Maximum Matchings and How to Find Them

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Department of Mathematics
William & Mary
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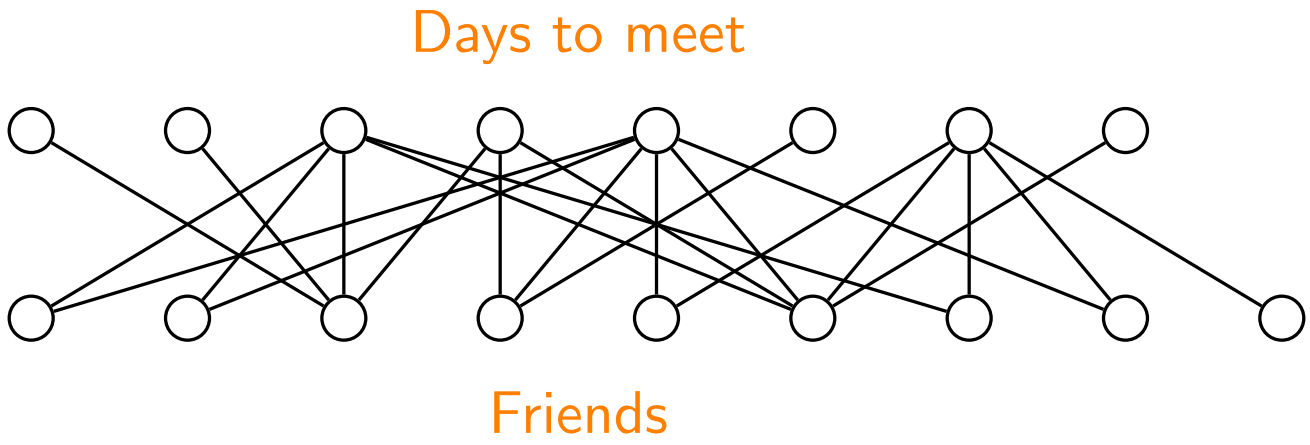

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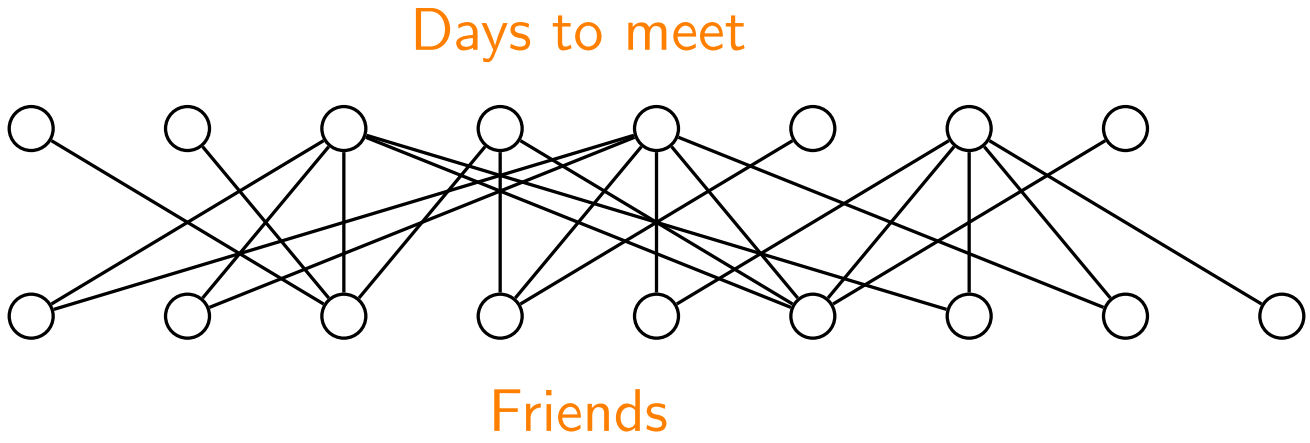

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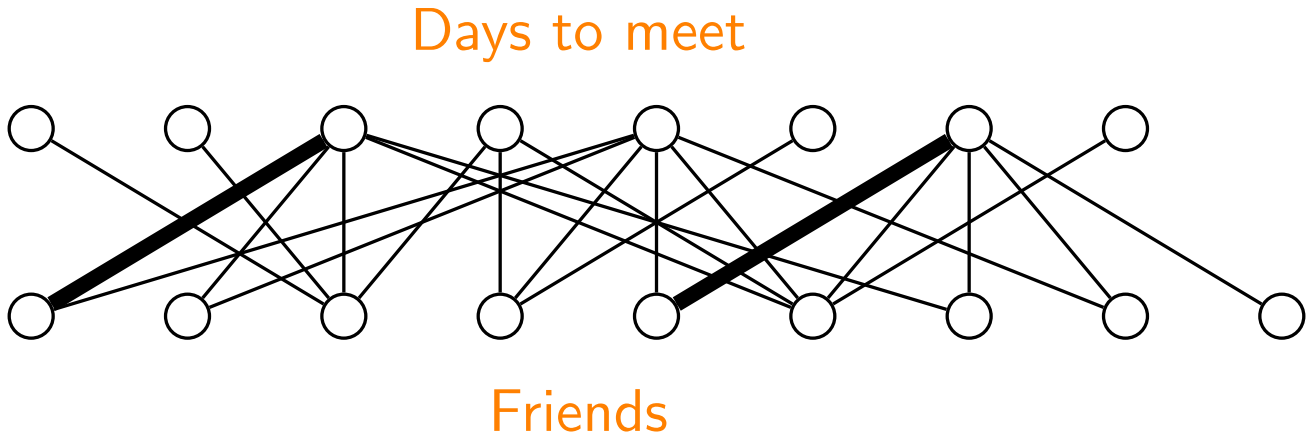

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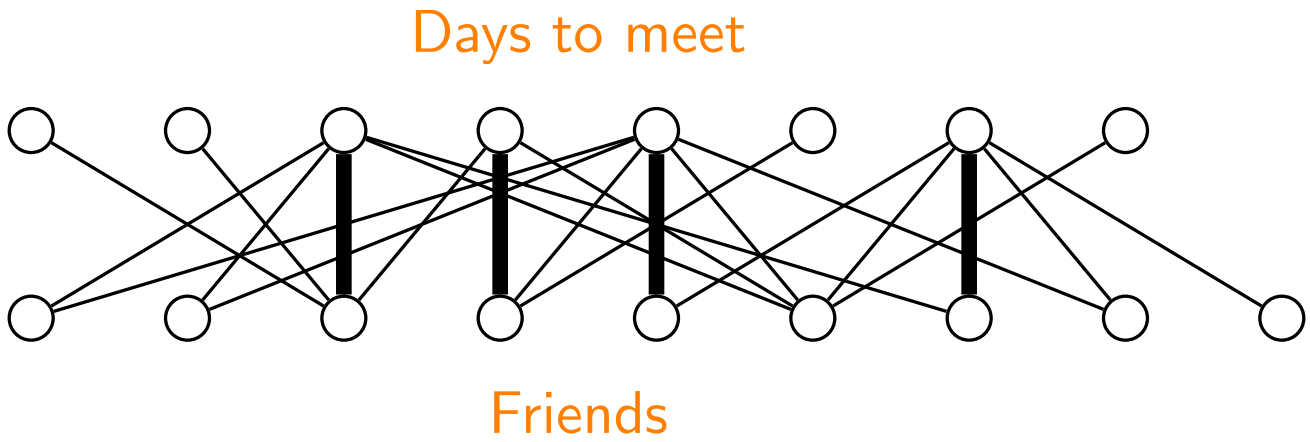

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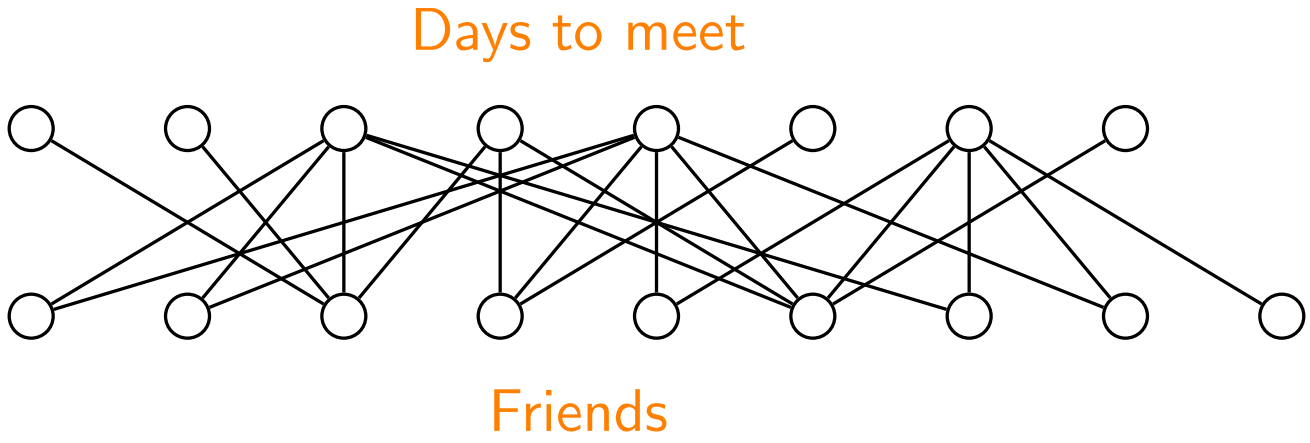

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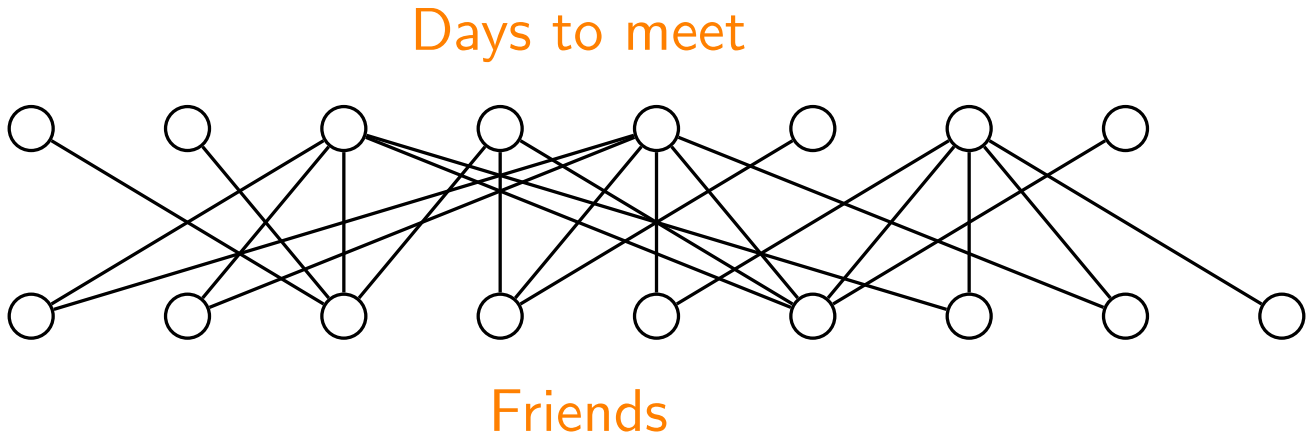

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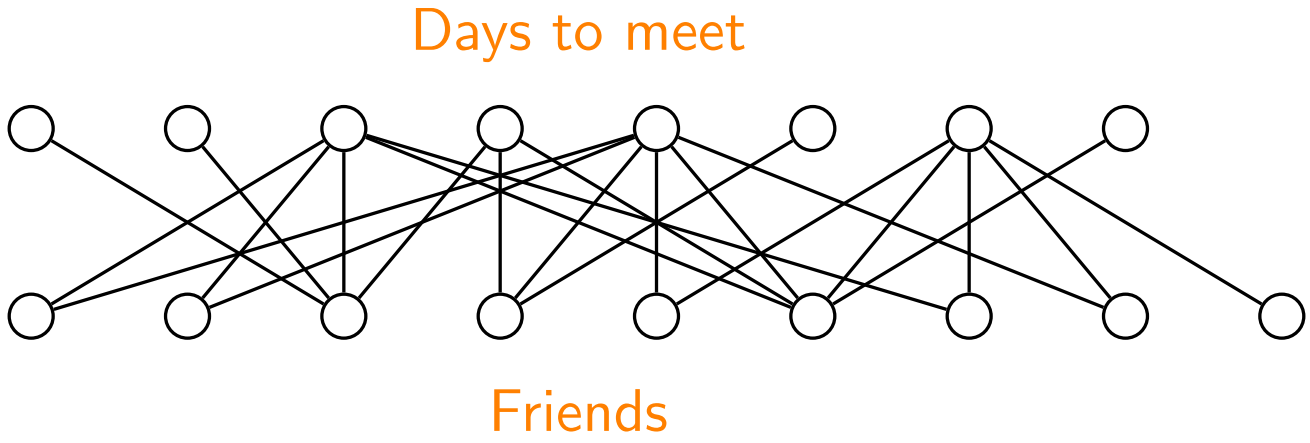

