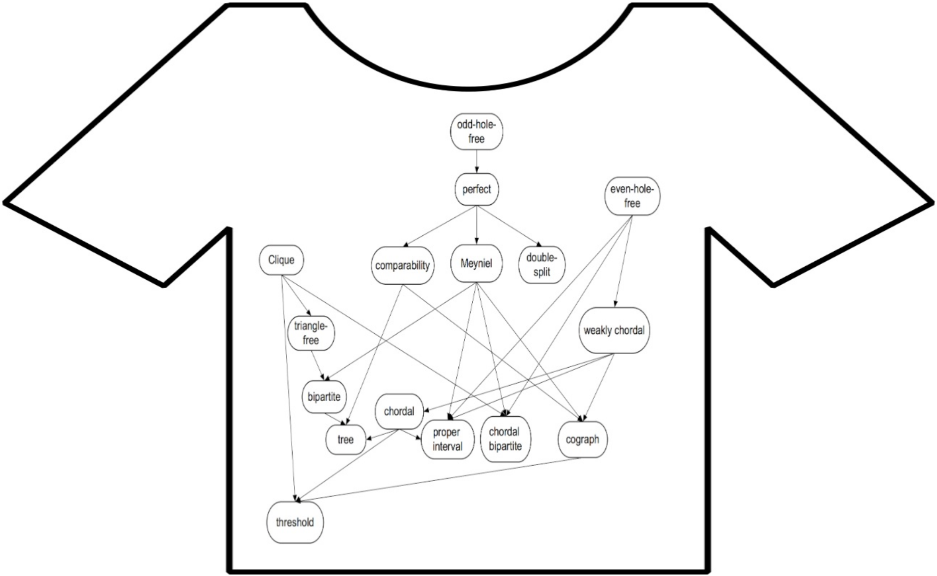
Graphs with $\chi = \Delta$ have big cliques

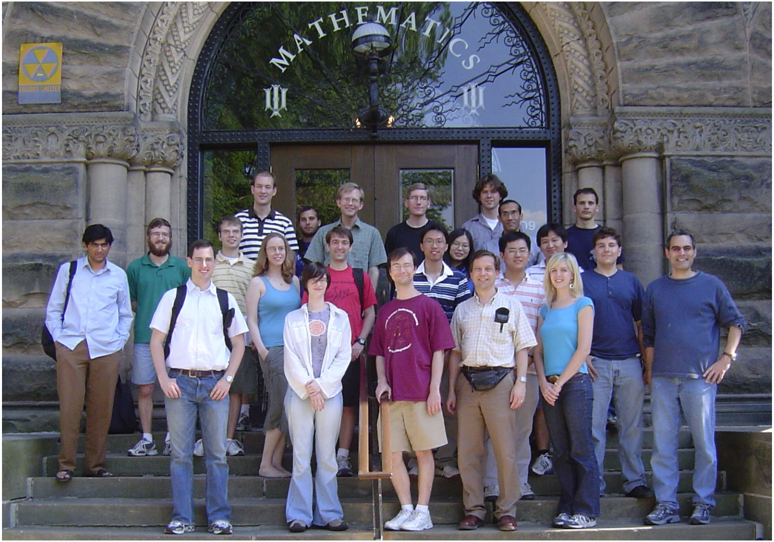
Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

