Reconfiguration of Colorings and List Colorings

Daniel W. Cranston
dcransto@gmail.com

William & Mary
28 March 2025
Mathematics Colloquium

What is Reconfiguration?

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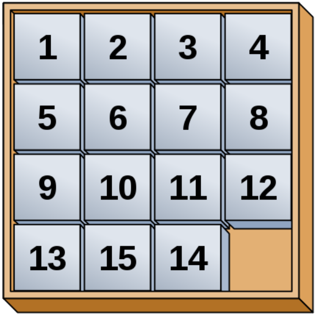

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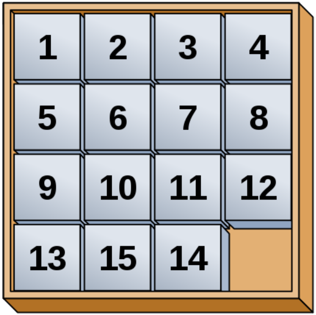

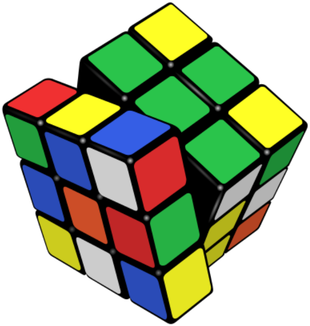

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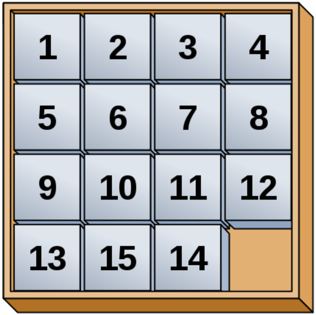

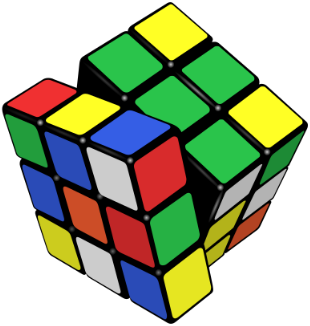

Image credit: Wikipedia

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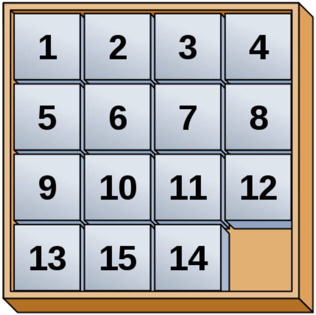

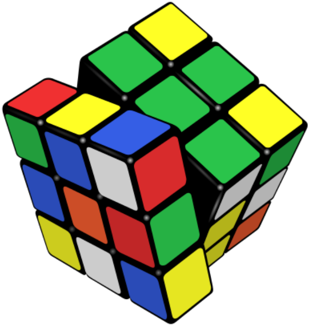

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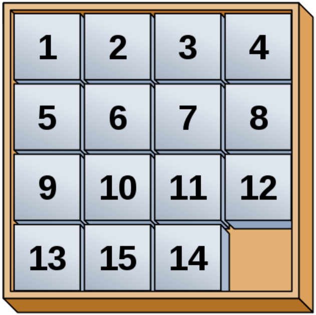

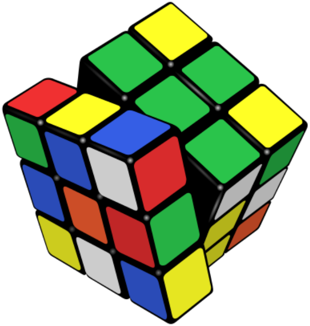

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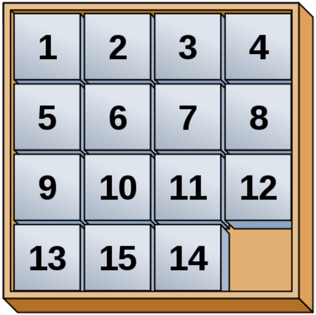

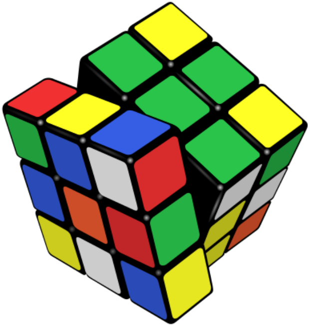

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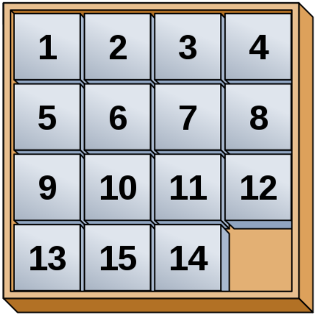

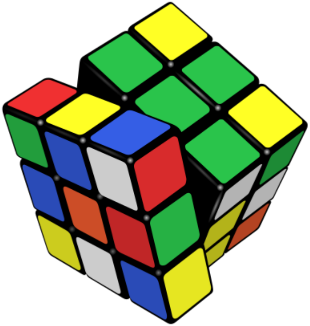

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- If so, how many moves do you need?