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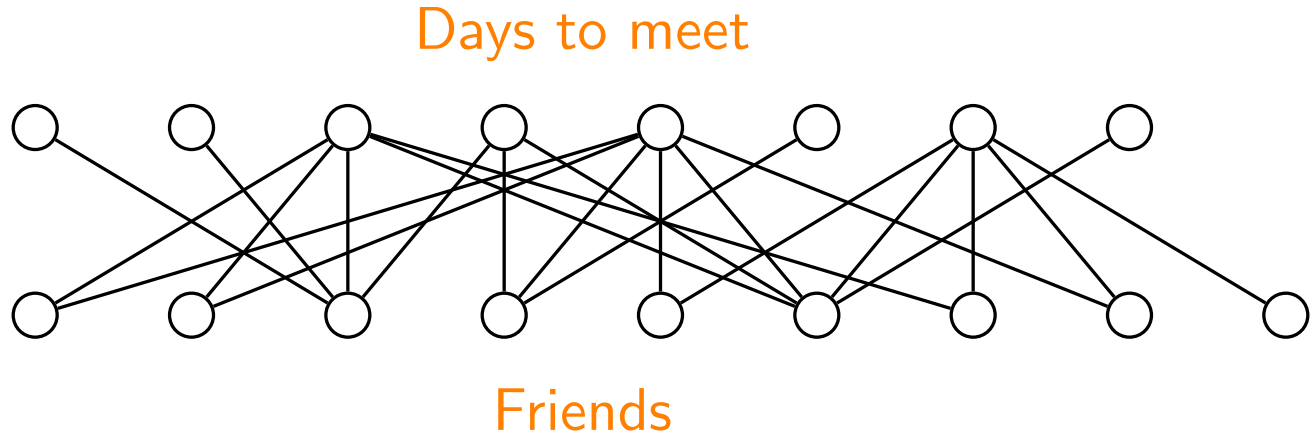

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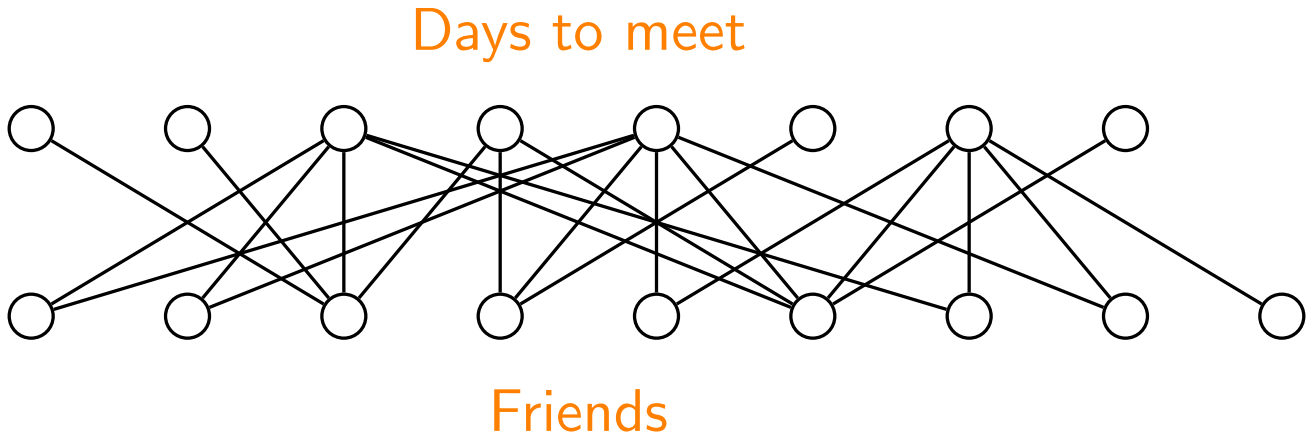

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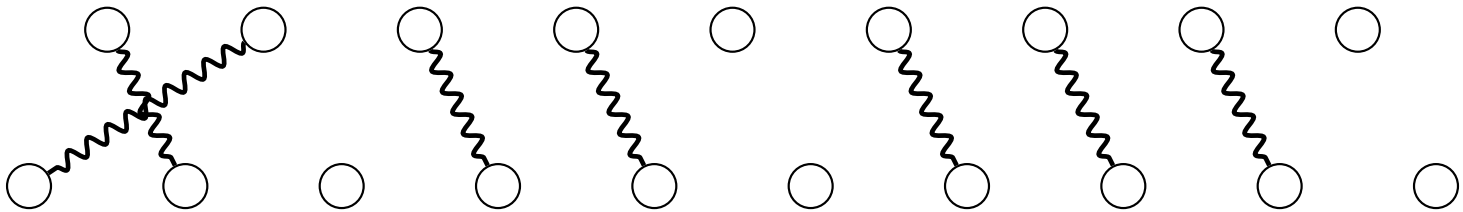

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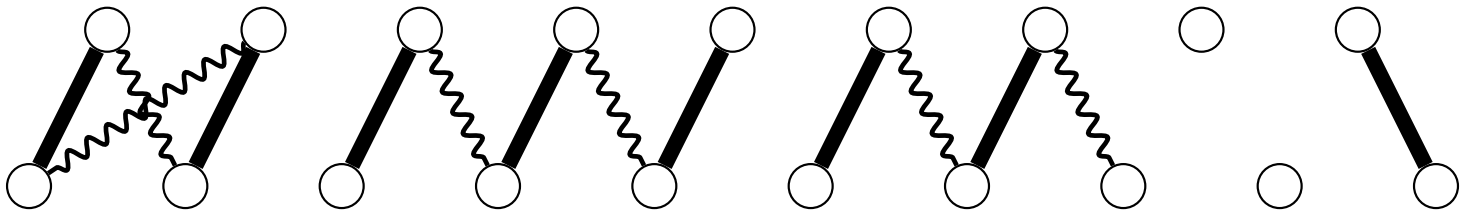

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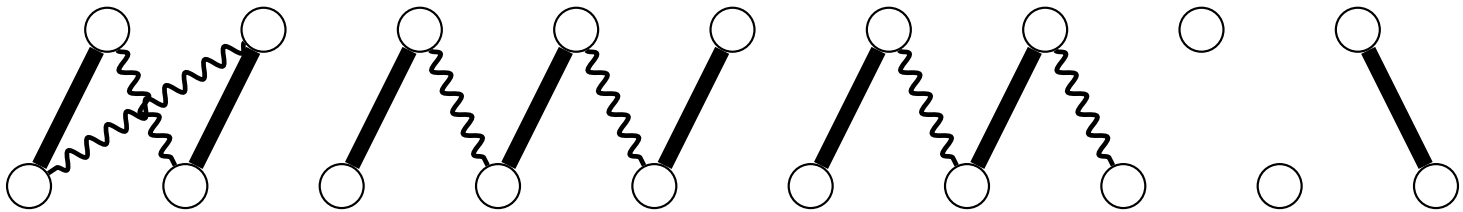

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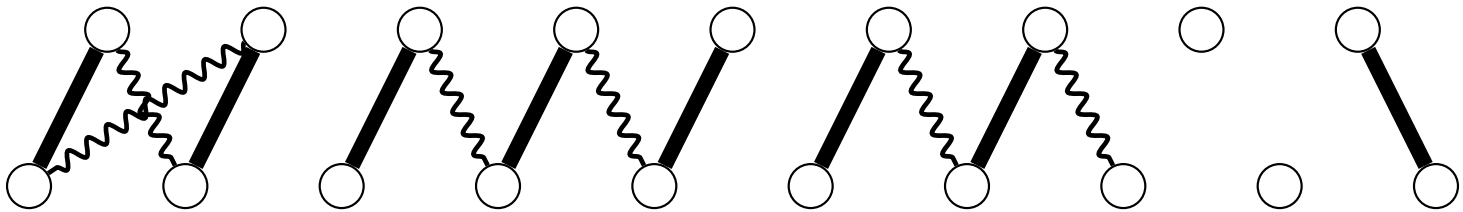

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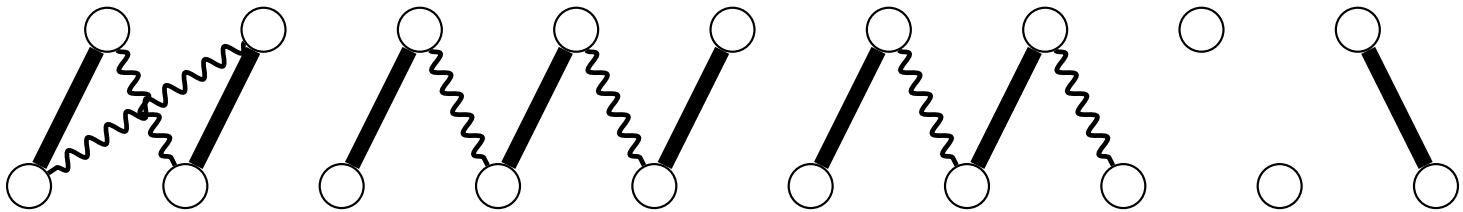

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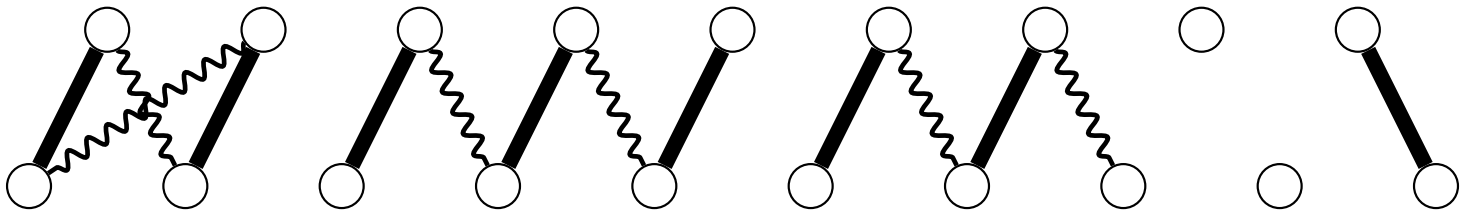

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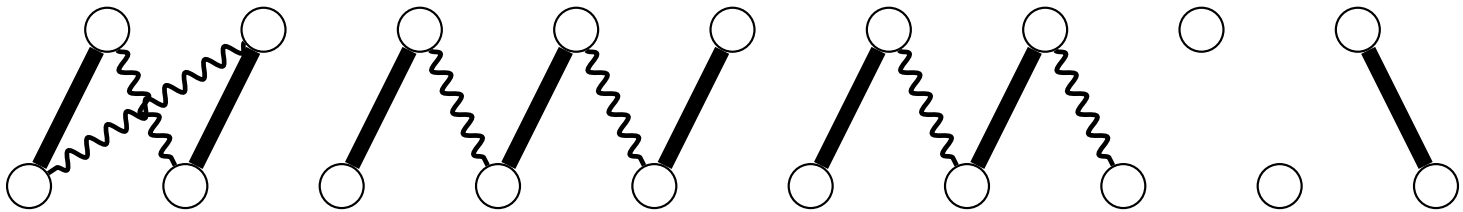

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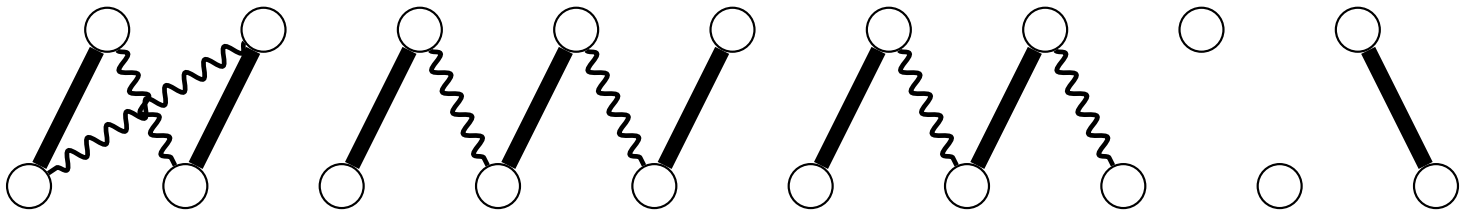

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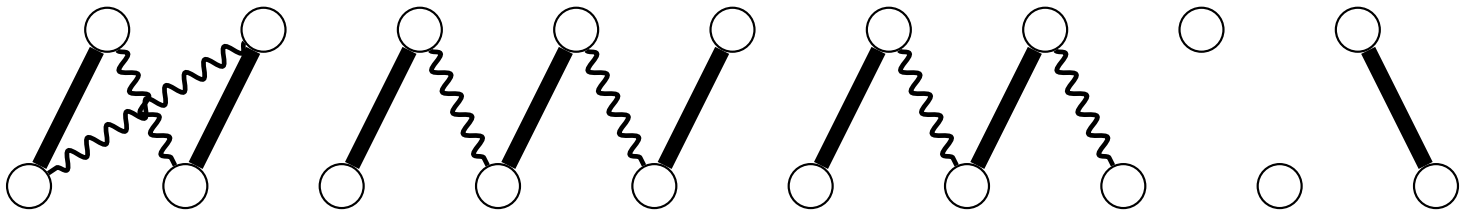

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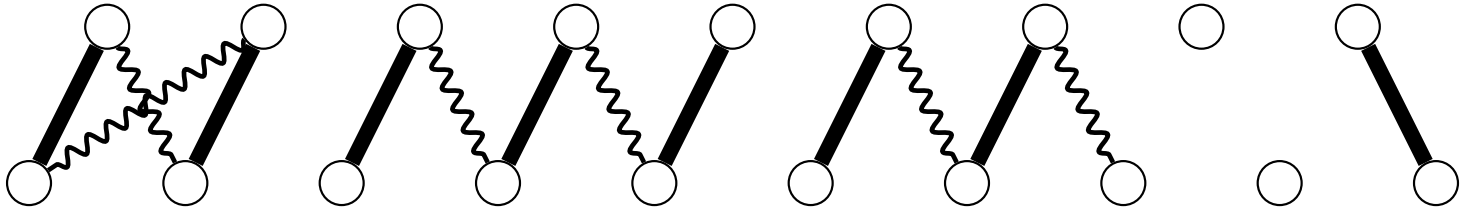