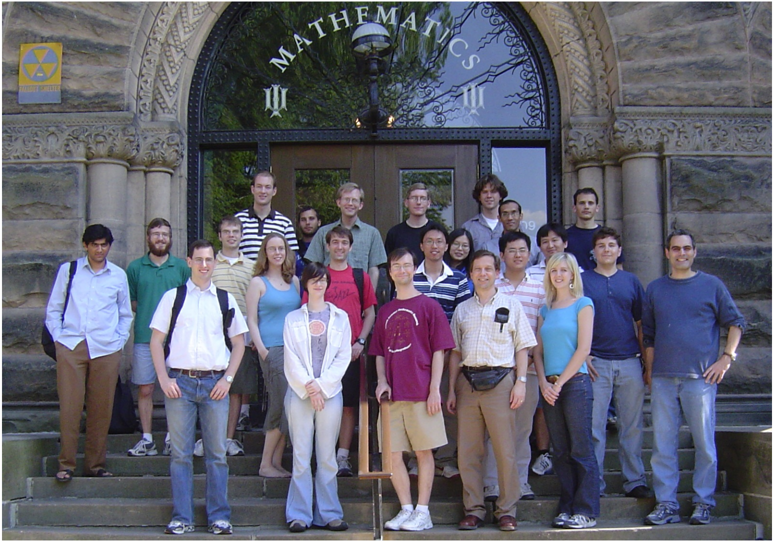
Joint with Landon Rabern
Slides available on my webpage

West Fest
A celebration of Doug's 60th birthday!
20 June 2014



Doug’s big reveal mid-lecture.


FALLOUT SHELTER
MATHEMATICS



You put 40 problems, 30 students, and a few faculty in a room;
mix thoroughly, then wait for papers to precipitate out.

–Doug explaining REGS

Coloring graphs with roughly Δ colors

Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?

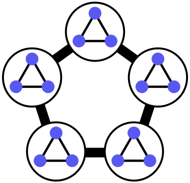
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



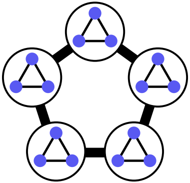
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$

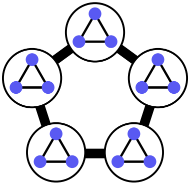
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$, $\omega = 6$

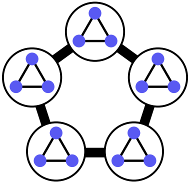
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$, $\omega = 6$, $\alpha = 2$

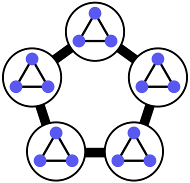
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$, $\omega = 6$, $\alpha = 2$
$\chi = \lceil 15/2 \rceil = 8$

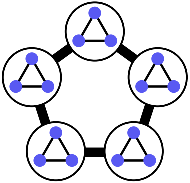
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

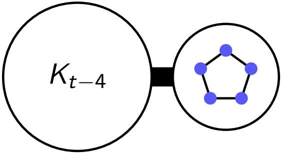
Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$, $\omega = 6$, $\alpha = 2$
$\chi = \lceil 15/2 \rceil = 8$

Why $\Delta - 1$?



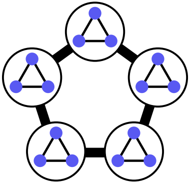
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

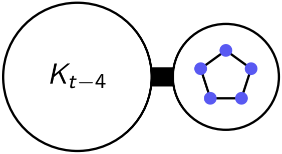
Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$, $\omega = 6$, $\alpha = 2$
$\chi = \lceil 15/2 \rceil = 8$

Why $\Delta - 1$?



$\Delta = t$

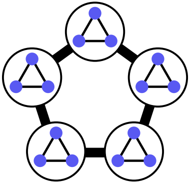
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

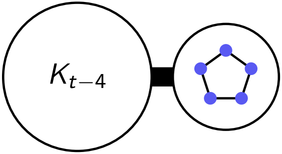
Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$, $\omega = 6$, $\alpha = 2$
$\chi = \lceil 15/2 \rceil = 8$

Why $\Delta - 1$?



$\Delta = t$, $\omega = t - 2$

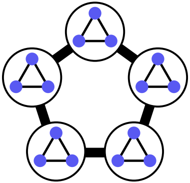
Coloring graphs with roughly Δ colors

Prop: For all G we have $\chi \leq \Delta + 1$.

Thm [Brooks 1941]**:**
If $\Delta \geq 3$ and $\omega \leq \Delta$ then $\chi \leq \Delta$.

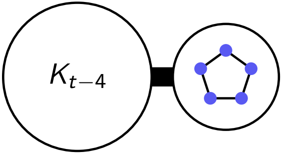
Borodin-Kostochka Conj. (B-K) [1977]**:**
If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

Why $\Delta \geq 9$?