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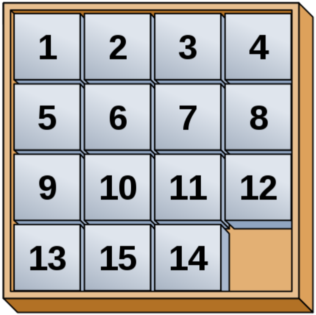

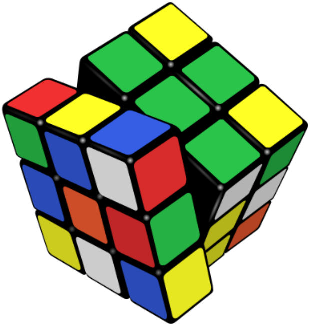

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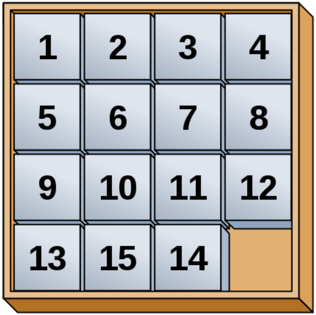

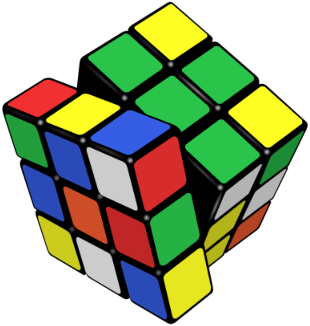

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Move from one instance to another by a sequence of small steps?

- Is it always possible?
- If so, how many moves do you need?
- Can you quickly find a short sequence from one to another?
- Can you quickly sample from all instances (nearly) uniformly?

What is Coloring Reconfiguration?

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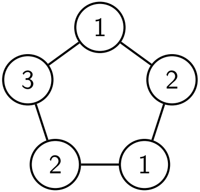

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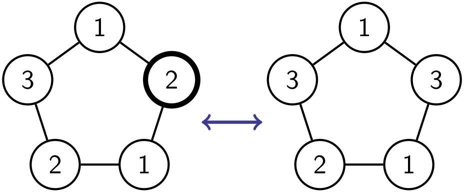

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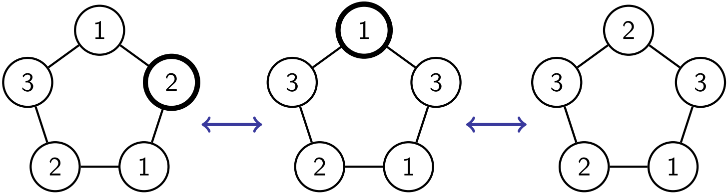

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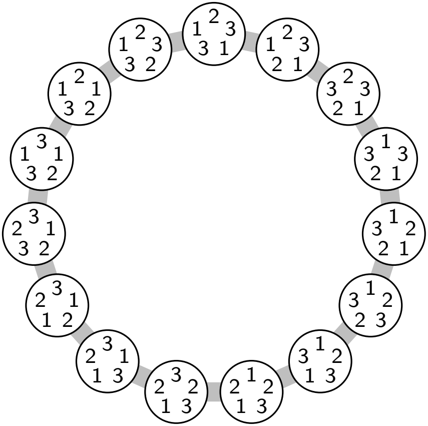

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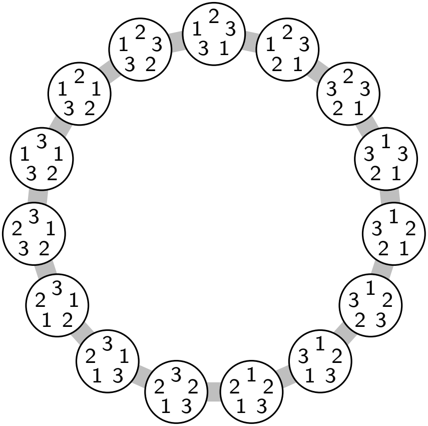