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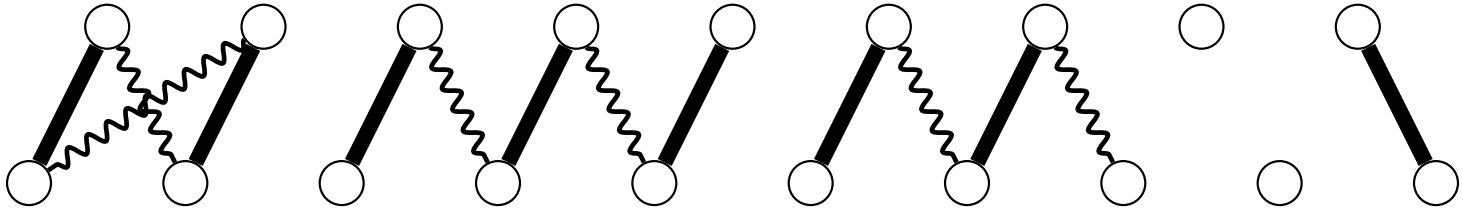

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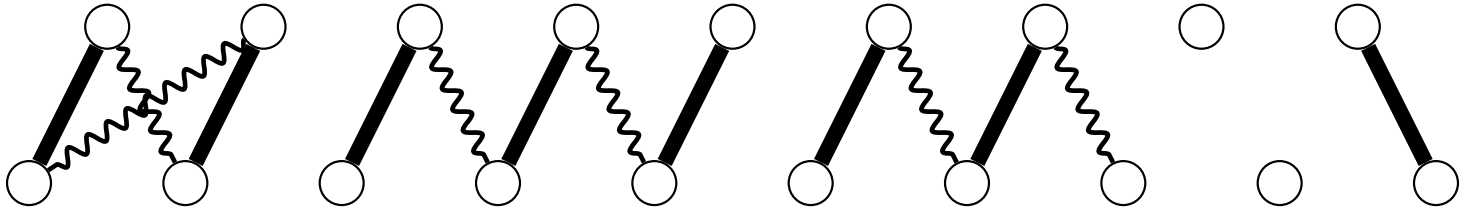

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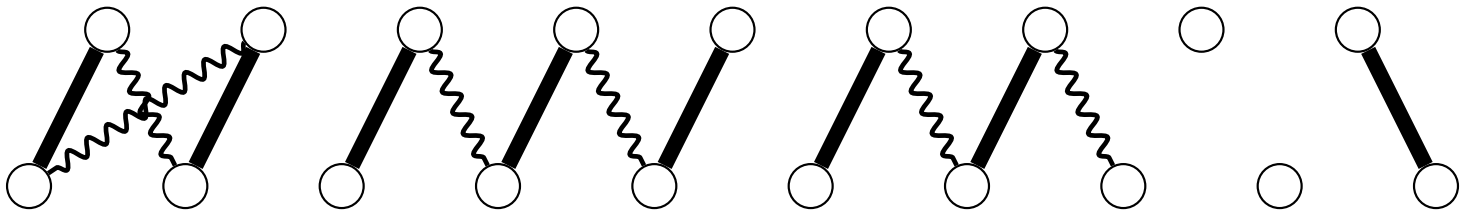

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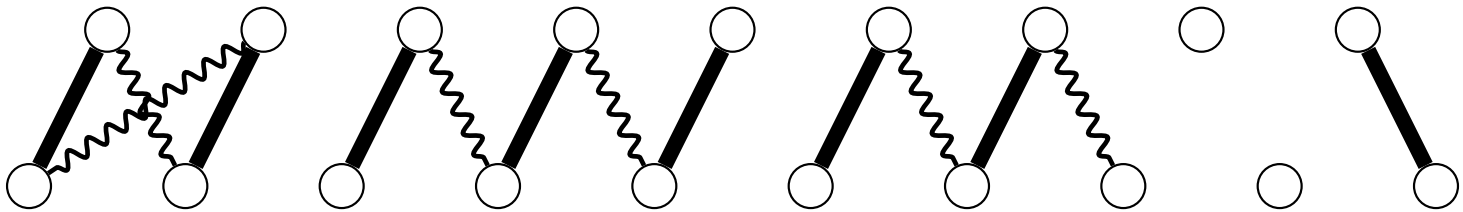

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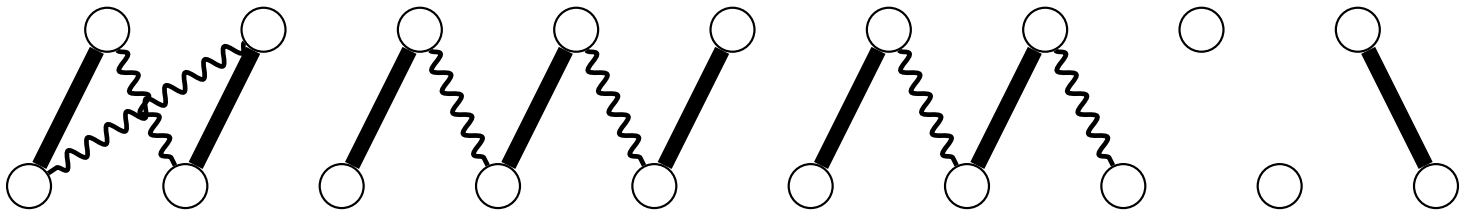

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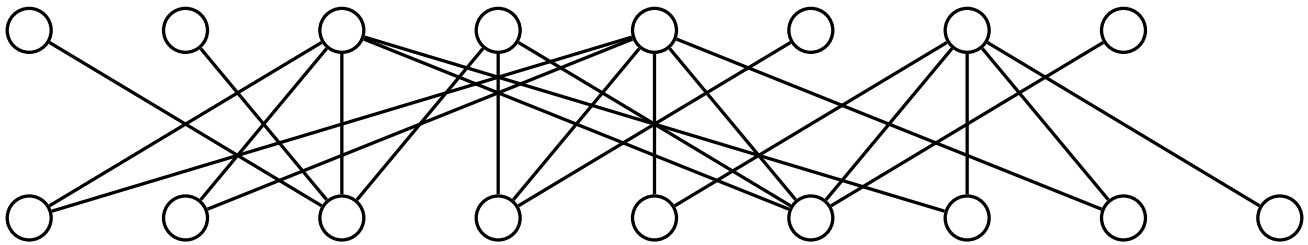

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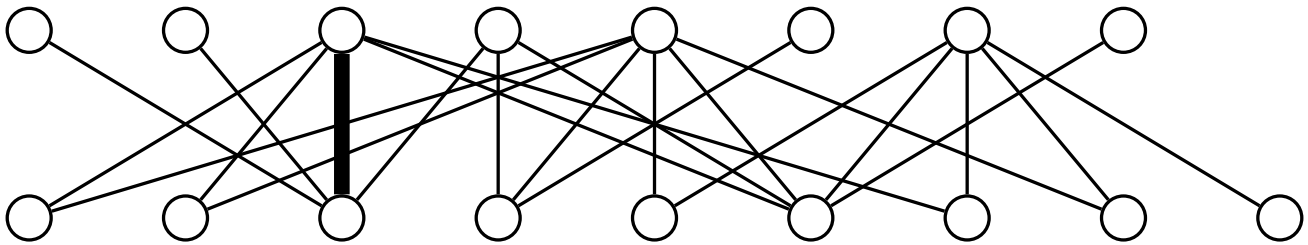

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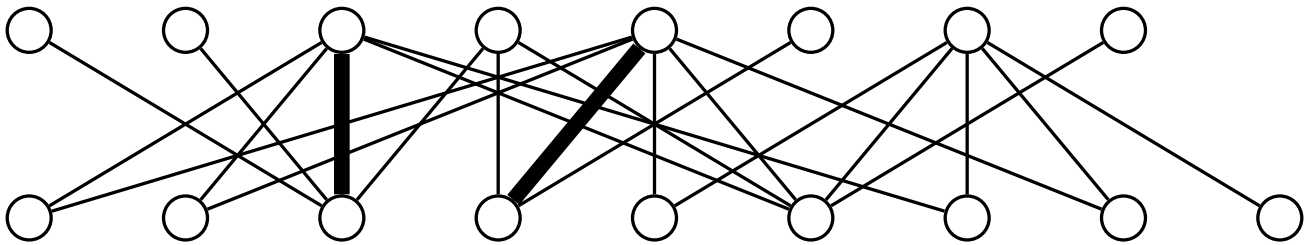

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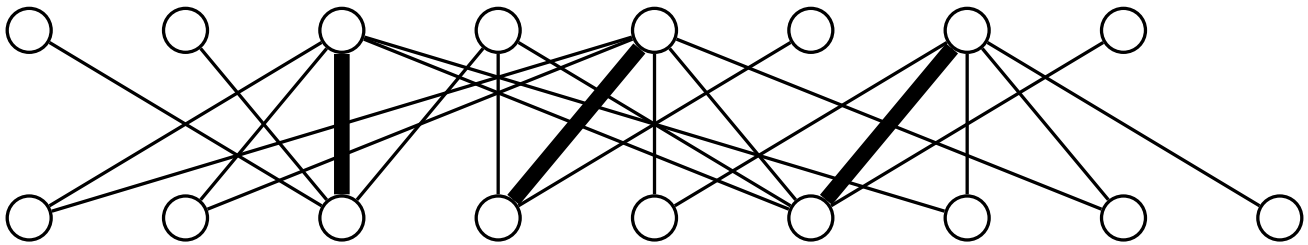

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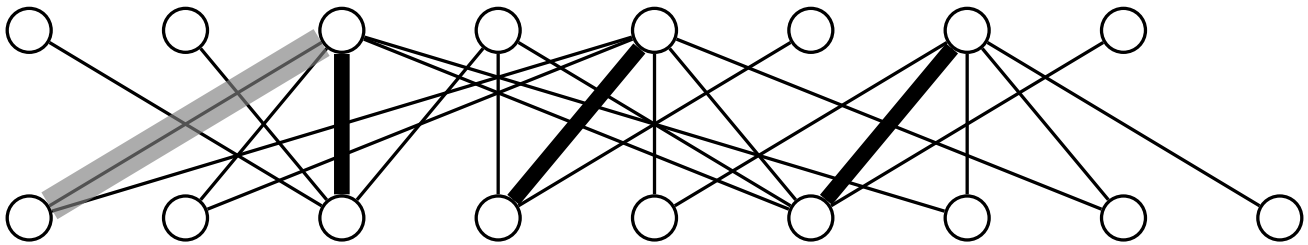

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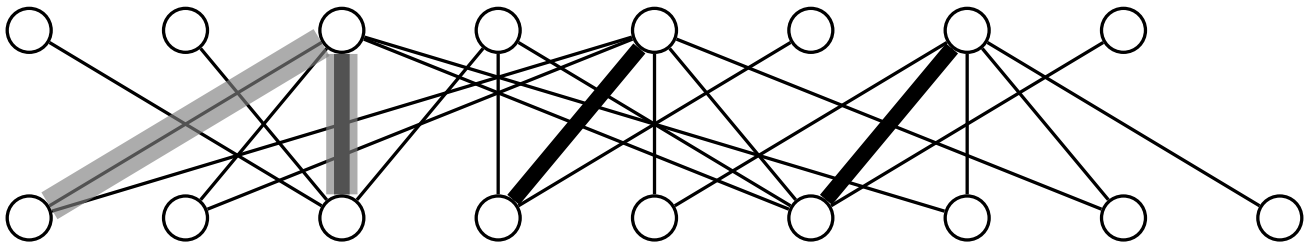

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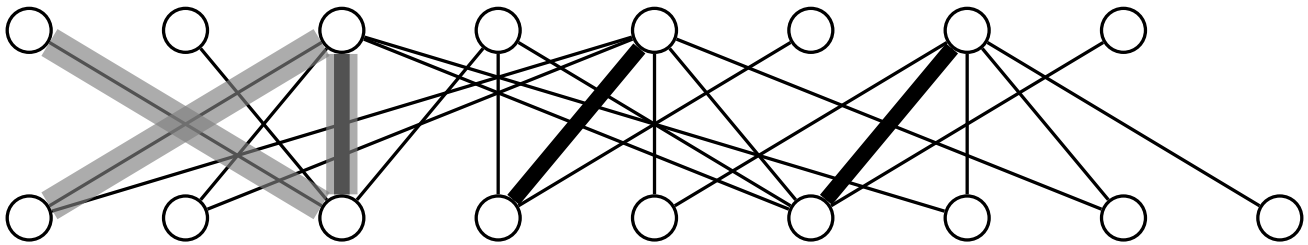

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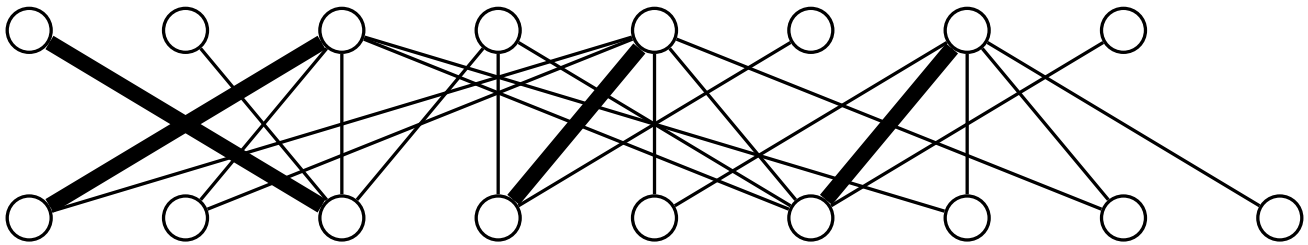

