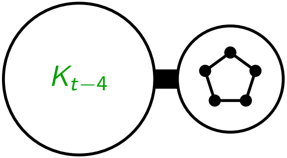

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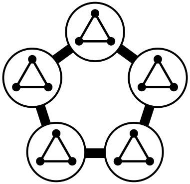

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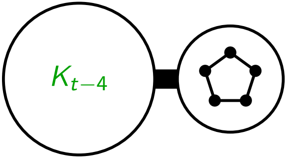

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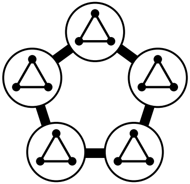

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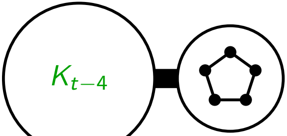

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$\chi(G) = (t-4)+3 = t-1$

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Small Pot Lemma (SPL):
For any graph G and function f (such that $f(v) < |G|$ for all $v \in G$), to verify that G is f-choosable, it suffices to consider all list assignments L such that $|L(v)| = f(v)$ and $|\cup_{v \in G} L(v)| < |G|$.

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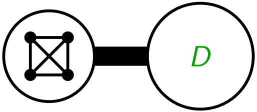

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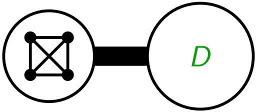

Lemma 1: If G has a $(\Delta - 1)$-clique K,
then each vertex outside K has at most 1 neighbor in K.

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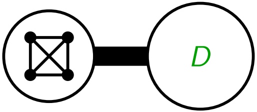

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Main Result

B-K Conj: Every graph with $\chi(G) = \Delta(G) \geq 9$ has a K_Δ.

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Main Thm: Borodin-Kostochka Conjecture is equivalent to ours.

Pf sketch that $K_4 * E_{\Delta-4}$ implies K_Δ:
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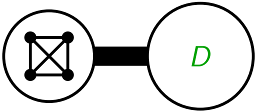

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We contradict either Lemma 1 or Lemma 2.

A key ingredient

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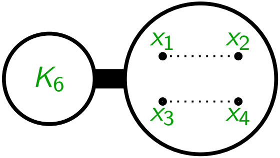

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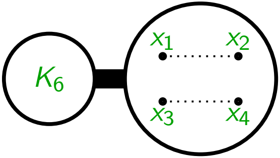

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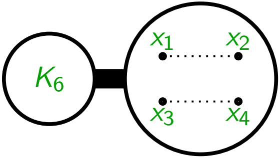

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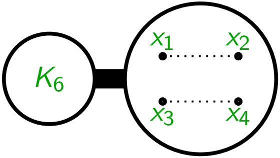

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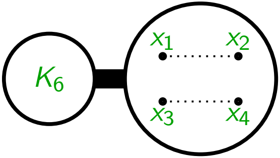

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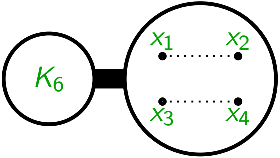

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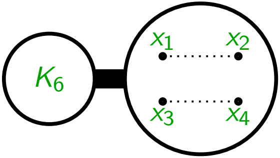

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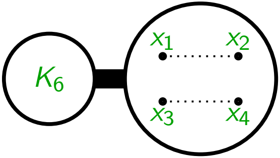

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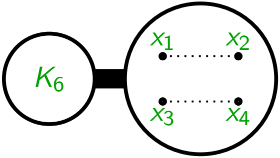

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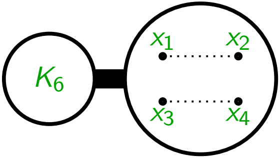

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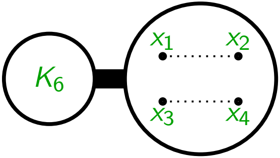

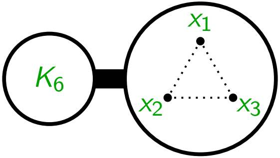

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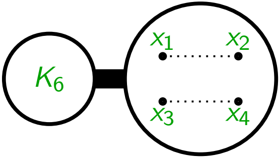

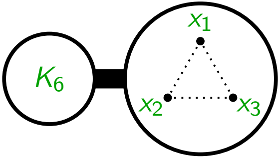

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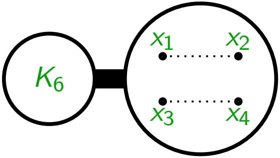

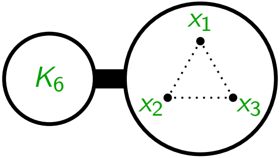

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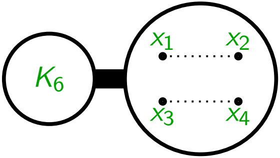

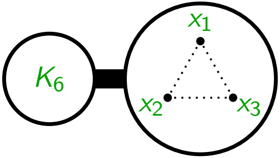

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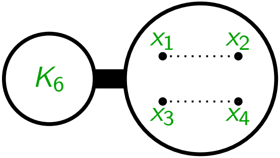

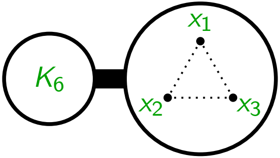

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Main Result (recap)

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Our Conj: Every graph with $\chi(G) = \Delta(G) \geq 9$ has a $K_3 * E_{\Delta-3}$.

Main Thm: Borodin-Kostochka Conjecture is equivalent to ours.

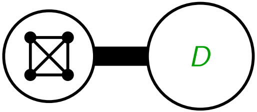
Main Result (recap)

B-K Conj: Every graph with $\chi(G) = \Delta(G) \geq 9$ has a K_Δ.

Our Conj: Every graph with $\chi(G) = \Delta(G) \geq 9$ has a $K_3 * E_{\Delta-3}$.

Main Thm: Borodin-Kostochka Conjecture is equivalent to ours.

Pf sketch that $K_4 * E_{\Delta-4}$ implies K_Δ:
Suppose not. Choose c/e G to minimize $|G|$.
Note that G has no d_1-choosable induced subgraph.



Lemma 1: If G has a $(\Delta - 1)$-clique K,
then each vertex outside K has at most 1 neighbor in K.

Lemma 2: If $|D| \geq 5$,
then $K_4 * D$ is d_1-choosable unless D is almost complete.

We contradict either Lemma 1 or Lemma 2.