$\Delta = 8$, $\omega = 6$, $\alpha = 2$
$\chi = \lceil 15/2 \rceil = 8$

Why $\Delta - 1$?



$\Delta = t$, $\omega = t - 2$
$\chi = (t - 4) + 3 = t - 1$

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98]

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98] and likely $\Delta \geq 10^{6}$ suffices

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98]
  and likely $\Delta \geq 10^{6}$ suffices
- Finding big cliques: If $\chi = \Delta$,

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98]
  and likely $\Delta \geq 10^6$ suffices

- Finding big cliques: If $\chi = \Delta$,
  - then $\omega \geq \lfloor \frac{\Delta+1}{2} \rfloor$ [Borodin-Kostochka '77]

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98] and likely $\Delta \geq 10^{6}$ suffices
- Finding big cliques: If $\chi = \Delta$,
    - then $\omega \geq \lfloor \frac{\Delta+1}{2} \rfloor$ [Borodin-Kostochka '77]
    - then $\omega \geq \lfloor \frac{2\Delta+1}{3} \rfloor$ [Mozhan '83]

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98] and likely $\Delta \geq 10^6$ suffices
- Finding big cliques: If $\chi = \Delta$,
  - then $\omega \geq \lfloor \frac{\Delta+1}{2} \rfloor$ [Borodin-Kostochka '77]
  - then $\omega \geq \lfloor \frac{2\Delta+1}{3} \rfloor$ [Mozhan '83]
  - then $\omega \geq \Delta - 28$ [Kostochka '80]

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98] and likely $\Delta \geq 10^{6}$ suffices

- Finding big cliques: If $\chi = \Delta$,
  - then $\omega \geq \lfloor \frac{\Delta+1}{2} \rfloor$ [Borodin-Kostochka '77]
  - then $\omega \geq \lfloor \frac{2\Delta+1}{3} \rfloor$ [Mozhan '83]
  - then $\omega \geq \Delta - 28$ [Kostochka '80]
  - then $\omega \geq \Delta - 3$ when $\Delta \geq 31$ [Mozhan '87]

Previous Results

- B-K Conjecture is true for claw-free graphs [C.-Rabern '13]
- B-K Conjecture is true when $\Delta \geq 10^{14}$ [Reed '98] and likely $\Delta \geq 10^{6}$ suffices
- Finding big cliques: If $\chi = \Delta$,
  - then $\omega \geq \lfloor \frac{\Delta+1}{2} \rfloor$ [Borodin-Kostochka '77]
  - then $\omega \geq \lfloor \frac{2\Delta+1}{3} \rfloor$ [Mozhan '83]
  - then $\omega \geq \Delta - 28$ [Kostochka '80]
  - then $\omega \geq \Delta - 3$ when $\Delta \geq 31$ [Mozhan '87]
  - then $\omega \geq \Delta - 3$ when $\Delta \geq 13$ [C.-Rabern '13+]

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Lemma 2: If G has $\chi = \Delta = 13$, then G contains K_{10}.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Lemma 2: If G has $\chi = \Delta = 13$, then G contains K_{10}.

Main Theorem: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Lemma 2: If G has $\chi = \Delta = 13$, then G contains K_{10}.

Main Theorem: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.
Proof: Let G be minimal counterexample. $\Delta \geq 14$ by Lemma 2.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Lemma 2: If G has $\chi = \Delta = 13$, then G contains K_{10}.

Main Theorem: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

Proof: Let G be minimal counterexample. $\Delta \geq 14$ by Lemma 2. If $\omega = \Delta - 4$, then let I be a hitting set expanded to be a maximal independent set; otherwise let I be any maximal independent set.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Lemma 2: If G has $\chi = \Delta = 13$, then G contains K_{10}.