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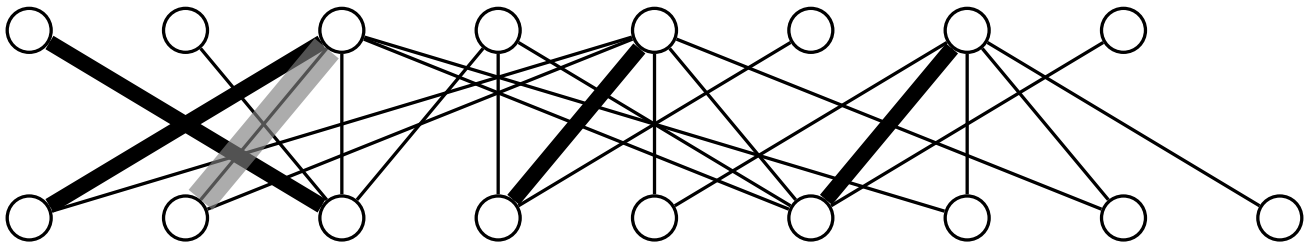

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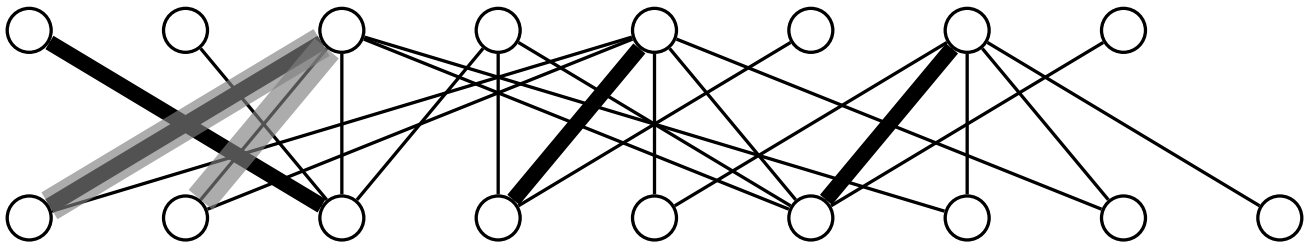

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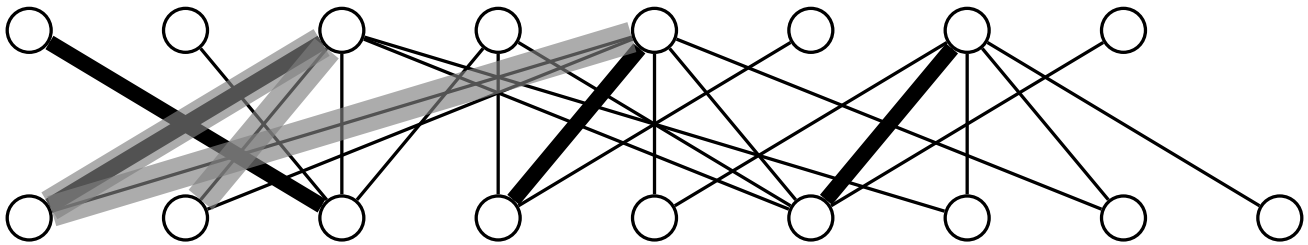

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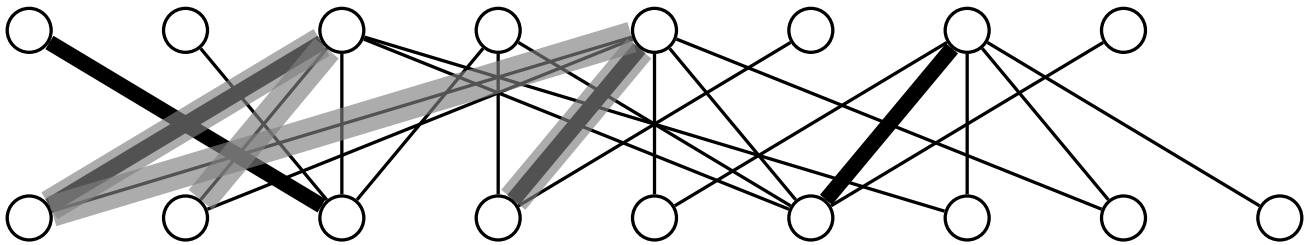

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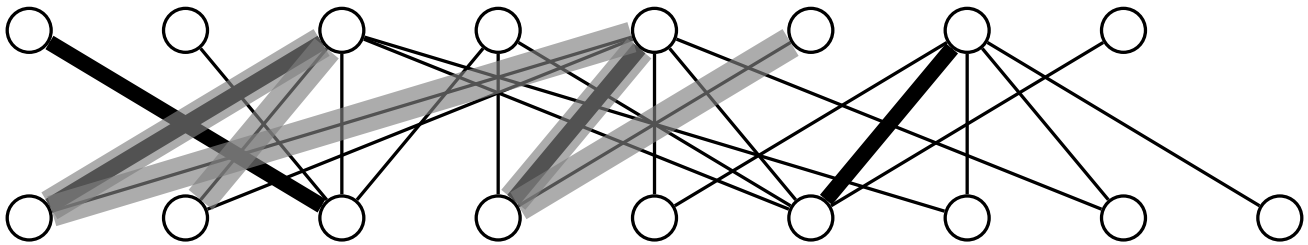

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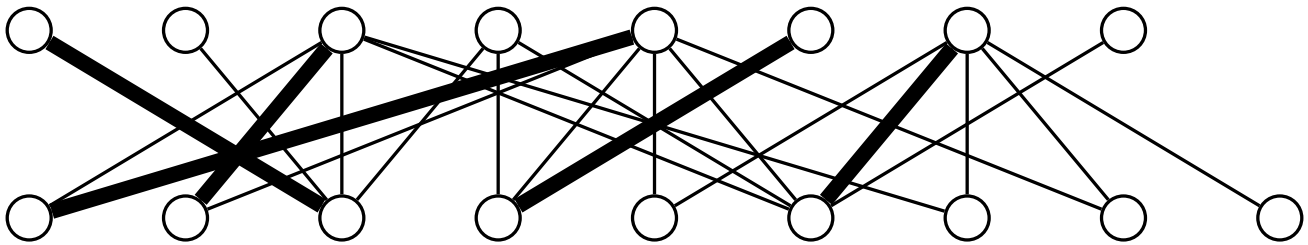

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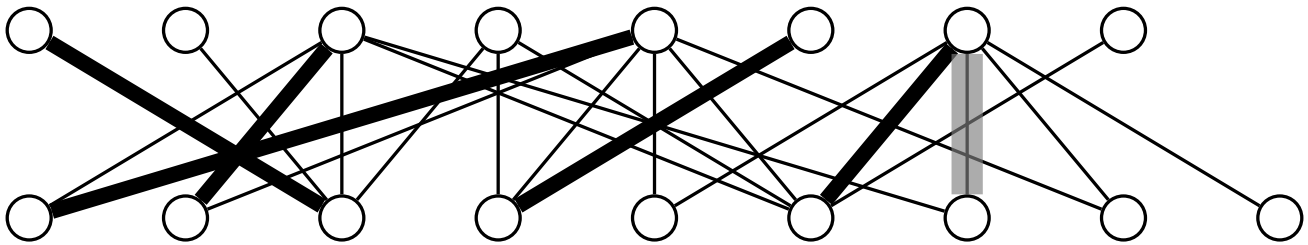

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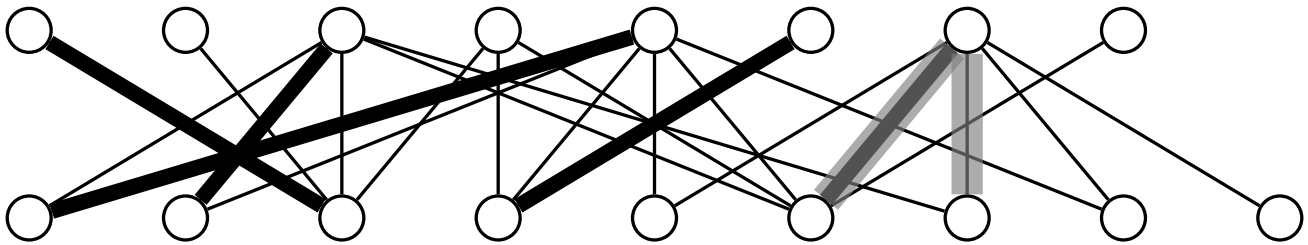

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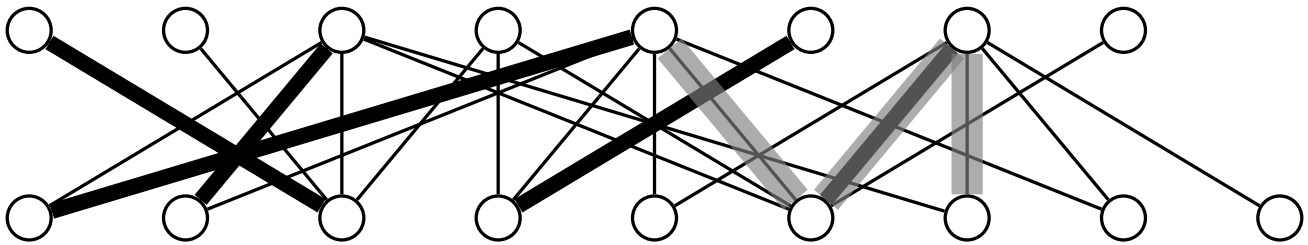

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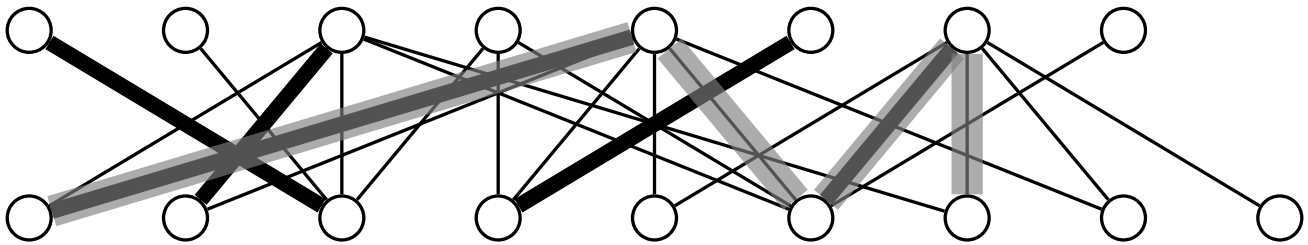

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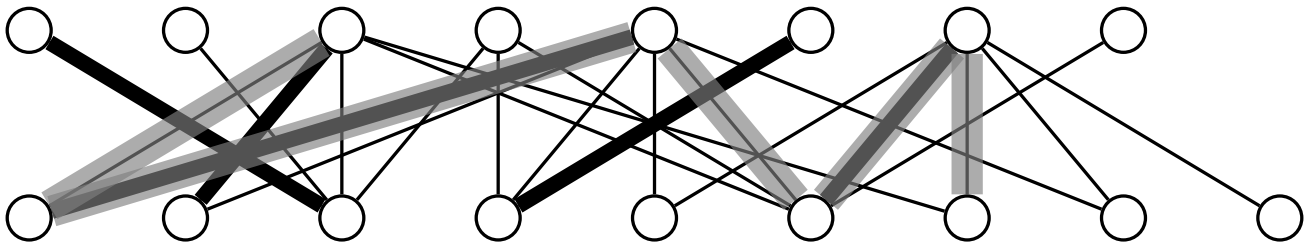

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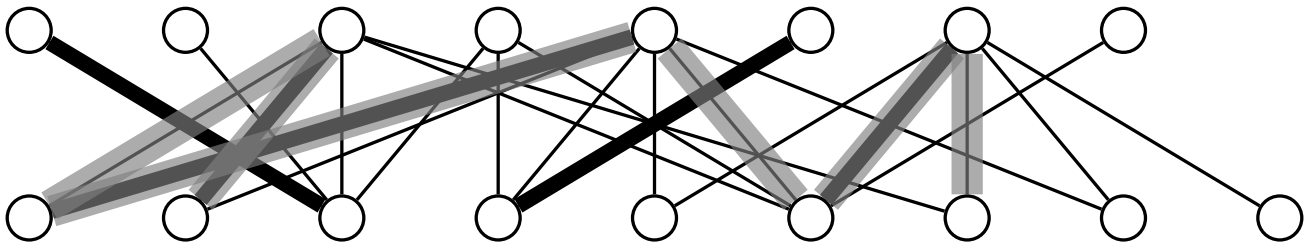

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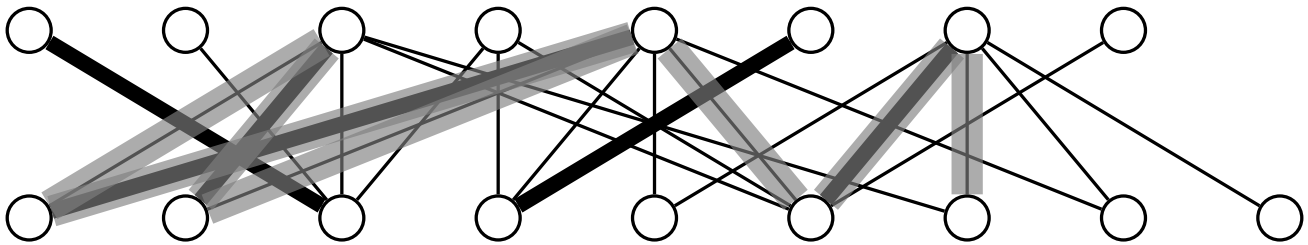

