An Elementary Proof of Bertrand's Postulate

Daniel W. Cranston
Virginia Commonwealth University

Davidson Math Coffee
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So, we need bounds on $f(p)$...

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So we need bounds on $\prod_{p\leq\frac{2n}{3}} p$ in terms of 4^x...

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Pf: If q is max prime $\leq x$, then suffices to show: $\prod_{p \leq q} p \leq 4^{q-1}$, since then $\prod_{p \leq x} p = \prod_{p \leq q} p \leq 4^{q-1} \leq 4^{x-1}$.

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Consider $q = 2m + 1$.

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For every positive n, there exists a prime p s.t. $n < p \leq 2n$.

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