Main Theorem: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

Proof: Let G be minimal counterexample. $\Delta \geq 14$ by Lemma 2.
If $\omega = \Delta - 4$, then let I be a hitting set expanded to be a maximal independent set; otherwise let I be any maximal independent set.

- If $\Delta(G - I) \leq \Delta(G) - 3$, then win by greedy coloring.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Lemma 2: If G has $\chi = \Delta = 13$, then G contains K_{10}.

Main Theorem: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

Proof: Let G be minimal counterexample. $\Delta \geq 14$ by Lemma 2. If $\omega = \Delta - 4$, then let I be a hitting set expanded to be a maximal independent set; otherwise let I be any maximal independent set.

- If $\Delta(G - I) \leq \Delta(G) - 3$, then win by greedy coloring.
- If $\Delta(G - I) = \Delta(G) - 2$, then win by Brooks' Theorem.

Main Theorem

Def: A hitting set is independent set intersecting every maximum clique.

Lemma 1: Every G with $\chi = \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.

Lemma 2: If G has $\chi = \Delta = 13$, then G contains K_{10}.

Main Theorem: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

Proof: Let G be minimal counterexample. $\Delta \geq 14$ by Lemma 2.
If $\omega = \Delta - 4$, then let I be a hitting set expanded to be a maximal independent set; otherwise let I be any maximal independent set.

- If $\Delta(G - I) \leq \Delta(G) - 3$, then win by greedy coloring.
- If $\Delta(G - I) = \Delta(G) - 2$, then win by Brooks' Theorem.
- If $\Delta(G - I) = \Delta(G) - 1$, then $G - I$ is a smaller counterexample, contradiction!

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur,

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \geq 89$ and $\omega = \Delta - 4$ has a hitting set I.

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \geq 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one.

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \le p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \le 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \ge 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one. To form I, choose one vertex from each S_i randomly.

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \geq 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one. To form I, choose one vertex from each S_i randomly. For each edge uv with endpoints u, v in distinct S_i, event E_{uv} is that u, v both chosen for I.

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \geq 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one. To form I, choose one vertex from each S_i randomly. For each edge uv with endpoints u, v in distinct S_i, event E_{uv} is that u, v both chosen for I. $\Pr(E_{uv}) = \frac{1}{|S_u|}\frac{1}{|S_v|} = k^{-2}$.

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \geq 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one. To form I, choose one vertex from each S_i randomly.
For each edge uv with endpoints u, v in distinct S_i, event
E_{uv} is that u, v both chosen for I. $\Pr(E_{uv}) = \frac{1}{|S_u|}\frac{1}{|S_v|} = k^{-2}$.
E_{uv} is independent of all but $2k(\Delta - (k-1)) = 20k$ events.

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \le p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \le 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \ge 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one. To form I, choose one vertex from each S_i randomly.
For each edge uv with endpoints u, v in distinct S_i, event
E_{uv} is that u, v both chosen for I. $\Pr(E_{uv}) = \frac{1}{|S_u|}\frac{1}{|S_v|} = k^{-2}$.
E_{uv} is independent of all but $2k(\Delta - (k-1)) = 20k$ events.
Finally, $4(20k)k^{-2} \le 1$

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \geq 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one. To form I, choose one vertex from each S_i randomly.
For each edge uv with endpoints u, v in distinct S_i, event
E_{uv} is that u, v both chosen for I. $\Pr(E_{uv}) = \frac{1}{|S_u|}\frac{1}{|S_v|} = k^{-2}$.
E_{uv} is independent of all but $2k(\Delta - (k-1)) = 20k$ events.
Finally, $4(20k)k^{-2} \leq 1 \Leftrightarrow k \geq 80$

Random Hitting Sets

Lovász Local Lemma: Suppose we do a random experiment.
Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that

- $\Pr(E_i) \leq p < 1$ for all i, and
- each E_i is mutually independent of all but d events.

If $4dp \leq 1$, then with positive probability no bad events occur, so we win!