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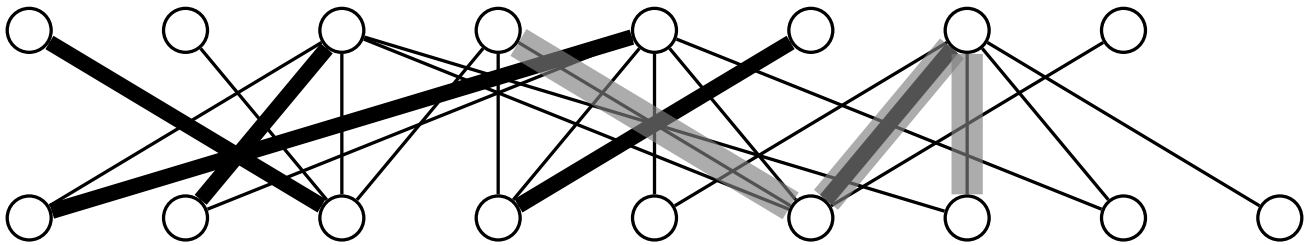

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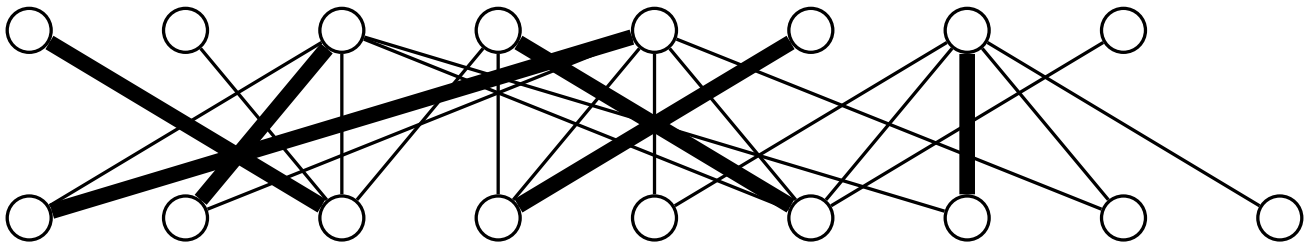

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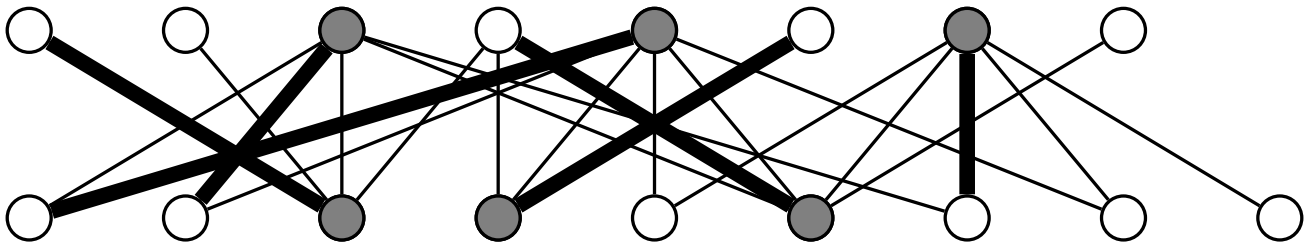

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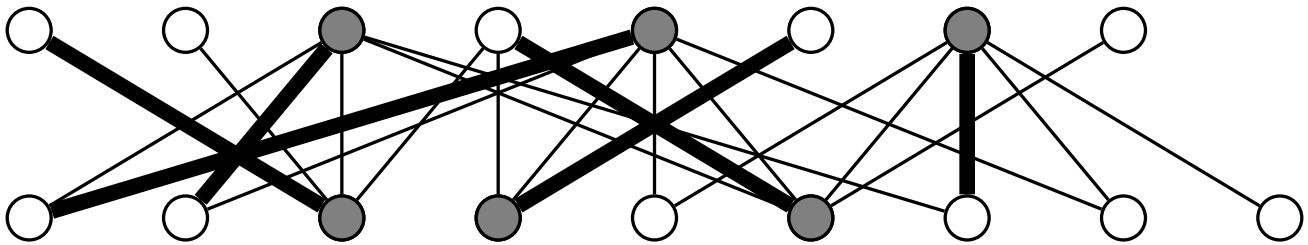

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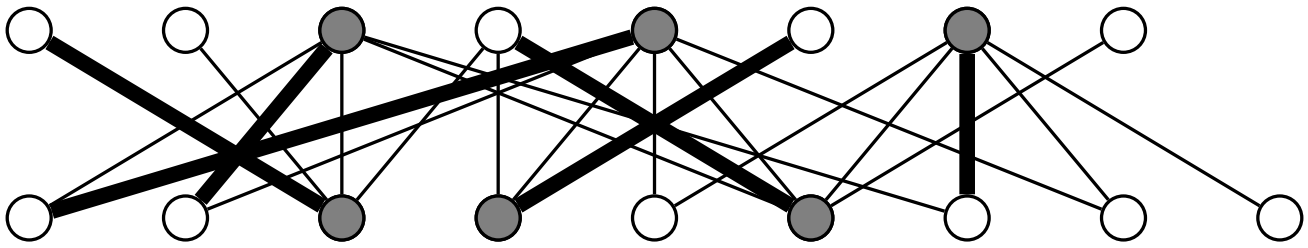

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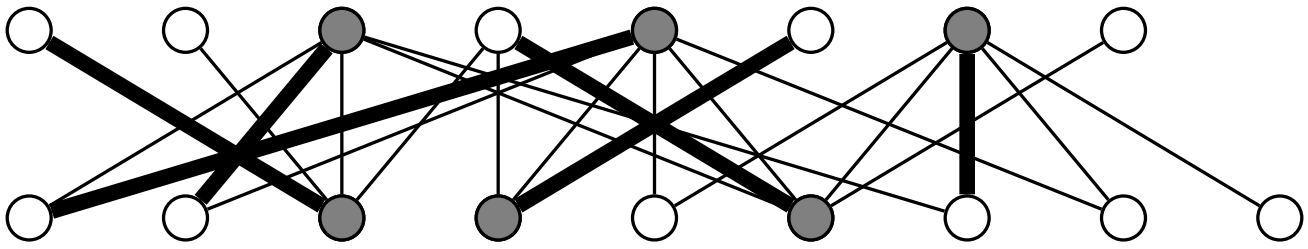

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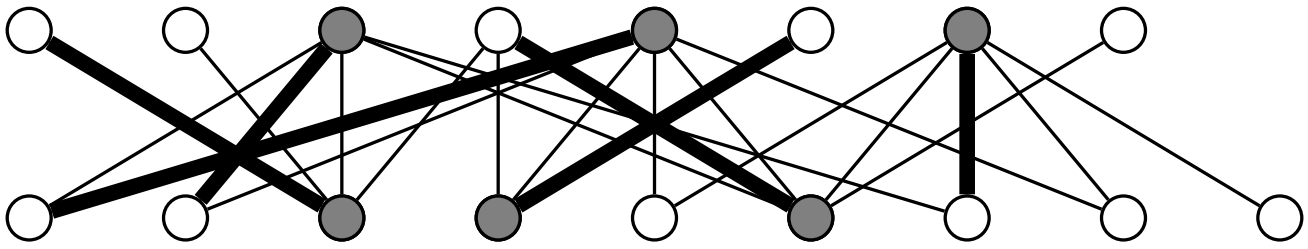

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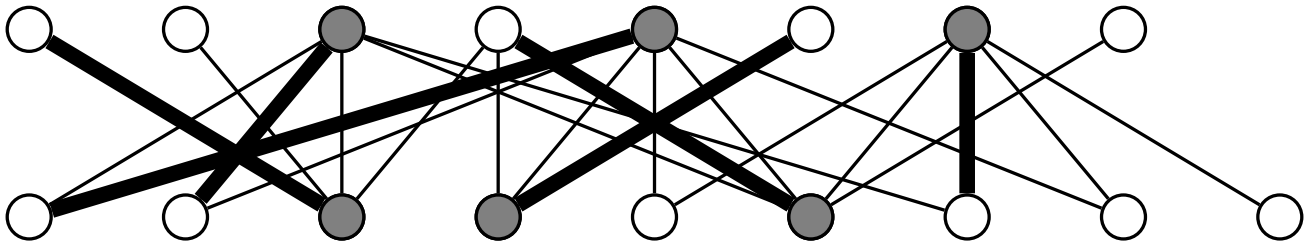

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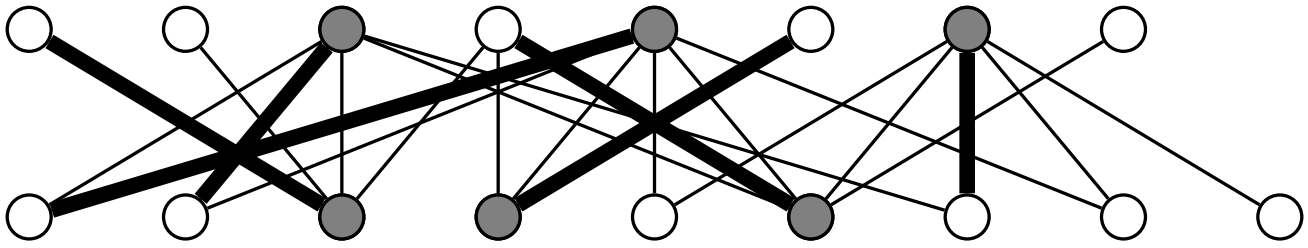

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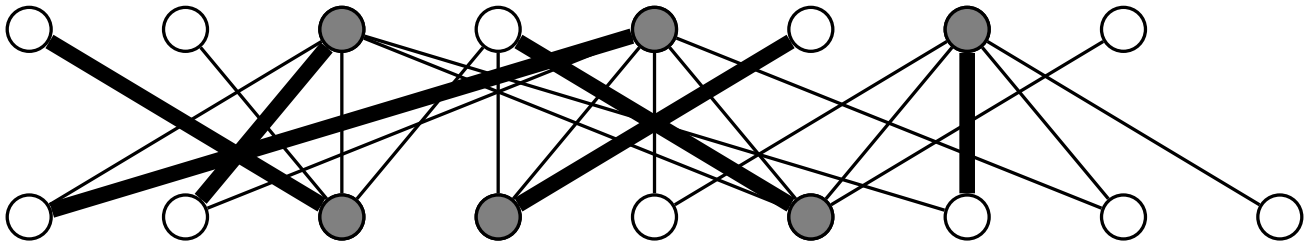

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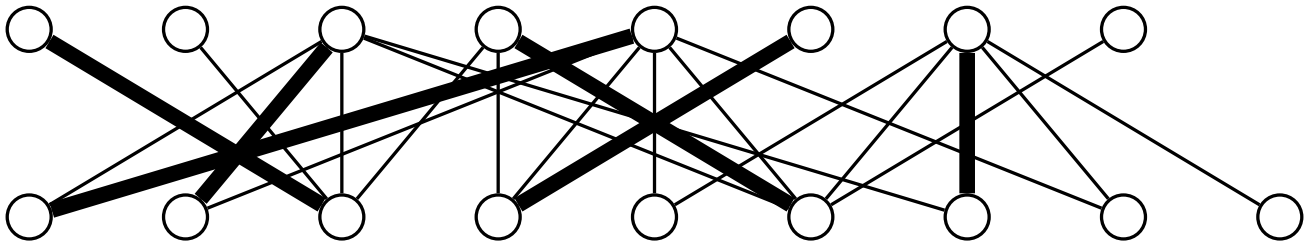

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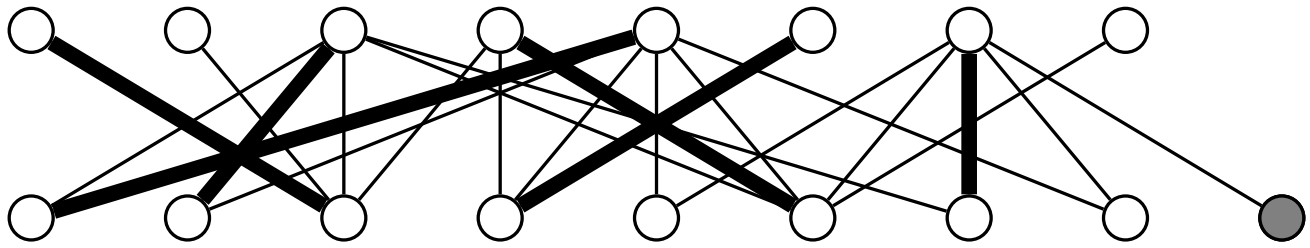

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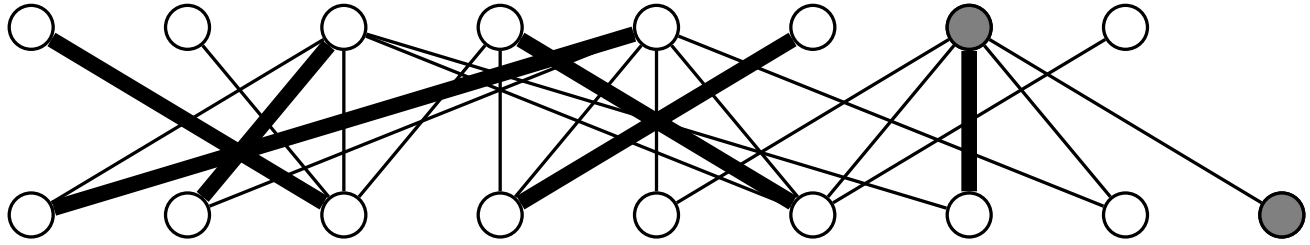