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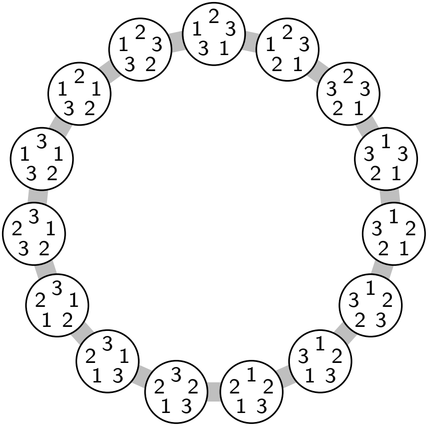

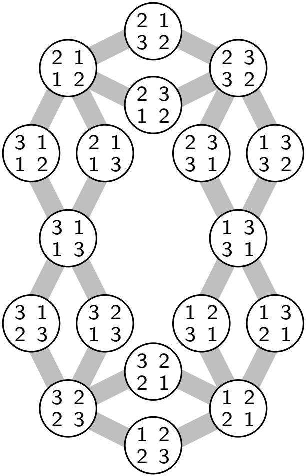

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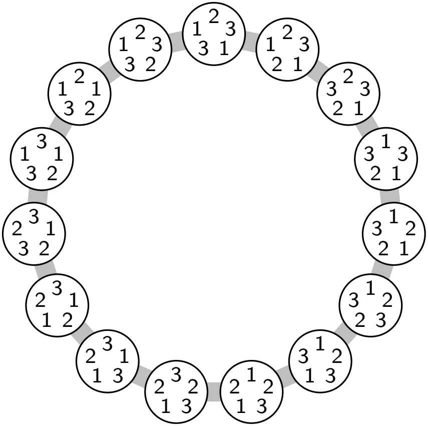

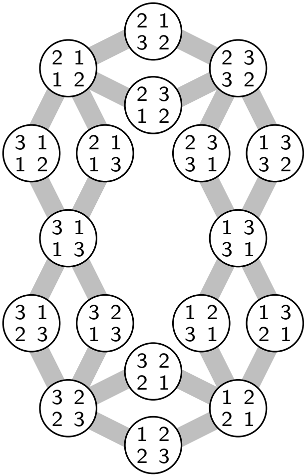

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- “Reconfiguration graphs” of 3-colorings of 5-cycle and 4-cycle.

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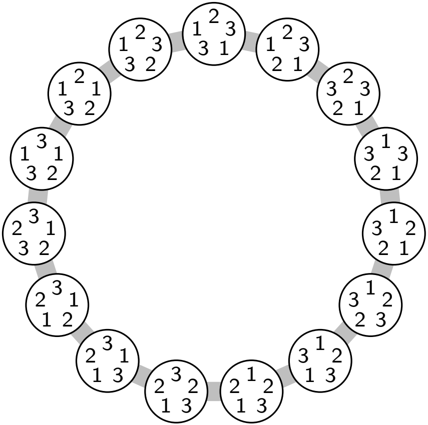

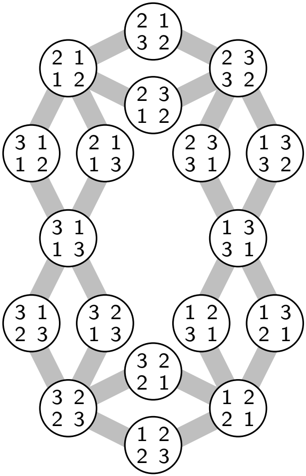

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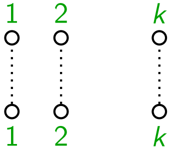
- “Reconfiguration graphs” of 3-colorings of 5-cycle and 4-cycle.
- Can ask all the same questions from the previous page.

Enlightening Examples



$G_k := K_{k,k} - kK_2$

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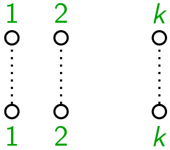

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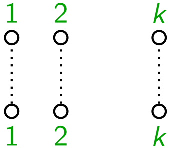

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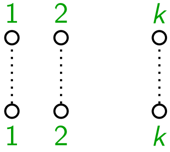

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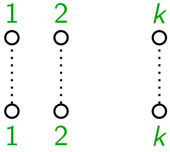

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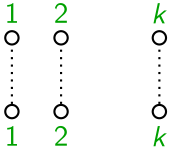

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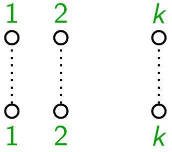

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Bounds on $\text{diam}(\mathcal{C}_k(G))$?