Lemma 1': Every G with $\chi = \Delta \geq 89$ and $\omega = \Delta - 4$ has a hitting set I.
Proof: Get disjoint cliques $S_1, S_2, \ldots$ of size $k := \Delta - 9$ so each maximum clique contains one. To form I, choose one vertex from each S_i randomly.
For each edge uv with endpoints u, v in distinct S_i, event
E_{uv} is that u, v both chosen for I. $\Pr(E_{uv}) = \frac{1}{|S_u|}\frac{1}{|S_v|} = k^{-2}$.
E_{uv} is independent of all but $2k(\Delta - (k-1)) = 20k$ events.
Finally, $4(20k)k^{-2} \leq 1 \Leftrightarrow k \geq 80 \Leftrightarrow \Delta \geq 89$.

What next?

What next?

The four-colour theorem is the tip of the iceberg, the thin end of the wedge, and the first cuckoo of Spring. –*William Tutte*

What next?

The four-colour theorem is the tip of the iceberg, the thin end of the wedge, and the first cuckoo of Spring. –*William Tutte*

Reed's Conjecture: $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

What next?

The four-colour theorem is the tip of the iceberg, the thin end of the wedge, and the first cuckoo of Spring. *–William Tutte*

Reed's Conjecture: $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

Theorem (Reed): There exists $\epsilon > 0$ such that $\chi \leq \lceil \epsilon\omega + (1-\epsilon)(\Delta+1) \rceil$.

What next?

The four-colour theorem is the tip of the iceberg, the thin end of the wedge, and the first cuckoo of Spring. *–William Tutte*

Reed's Conjecture: $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

Theorem (Reed): There exists $\epsilon > 0$ such that $\chi \leq \lceil \epsilon\omega + (1-\epsilon)(\Delta+1) \rceil$. Conjectured that $\epsilon = \frac{1}{2}$ works.

In Review

B-K Conj: Every graph with $\chi = \Delta \geq 9$ contains K_Δ.

- If true, then best possible.
- True for claw-free graphs, and also for large Δ.

In Review

B-K Conj: Every graph with $\chi = \Delta \geq 9$ contains K_Δ.

- If true, then best possible.
- True for claw-free graphs, and also for large Δ.

Main Result: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

- Hitting sets reduce to the case $\Delta = 13$.
  - Local Lemma for $\Delta \geq 89$.
  - Smaller Δ are trickier, but it works for $\Delta \geq 14$.
- If $\Delta = 13$, then $\chi \leq 12$ or G has K_{10}.

In Review

B-K Conj: Every graph with $\chi = \Delta \geq 9$ contains K_Δ.

- If true, then best possible.
- True for claw-free graphs, and also for large Δ.

Main Result: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

- Hitting sets reduce to the case $\Delta = 13$.
  - Local Lemma for $\Delta \geq 89$.
  - Smaller Δ are trickier, but it works for $\Delta \geq 14$.
- If $\Delta = 13$, then $\chi \leq 12$ or G has K_{10}.

The Iceberg (Reed's Conj): $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

In Review

B-K Conj: Every graph with $\chi = \Delta \geq 9$ contains K_Δ.

- If true, then best possible.
- True for claw-free graphs, and also for large Δ.

Main Result: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

- Hitting sets reduce to the case $\Delta = 13$.
  - Local Lemma for $\Delta \geq 89$.
  - Smaller Δ are trickier, but it works for $\Delta \geq 14$.
- If $\Delta = 13$, then $\chi \leq 12$ or G has K_{10}.

The Iceberg (Reed's Conj): $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

In Review

B-K Conj: Every graph with $\chi = \Delta \geq 9$ contains K_Δ.

- If true, then best possible.
- True for claw-free graphs, and also for large Δ.

Main Result: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

- Hitting sets reduce to the case $\Delta = 13$.
  - Local Lemma for $\Delta \geq 89$.
  - Smaller Δ are trickier, but it works for $\Delta \geq 14$.
- If $\Delta = 13$, then $\chi \leq 12$ or G has K_{10}.

The Iceberg (Reed's Conj): $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

In Review

B-K Conj: Every graph with $\chi = \Delta \geq 9$ contains K_Δ.