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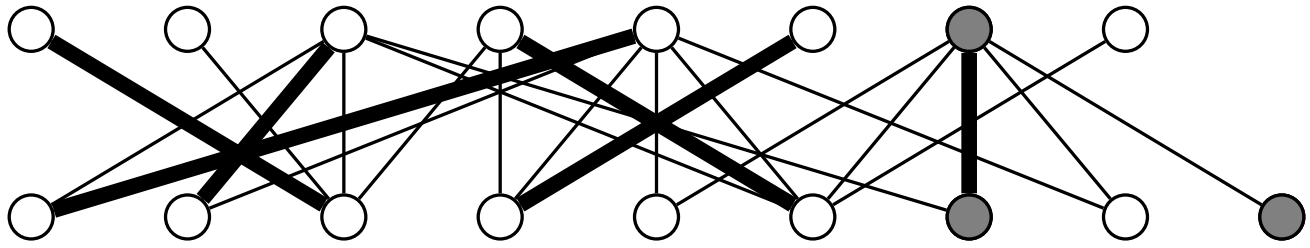

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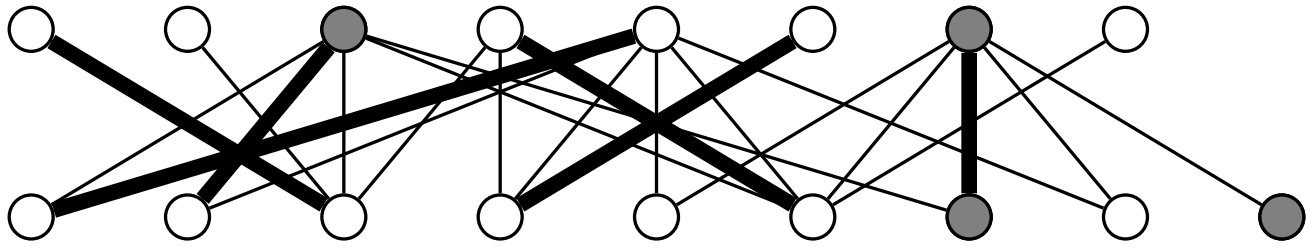

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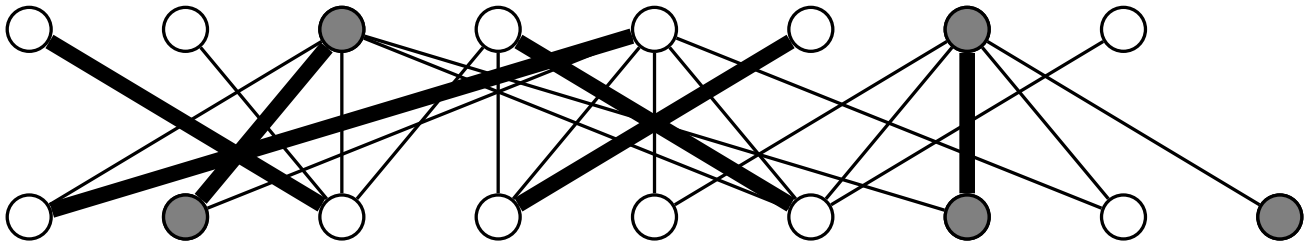

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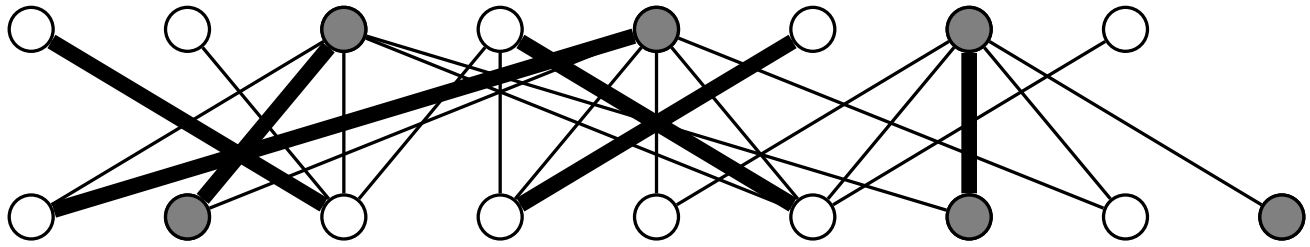

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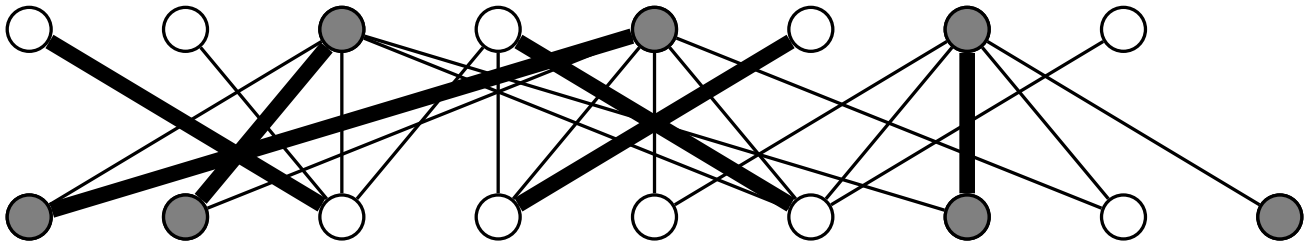

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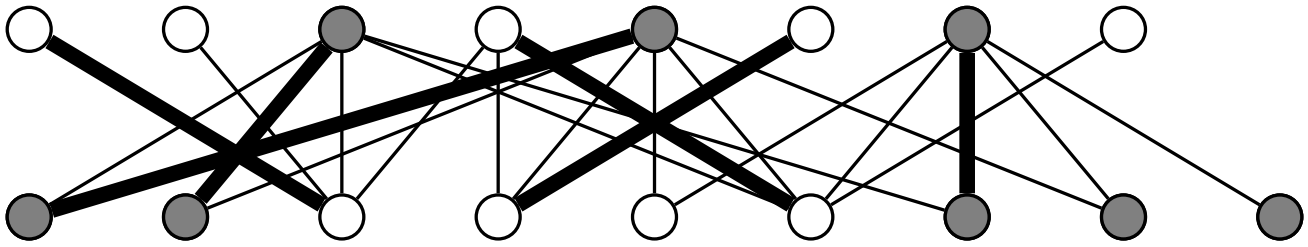

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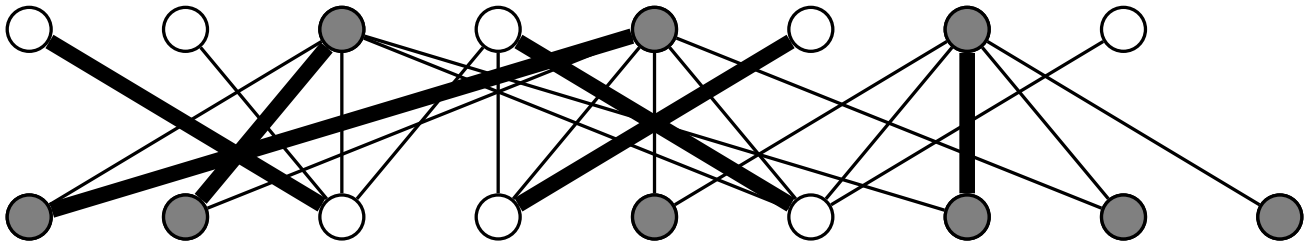

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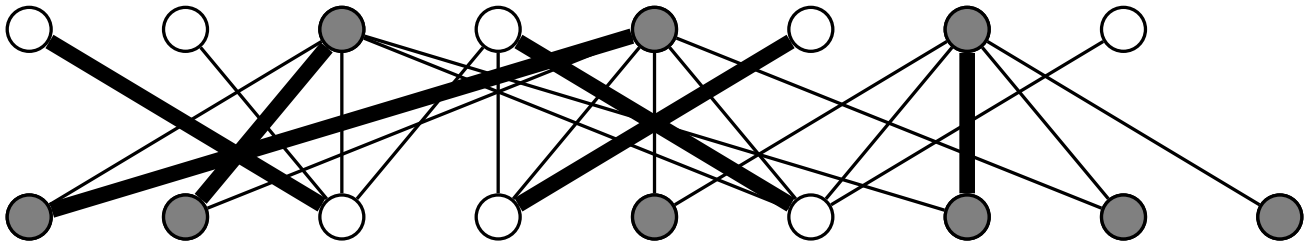

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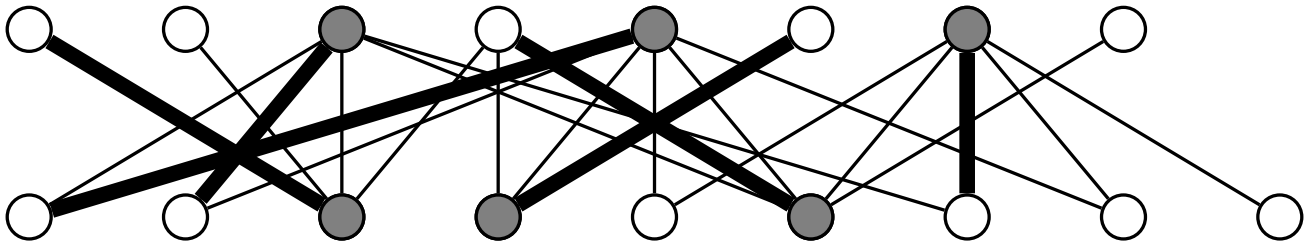

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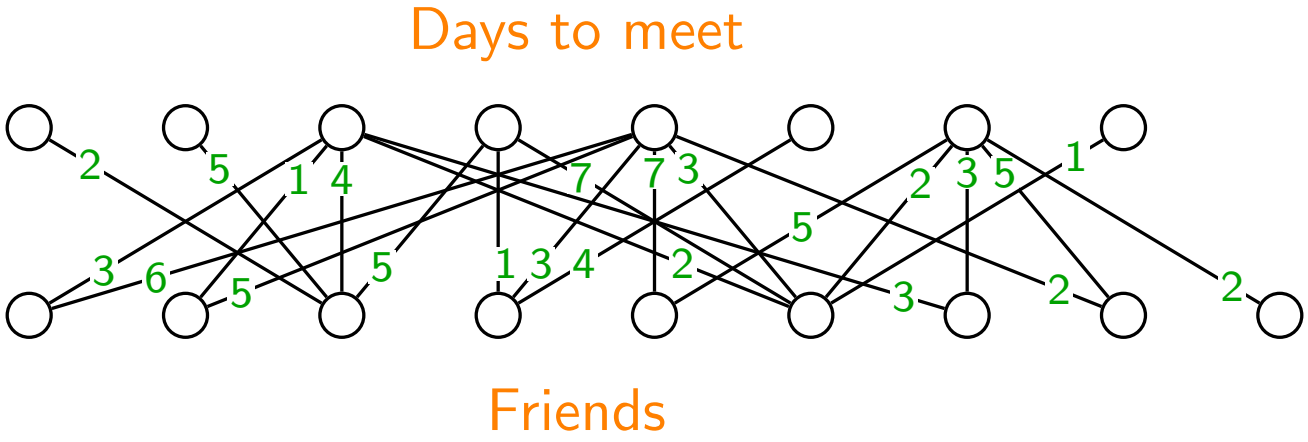

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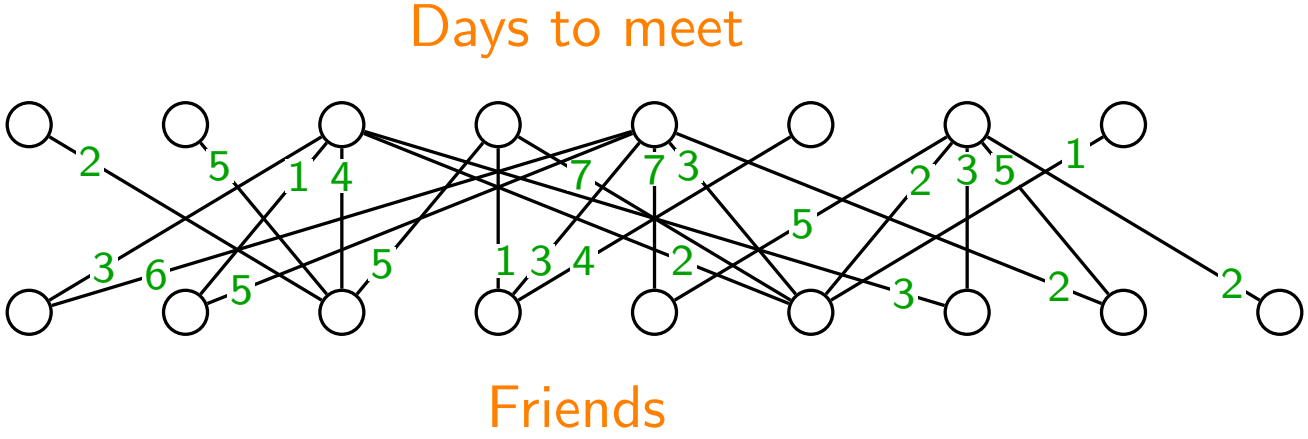

