Antimagic Labelings of Regular Bipartite Graphs

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Antimagic Labelings

Def. magic labeling: an injection from the edges of G to $\{1, 2, ..., |E|\}$ such that the sum of the labels incident to each vertex is the same

Def. a graph is magic if it has an magic labeling

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Pf. for $\Delta(G) = n - 1$. Let $d(v) = n - 1$. Label $G - v$ arbitrarily. Label the final star in order of partial sum.

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Thm. [Alon et al. 2004] Every complete partite graph other than K_2 is antimagic.

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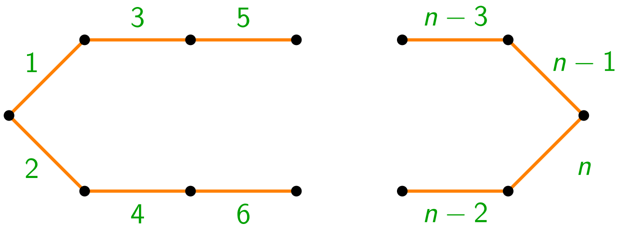

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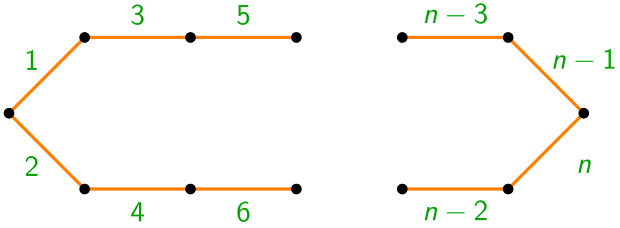

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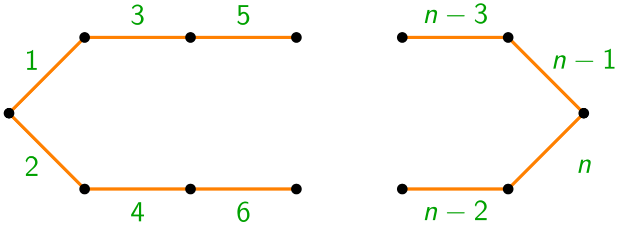

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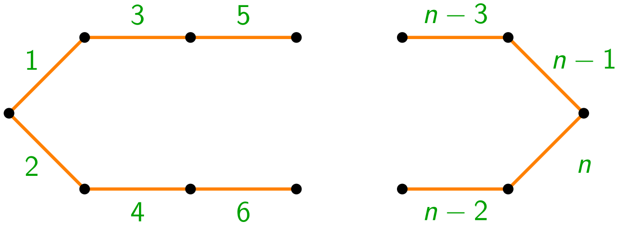

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Today, I'll prove the theorem for $k \geq 5$ odd.

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constructing an antimagic labeling: resolving all potential conflicts

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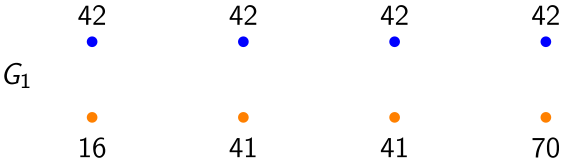

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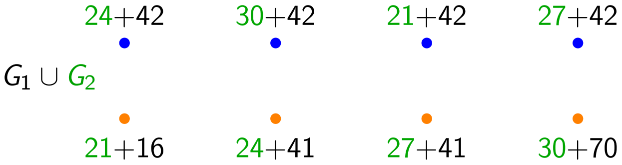

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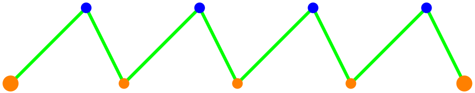

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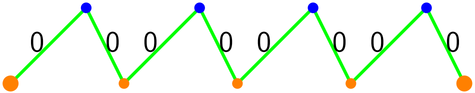

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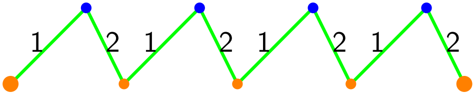

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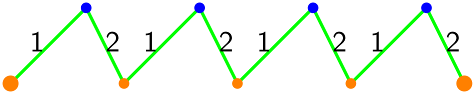

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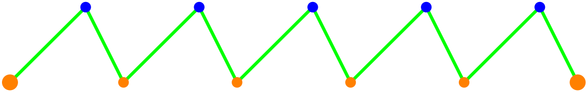

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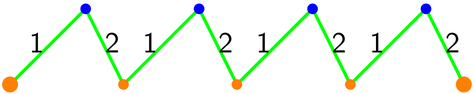

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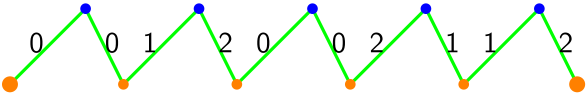

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Lem. Let G be a 3-regular bipartite graph with parts A and B. Let b_i be an ordering of the vertices of B. We can label G with the integers 1 through $3n$ so that at each b_i the sum is $3n + 3i$ and for each i exactly one vertex in A has sum $3n + 3i$.

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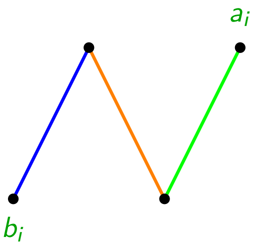

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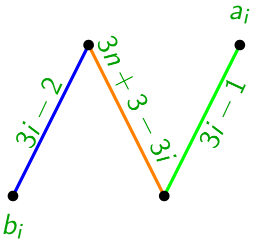

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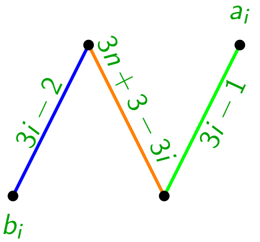

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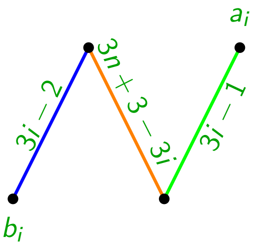

at b_i sum is $(3i - 2) + (3n + 3 - 3j) + (3j - 1) = 3n + 3i$
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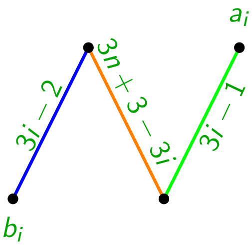

at b_i sum is $(3i-2) + (3n+3-3j) + (3j-1) = 3n+3i$
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Hence, every regular bipartite graph of odd degree ≥ 5 is antimagic.

Thank you!