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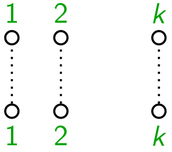

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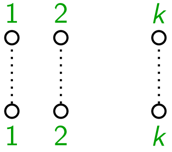

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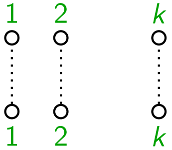

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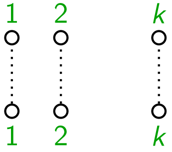

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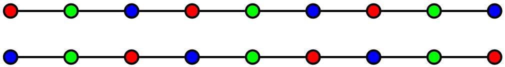

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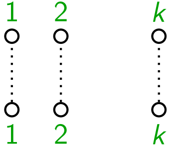

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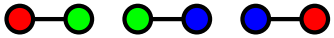
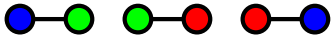
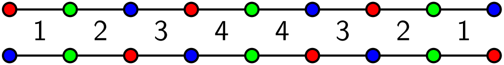
How about "nice" graphs? '+':  '−': 



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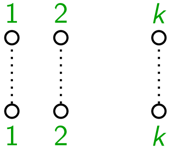

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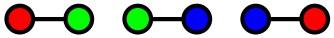
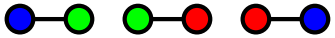
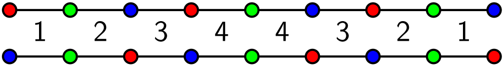
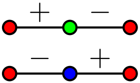
How about "nice" graphs? '+':  '−': 





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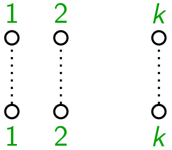

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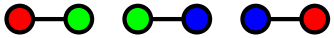
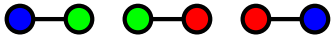
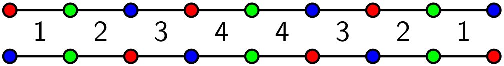
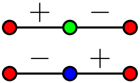
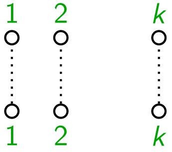
How about “nice” graphs? ‘+’:  ‘−’: 





- So $\text{diam}(\mathcal{C}_3(P_n)) = \Theta(n^2)$

Enlightening Examples

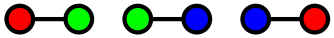
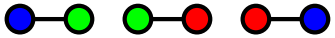


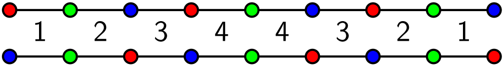
$G_k := K_{k,k} - kK_2$

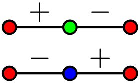
- $\mathcal{C}_k(G_k)$ is disconnected (frozen colorings)
- $\mathcal{C}_\ell(G_k)$ is connected when $3 \le \ell < k$.
  Pf: Each part has color repeated, only used on that part. So can reach 2-coloring.
- $\mathcal{C}_\ell(G_k)$ connected when $\ell > k$ (later)

Bounds on $\mathrm{diam}(\mathcal{C}_k(G))$?

- Trivial upper bound: $k^{|G|}$
- Non-trivial lower bound: $2^{\Theta(\sqrt{|G|})}$ (with 4 colors)
  - Encode an n-bit counter with $\Theta(n^2)$ vertices

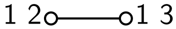
How about "nice" graphs? '+':  '−': 





- So $\mathrm{diam}(\mathcal{C}_3(P_n)) = \Theta(n^2)$ and $\mathrm{diam}(\mathcal{C}_k(P_n \vee K_{k-3})) = \Theta(n^2)$

What is List Coloring Reconfiguration?



- list-assignment L: each vertex v gets allowable colors $L(v)$

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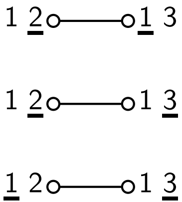

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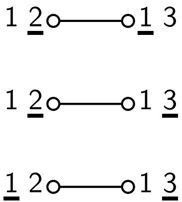

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- Given L-colorings α and β, can we change α to β by recoloring single vertices, keeping L-coloring at each step?