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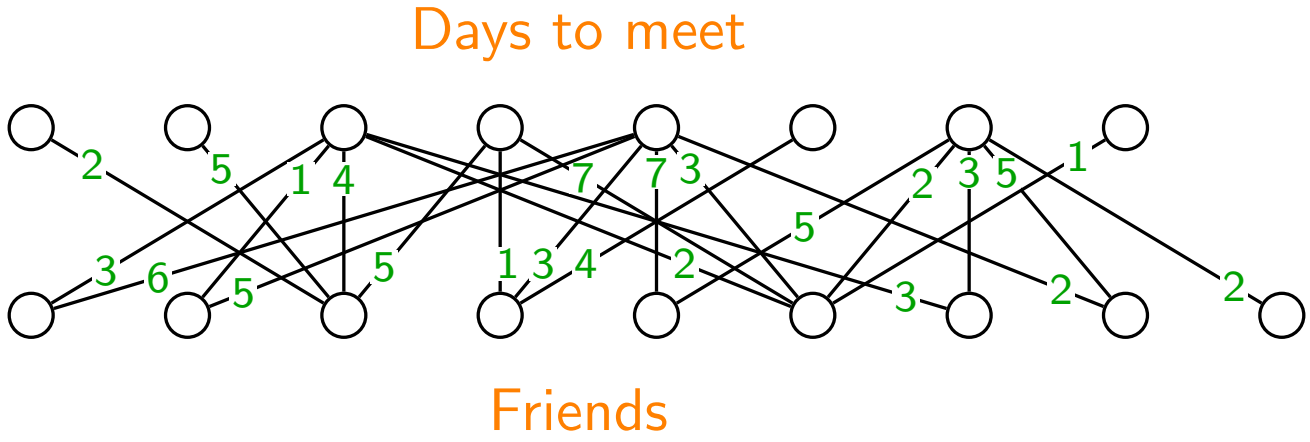

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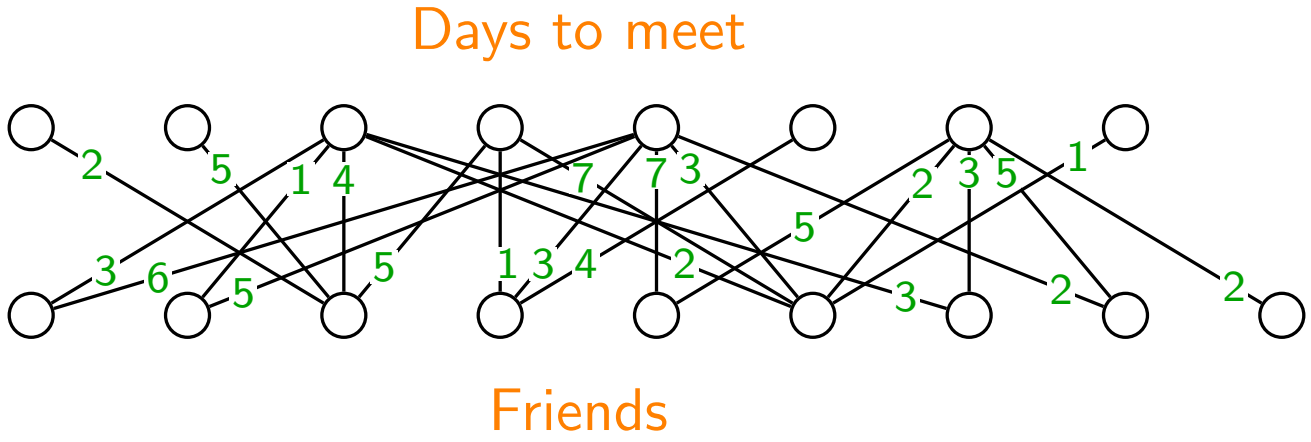

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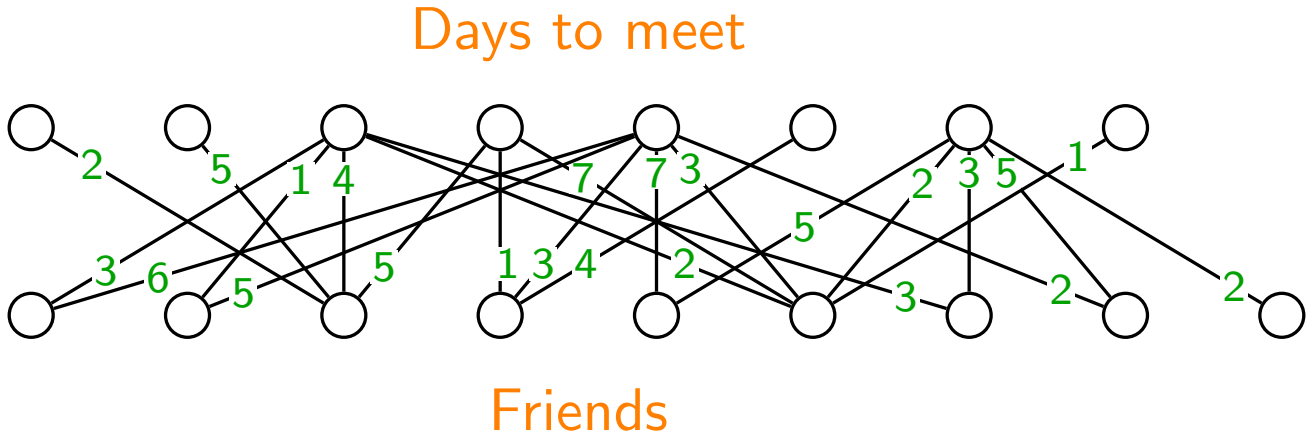

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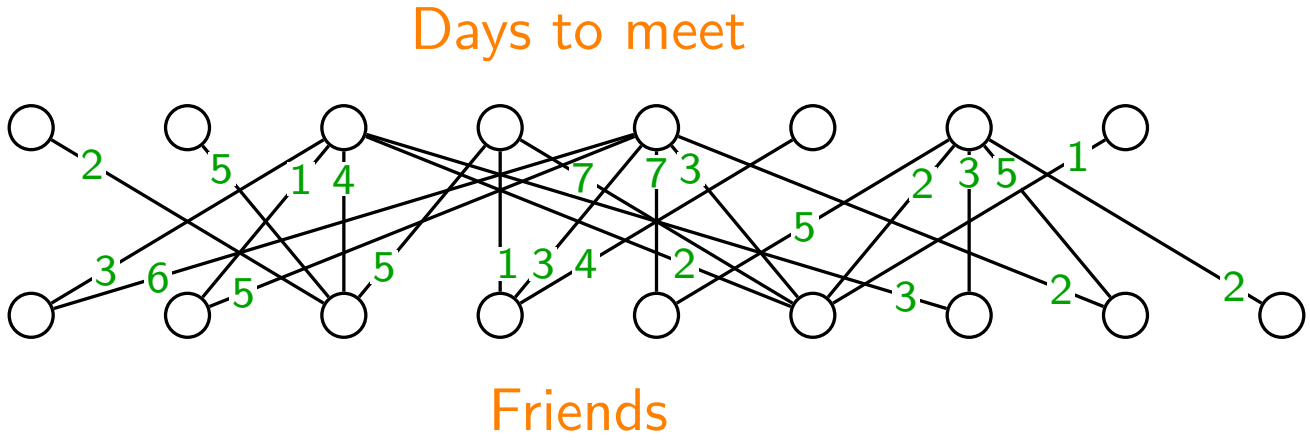

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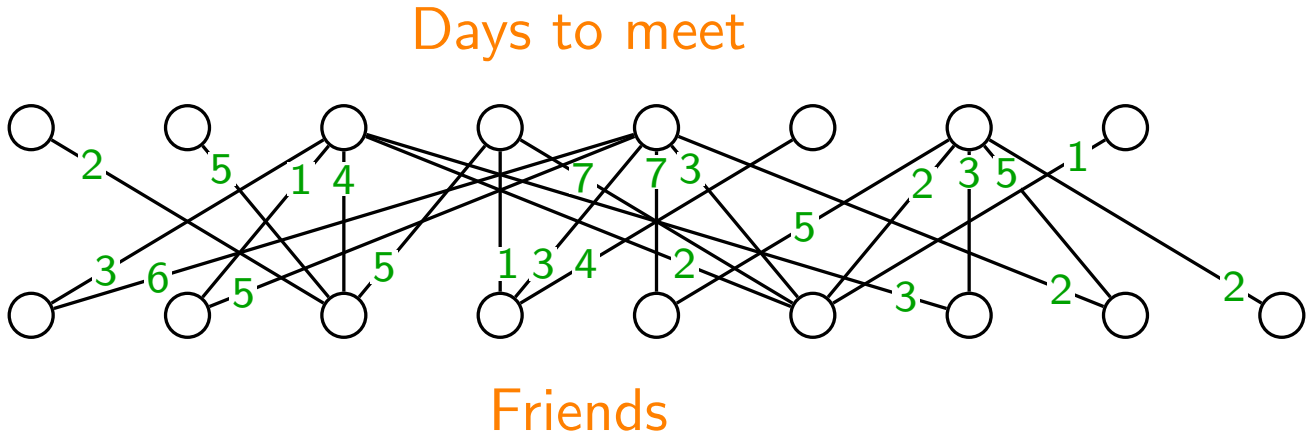

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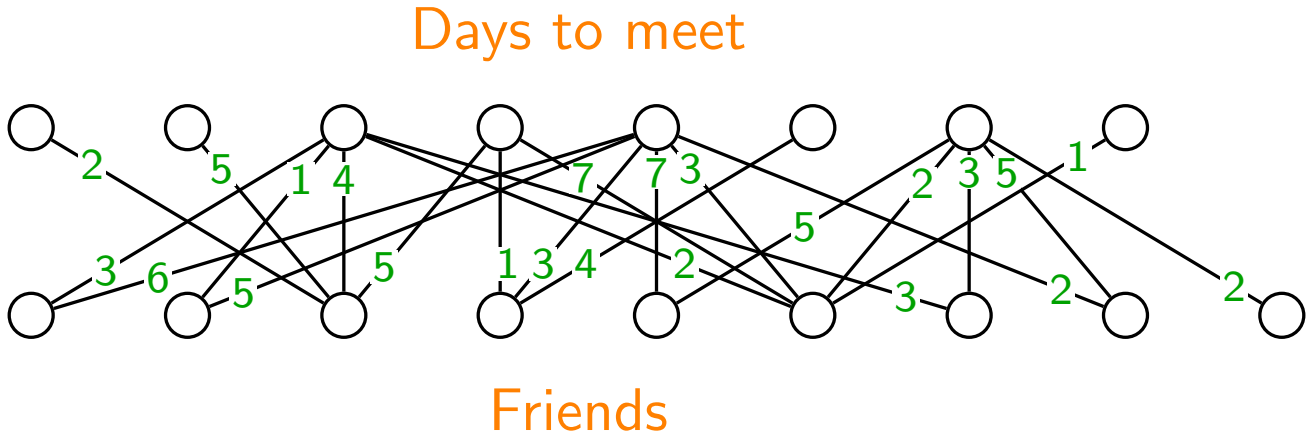

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Key Theorem: Fix a bipartite G with edge weights w. For every max. perfect matching M, there is a min. vertex cover f such that $\sum_{xy \in M} w(xy) = \sum_{v \in V(G)} f(v)$. Proves f and M are both optimal.

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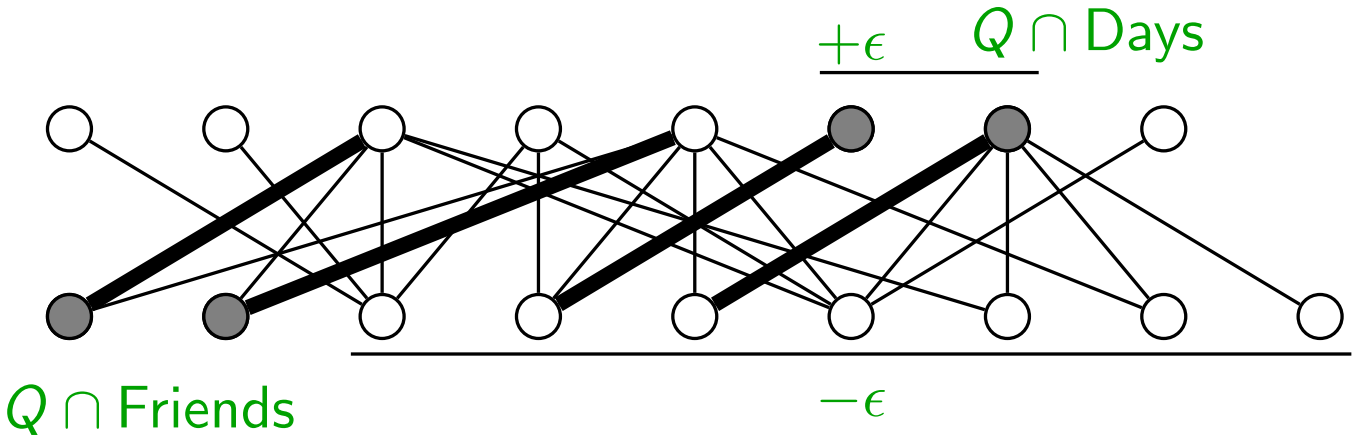