- If true, then best possible.
- True for claw-free graphs, and also for large Δ.

Main Result: Every graph with $\chi = \Delta \geq 13$ contains $K_{\Delta-3}$.

- Hitting sets reduce to the case $\Delta = 13$.
  - Local Lemma for $\Delta \geq 89$.
  - Smaller Δ are trickier, but it works for $\Delta \geq 14$.
- If $\Delta = 13$, then $\chi \leq 12$ or G has K_{10}.

The Iceberg (Reed's Conj): $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$.

Clubs and Clubhouses

Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \ldots, V_4$ and a vertex v with certain properties.

Clubs and Clubhouses

Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \ldots, V_4$ and a vertex v with certain properties.
For each V_i, components of $G[V_i]$ are clubs meeting in clubhouse V_i.

Clubs and Clubhouses

Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \ldots, V_4$ and a vertex v with certain properties.
For each V_i, components of $G[V_i]$ are clubs meeting in clubhouse V_i.

- The club R containing v is a K_4.

Clubs and Clubhouses

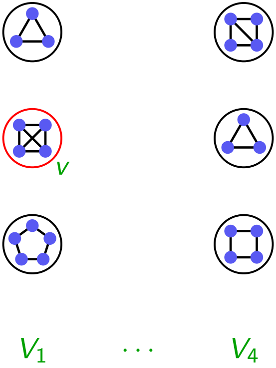
Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \ldots, V_4$ and a vertex v with certain properties.
For each V_i, components of $G[V_i]$ are clubs meeting in clubhouse V_i.

- The club R containing v is a K_4.
- All other clubs are 3-colorable.

Clubs and Clubhouses

Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \ldots, V_4$ and a vertex v with certain properties. For each V_i, components of $G[V_i]$ are clubs meeting in clubhouse V_i.

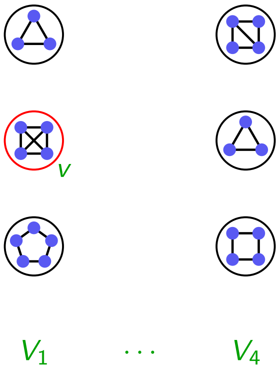
- The club R containing v is a K_4.
- All other clubs are 3-colorable.



Clubs and Clubhouses

Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \dots, V_4$ and a vertex v with certain properties. For each V_i, components of $G[V_i]$ are clubs meeting in clubhouse V_i.

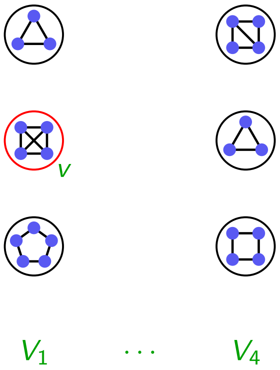
- The club R containing v is a K_4.
- All other clubs are 3-colorable.
- For $w \in V(R)$ and $j \in \{1, \dots, 4\}$: If $d_{V_j}(w) = 3$, then $G[V_j + w]$ has a K_4 component.



Clubs and Clubhouses

Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \ldots, V_4$ and a vertex v with certain properties. For each V_i, components of $G[V_i]$ are clubs meeting in clubhouse V_i.

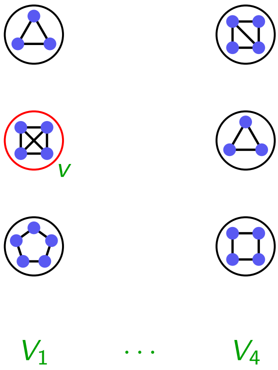
- The club R containing v is a K_4.
- All other clubs are 3-colorable.
- For $w \in V(R)$ and $j \in \{1, \ldots, 4\}$: If $d_{V_j}(w) = 3$, then $G[V_j + w]$ has a K_4 component.
- For $w \in V(R)$ and $j \in \{1, \ldots, 4\}$: If w has 2 neighbors in club S of clubhouse V_i, then $\chi(S + w) = 4$.