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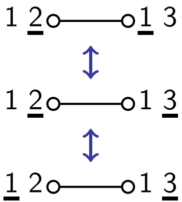

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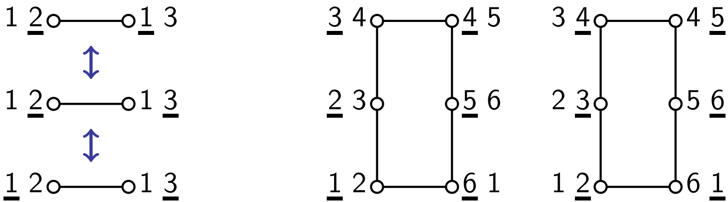

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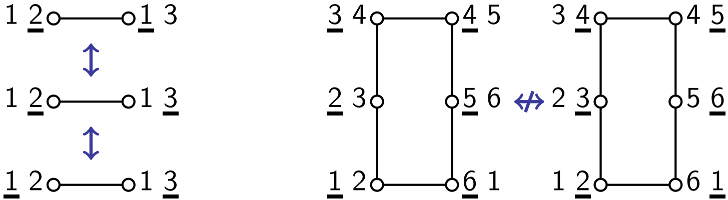

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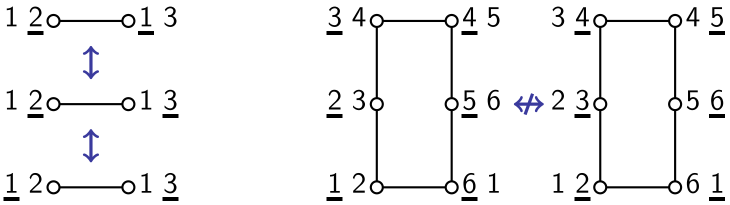

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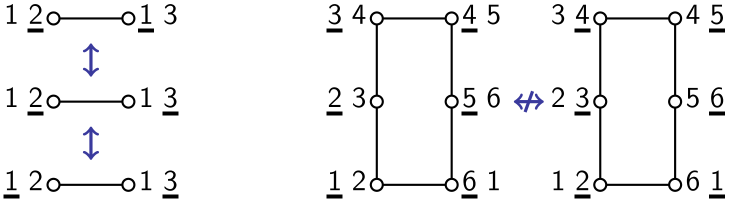

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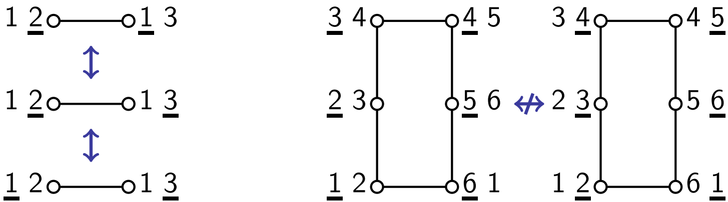

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- If so, how many steps are needed?
- Given list-assignment L, can we transform every L-coloring α into every L-coloring β?
- If so, how many steps are needed in the worst case?

Degenerate Graphs: Conjectures and Tools

Cereceda's Conj: For each $d \in \mathbb{Z}^+$, $\exists$ constant C_d s.t. if G is d-degenerate and $k \geq d+2$, then $\text{diam}(\mathcal{C}_k(G)) \leq C_d|G|^2$.

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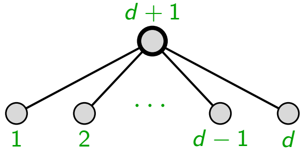

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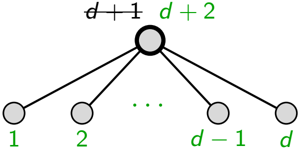

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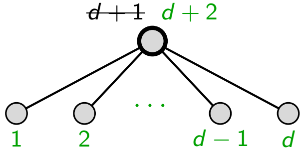

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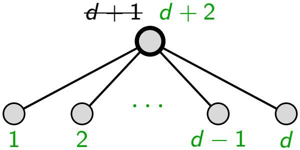

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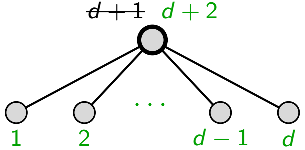

$c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, \dots$

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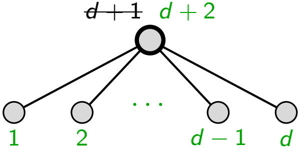

$$\downarrow$$
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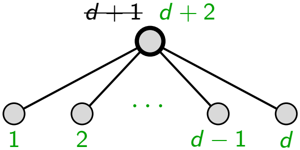

$$\begin{array}{cccccccccc} & & & \downarrow & & & & \downarrow & \\ c_1, & c_2, & c_3, & c_4, & c_5, & c_6, & c_7, & c_8, & \dots \end{array}$$

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Thm:[B-P] Fix $k > d$. For all $\epsilon > 0$, $\exists\ C_{d,\epsilon}$ s.t. if $|L(v)| \geq k$ for all v and $\text{mad}(G) \leq d - \epsilon$, then $\text{diam}(\mathcal{C}_L(G)) = O(n^{C_{d,\epsilon}})$.