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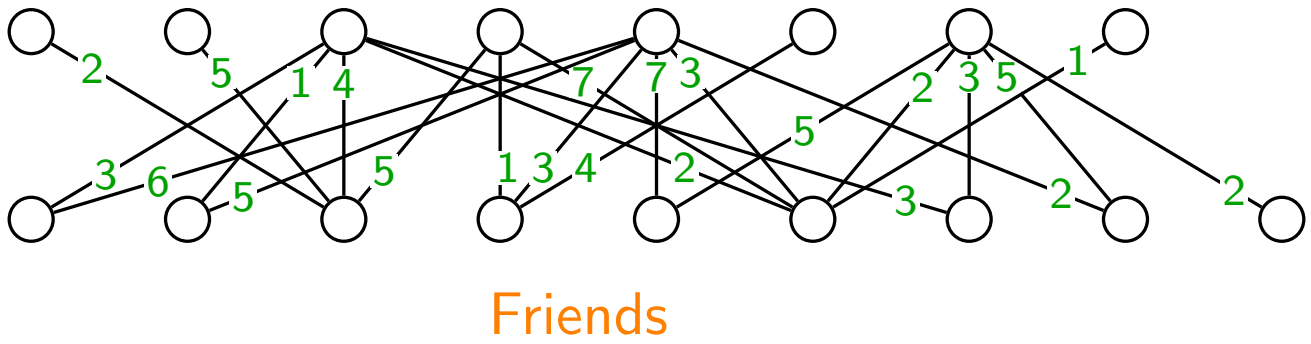

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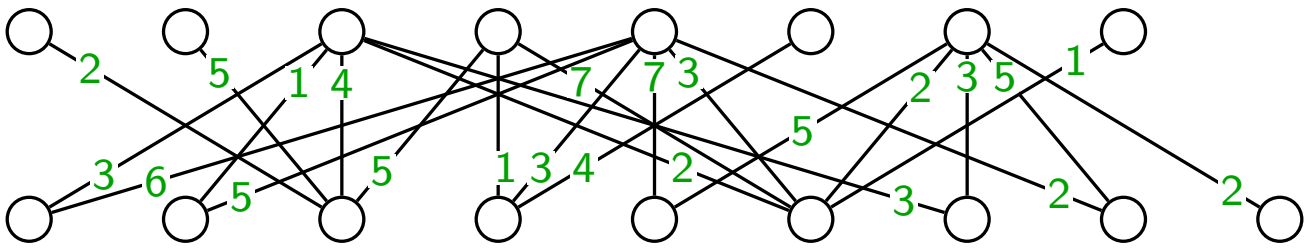

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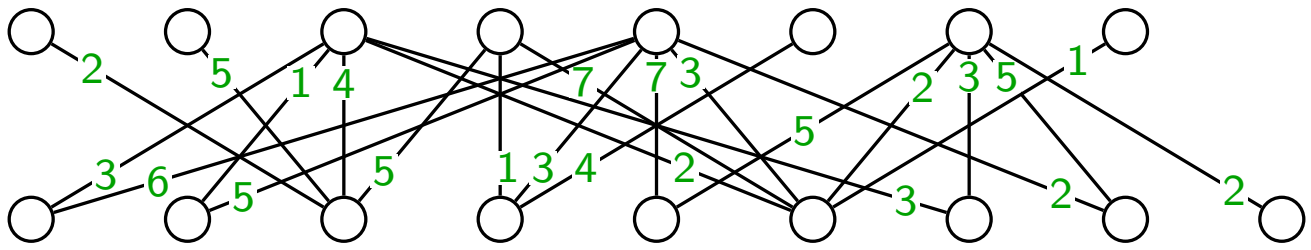

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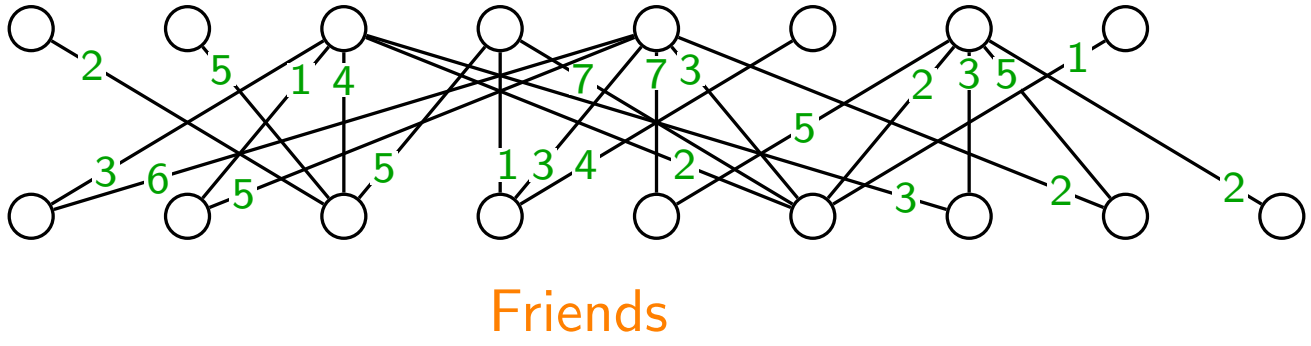

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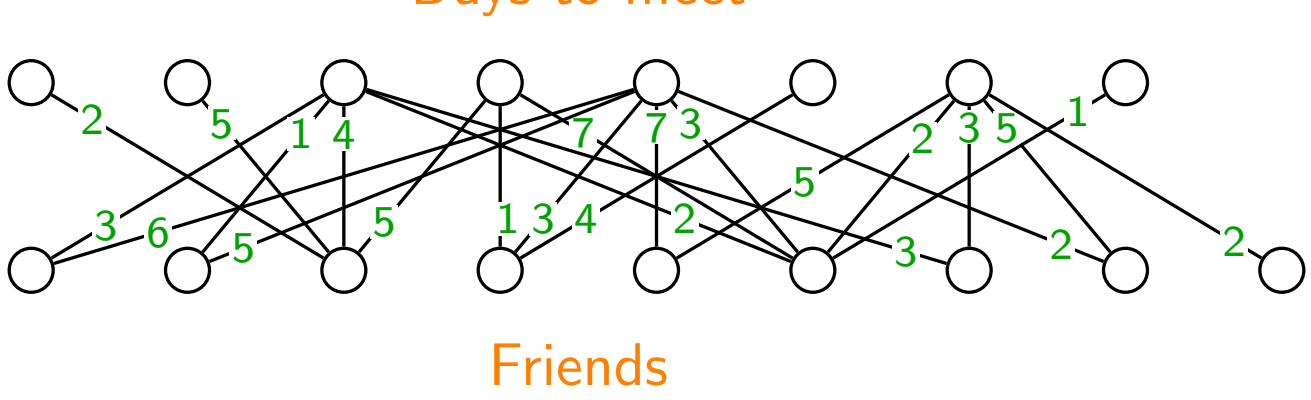

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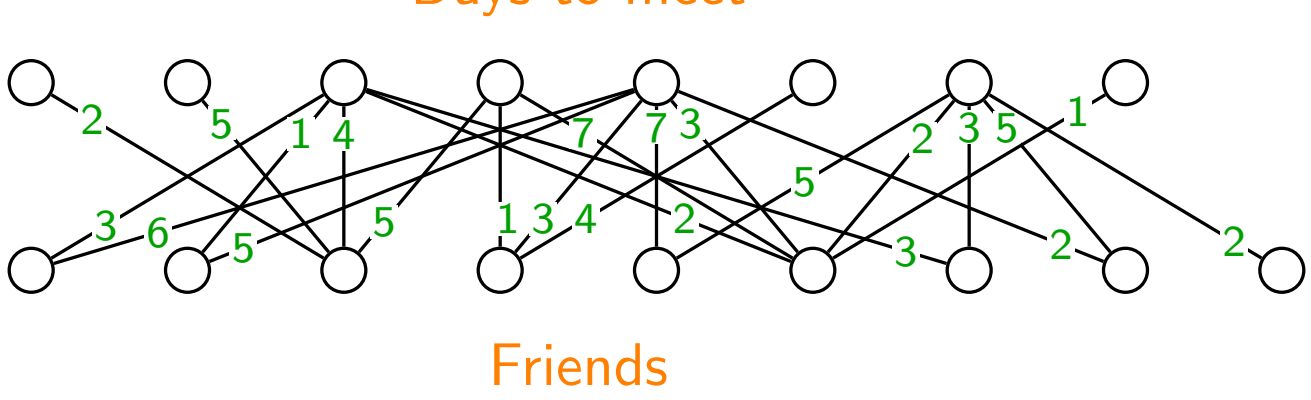

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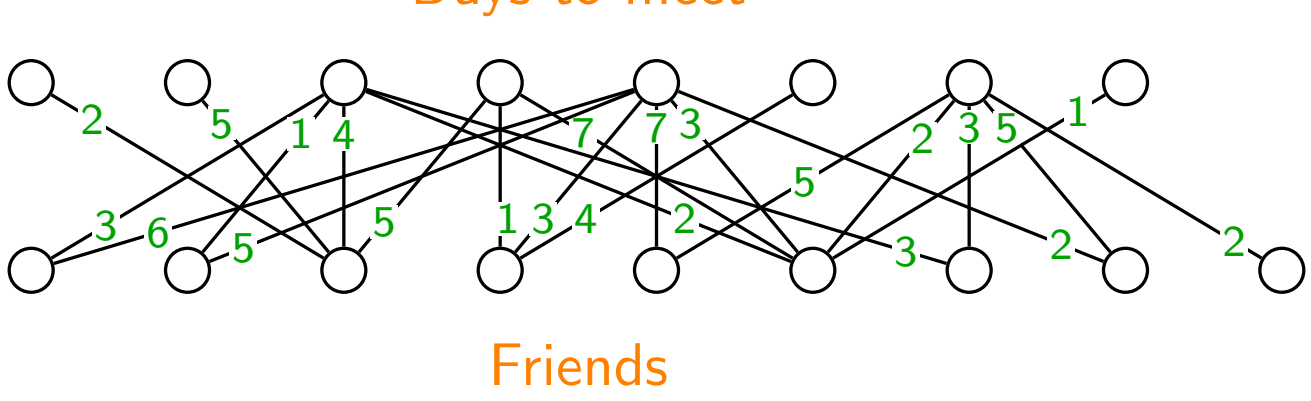

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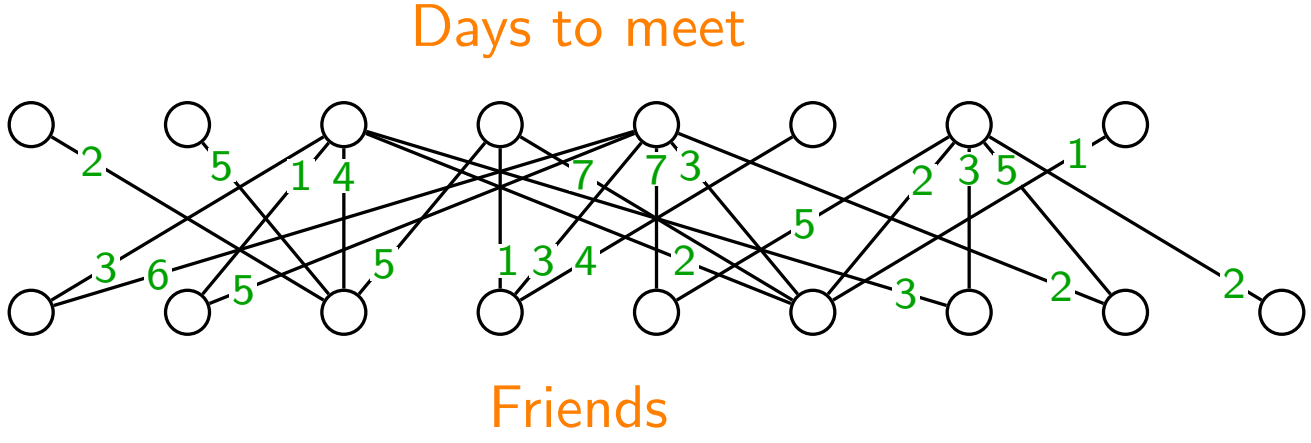

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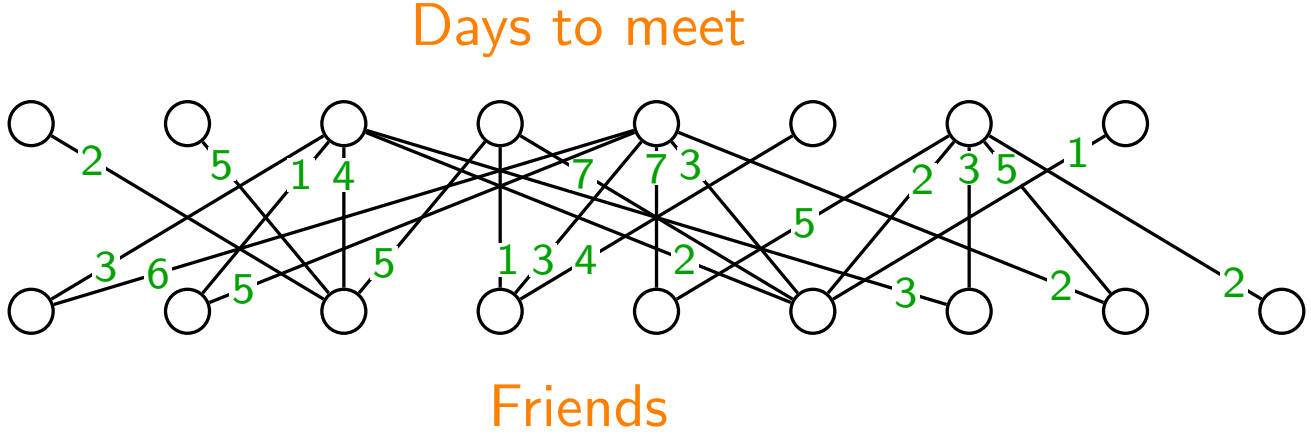

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