Clubs and Clubhouses

Def: A Mozhan Partition of a graph G with $\Delta = 13$ is a partition of V into clubhouses $V_1, \ldots, V_4$ and a vertex v with certain properties. For each V_i, components of $G[V_i]$ are clubs meeting in clubhouse V_i.

- The club R containing v is a K_4.
- All other clubs are 3-colorable.
- For $w \in V(R)$ and $j \in \{1, \ldots, 4\}$: If $d_{V_j}(w) = 3$, then $G[V_j + w]$ has a K_4 component.
- For $w \in V(R)$ and $j \in \{1, \ldots, 4\}$: If w has 2 neighbors in club S of clubhouse V_i, then $\chi(S + w) = 4$.



Lem: Every Δ-critical graph with $\Delta = 13$ has a Mozhan partition.

The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options.

The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other.

The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

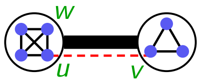
Claim 1: No clubs become (in)complete to each other.

The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.

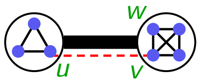


The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.

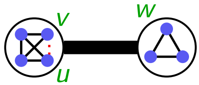


The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



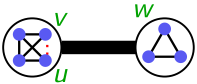
The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.

Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.



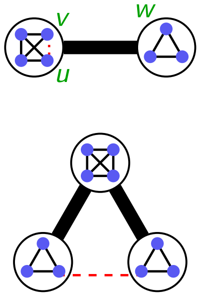
The Vertex Shuffle

Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.

Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.

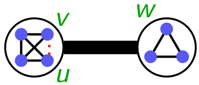


The Vertex Shuffle

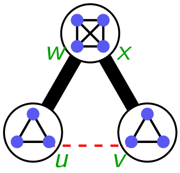
Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.

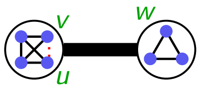


The Vertex Shuffle

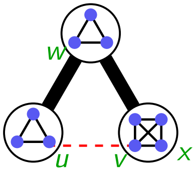
Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.

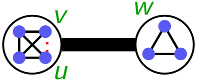


The Vertex Shuffle

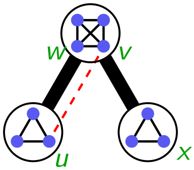
Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.

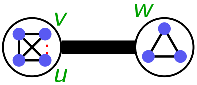


The Vertex Shuffle

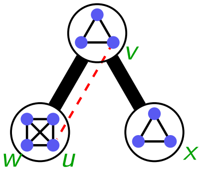
Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.

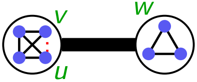


The Vertex Shuffle

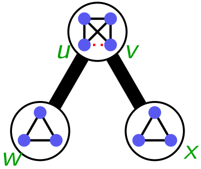
Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.

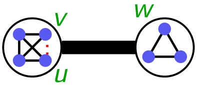


The Vertex Shuffle

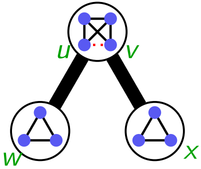
Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.



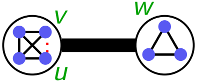
Claim 3: Each club is active at most three times.

The Vertex Shuffle

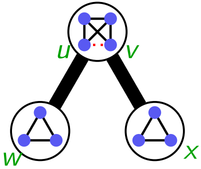
Lemma 2: If G has $\chi = \Delta = 13$, then G has a K_{10}.

Pf Idea: Start with a Mozhan partition of G. Repeatedly send a member of the active K_4 to a clubhouse where it has only 3 neighbors (forming a new K_4), always at least 2 options. Move each vertex only once. Never move between clubs joined to each other. Find either a 12-coloring or K_{10}.

Claim 1: No clubs become (in)complete to each other.



Claim 2: If G has K_4 joined to K_3's in two other clubhouses, then G has K_{10}.



Claim 3: Each club is active at most three times.

Claim 4: G contains K_{10}.