Degenerate Graphs: Applications of Key Lemma

Key Lem: Fix G, L, v, and L-colorings α and β. Let $G' := G - v$, $\alpha' := \alpha_{\restriction G'}$, $\beta' := \beta_{\restriction G'}$. If we can transform α' to β' only recoloring $N(v)$ at most s times, then we can tranform α to β only recoloring v at most $\lceil \frac{s}{|L(v)|-d(v)-1} \rceil + 1$ times. **Pf:** Above, more carefully.

Thm:[Cereceda] If G is d-degenerate and $|L(v)| \geq 2d+1$ for all v, then $\mathrm{diam}(\mathcal{C}_L(G)) \leq \binom{n+1}{2}$.
Pf: By induction, v_i recolored $\leq i$ times. By IH, true for $G - v_n$. Extend, recoloring v_n at most $\lceil \frac{d(n-1)}{2d+1-d-1} \rceil + 1 = n$ times.

Thm:[Bousquet-Perarnau] If G is d-degenerate and $|L(v)| \geq 2d+2$ for all v, then $\mathrm{diam}(\mathcal{C}_L(G)) \leq (d+1)n$.
Pf: By induction, v_i recolored $\leq d+1$ times. By IH, for $G - v_n$. Extend, recoloring v_n at most $\lceil \frac{d(d+1)}{2d+2-d-1} \rceil + 1 = d+1$ times.

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Pf: Like above, but delete many vertices at once.

Coloring Reconfig: Best Bounds for Degenerate Graphs

Thm:[Bousquet–Heinrich] Fix $\epsilon \in (0,1)$. There exist C_1, C_2, C_ϵ s.t. for all $d, k \in \mathbb{Z}^+$, if G is d-degenerate, then:

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Thm:[Bartier–Bousquet–Feghali–Heinrich–Moore–Pierron]
If G is planar with girth 5, then $\mathcal{C}_4(G)$ is connected.

Avoiding Frozen Colorings

Prop: Every graph G has list assignment L with $|L(v)| = d(v) + 1$ for all v and L-colorings α and β s.t. we cannot reach β from α.

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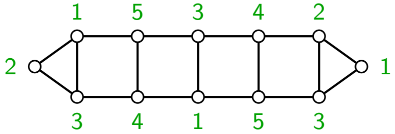

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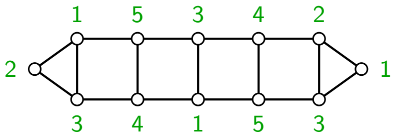

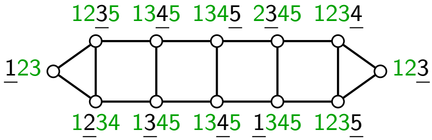

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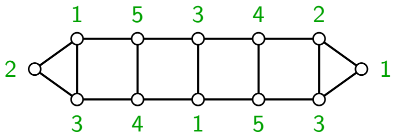

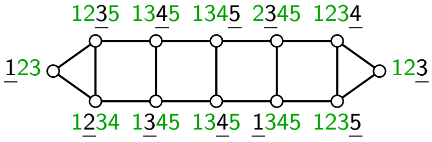

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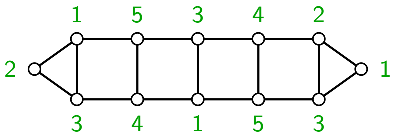

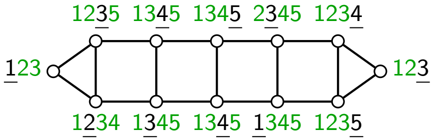

Prop: Fix an acyclic orientation D of a graph G and a list assignment L for G. If $|L(v)| \geq d_D(v) + 2$ for all $v \in V(G)$, then every two L-colorings α and β can reach each other by single vertex recolorings.

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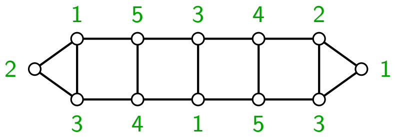

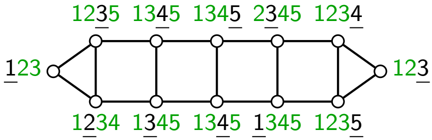

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Pf sketch: Induction on $|V(G)|$.

“Brooks’ Theorem” for List Coloring Reconfiguration

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List Coloring
(Static)

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	List Coloring (Static)	List Coloring Reconfiguration

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Refined Greedy

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Better Bound (Brooks’)		

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Thm: [Cambie–Cames van Batenburg–C.–Kang–van den Heuvel, Feghali–Johnson–Paulusma]

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Thm: [Cambie–Cames van Batenburg–C.–Kang–van den Heuvel, Feghali–Johnson–Paulusma] Let G be connected with $\Delta \geq 3$, and with list-assignment L s.t. $|L(v)| \geq d(v) + 1$ for all v.

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Thm: Let a graph G be 3-connected and regular. If α and β are unfrozen $(\Delta + 1)$-colorings of G, then α can reach β.

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Bonus Lem: Say G is 3-connected with $|L(v)| \geq d(v) + 1$ for all v. Let x_1, x_2 be at distance 2. If α and β are L-colorings with $\alpha(x_1) = \alpha(x_2) = \beta(x_1) = \beta(x_2)$, then α can reach β.

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Pf: Now a common neighbor y of x_1, x_2 effectively has an extra color.

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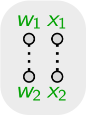
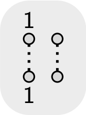
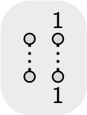




α



β

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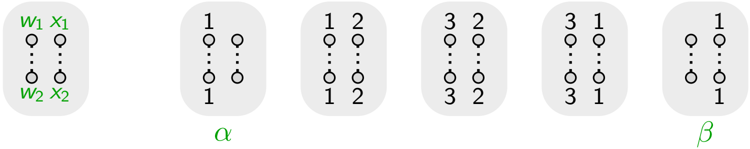

"Brooks' Theorem" for List Coloring Reconfiguration

Thm: Let a graph G be 3-connected and regular. If α and β are unfrozen $(\Delta + 1)$-colorings of G, then α can reach β.

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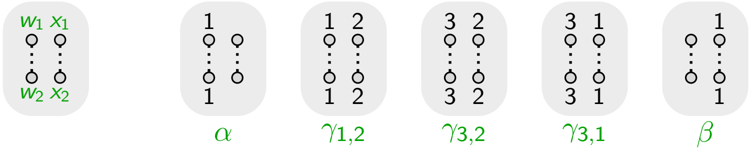

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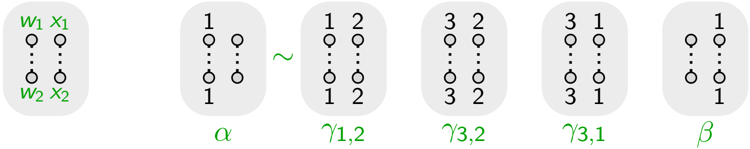

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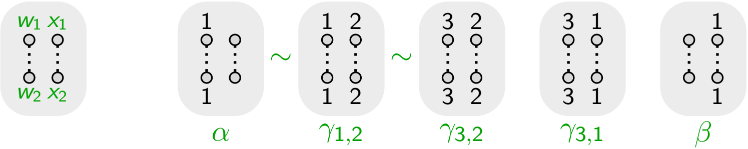

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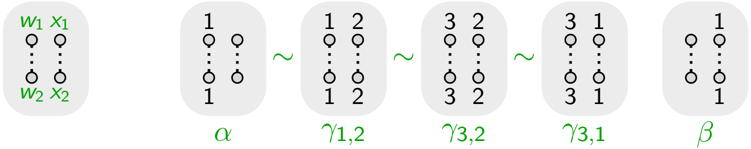

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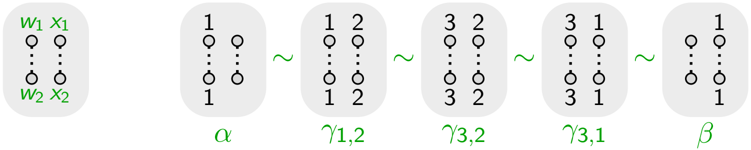

How Many Moves are Needed?

Prop: For every G and every f, with $f(v) \geq 2$ for all v, there is list assignment L with $|L(v)| = f(v)$ for all v and L-colorings α and β where changing α to β needs $n(G) + \mu(G)$ moves.

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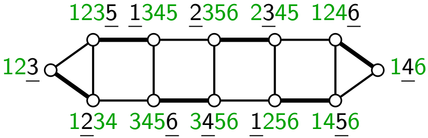

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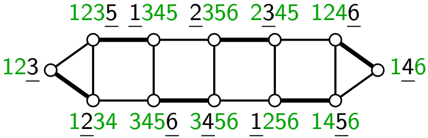

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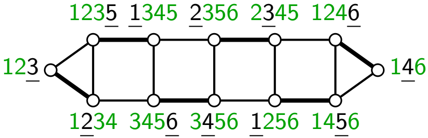

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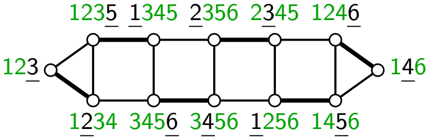

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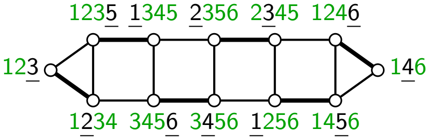

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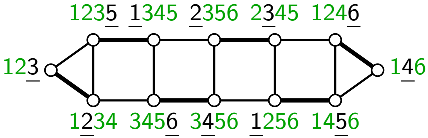

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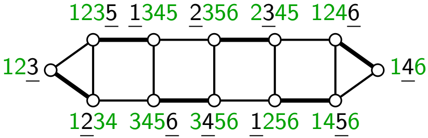

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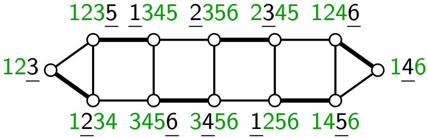

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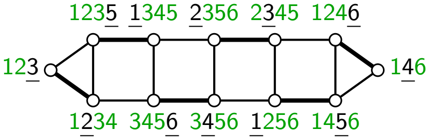

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Pf Sketch: Induction on number of colors k used by β. Base case, $k = 1$, is easy. By Pigeonhole, some color c used at least as often by β as by α. From α, recolor every vertex using c. Now recolor every vertex using c in β. Finish by induction.

Thm: Fix L with $|L(v)| \geq d(v) + 2$ for all v and L-colorings α and β. Can recolor α to β in at most $n(G) + 2\mu(G)$ steps.
Pf Sketch: Induction on $n(G)$. Find v with $\mu(G - v) = \mu(G) - 1$. If no such v exists, finish by lemma. By Pigeonhole, find c in $L(v)$ used at most once on $N(v)$ by α and β (in total). Assume α uses c once on $w \in N(v)$. From α: recolor w, recolor v with c. Finish on $G - v$ by induction. Recolor v to $\beta(v)$.

Two Proof Sketches

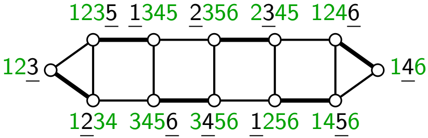
Lem: Fix list-assignment L with $|L(v)| \geq d(v) + 2$ for all v. Given L-colorings α and β, can recolor α to β in at most $2n(G) - 1$ steps.
Pf Sketch: Induction on number of colors k used by β. Base case, $k = 1$, is easy. By Pigeonhole, some color c used at least as often by β as by α. From α, recolor every vertex using c. Now recolor every vertex using c in β. Finish by induction.

Thm: Fix L with $|L(v)| \geq d(v) + 2$ for all v and L-colorings α and β. Can recolor α to β in at most $n(G) + 2\mu(G)$ steps.
Pf Sketch: Induction on $n(G)$. Find v with $\mu(G - v) = \mu(G) - 1$. If no such v exists, finish by lemma. By Pigeonhole, find c in $L(v)$ used at most once on $N(v)$ by α and β (in total). Assume α uses c once on $w \in N(v)$. From α: recolor w, recolor v with c. Finish on $G - v$ by induction. Recolor v to $\beta(v)$. Extra steps: $2 + 1 = n(G) - n(G - v) + 2(\mu(G) - \mu(G - v))$.

How Many Moves are Needed?

Prop: For every G and every f, with $f(v) \geq 2$ for all v, there is list assignment L with $|L(v)| = f(v)$ for all v and L-colorings α and β where changing α to β needs $n(G) + \mu(G)$ moves.

Pf: Every vert needs recolored; every edge of M needs extra step.



Conj:[Cambie–Cames van Batenburg–C.] For list assignment L with $|L(v)| \geq d(v) + 2$ for all v and L-colorings α and β, can always change α to β in at most $n(G) + \mu(G)$ steps.

Thm:[Cambie–Cames van Batenburg–C.] arXiv:2204.07928
(a) If $|L(v)| \geq 2d(v) + 1$, then $n(G) + \mu(G)$ steps suffice.
(b) If $|L(v)| \geq d(v) + 2$, then $n(G) + 2\mu(G)$ steps suffice.

Correspondence Coloring: $\mu(G) \to \tau(G)$. Conj. and Theorems