Circular Coloring of Planar Graphs

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Joint with Jiaao Li

CanaDAM, SFU
29 May 2019

Circular Coloring of Planar Graphs

Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.

Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$. That is, f preserves edges. In this case, $G \to H$.

Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$. That is, f preserves edges. In this case, $G \to H$.

Obs: H-coloring generalizes t-coloring ($H = K_t$).

Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

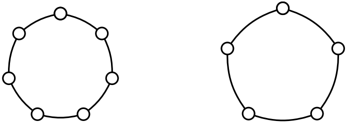
Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$. That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring.

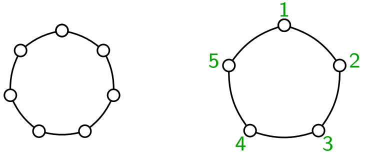


Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring.

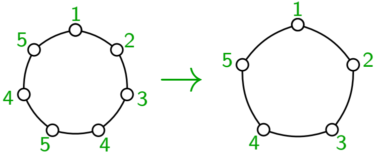


Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring. $C_7 \to C_5$

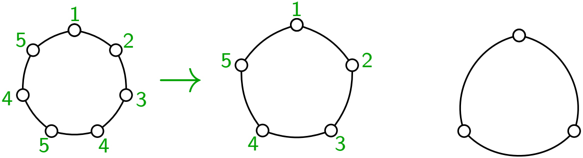


Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring. $C_7 \to C_5$

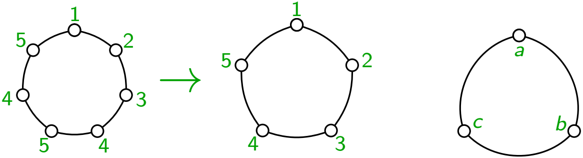


Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring. $C_7 \to C_5$

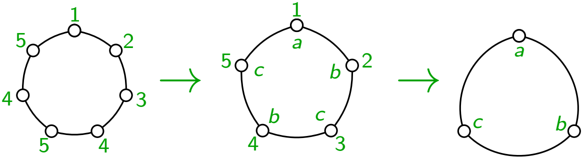


Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring. $C_7 \to C_5$ and $C_5 \to C_3$

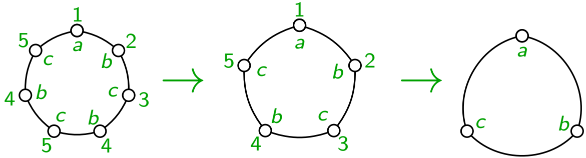


Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring. $C_7 \to C_5$ and $C_5 \to C_3$, so $C_7 \to C_3$.

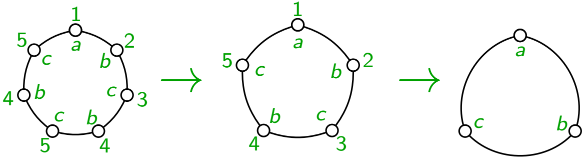


Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring. $C_7 \to C_5$ and $C_5 \to C_3$, so $C_7 \to C_3$.



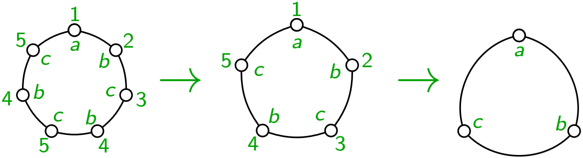
Proving $G \to C_7$ is stronger than $G \to C_5$,

Circular Coloring of Planar Graphs

Grötzsch's Thm: Every triangle-free planar graph is 3-colorable.

Def: For graphs G and H, we can H-color G if there exists $f : V(G) \to V(H)$ such that $f(v)f(w) \in E(H)$ for all $vw \in E(G)$.
That is, f preserves edges. In this case, $G \to H$.
Obs: H-coloring generalizes t-coloring ($H = K_t$).
Obs: Closed walks map to closed walks.

Ex: Circular Coloring. $C_7 \to C_5$ and $C_5 \to C_3$, so $C_7 \to C_3$.



Proving $G \to C_7$ is stronger than $G \to C_5$, since $C_5 \not\to C_7$.

The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$.

The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

Prop: $g(t) \leq 10t$

The Bigger Picture

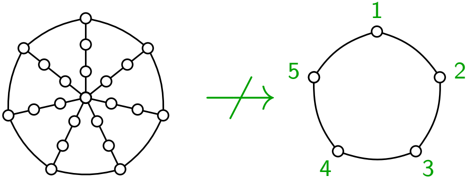
Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

Prop: $g(t) \leq 10t$ and [Devos] $g(t) \geq 4t$.

The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

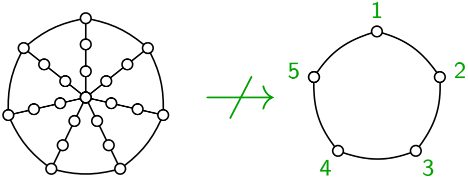
Prop: $g(t) \leq 10t$ and [Devos] $g(t) \geq 4t$.



The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

Prop: $g(t) \leq 10t$ and [Devos] $g(t) \geq 4t$.

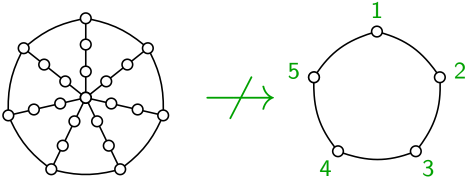


Planar Circular Coloring Conj: If G is planar with girth at least $4t$, then $G \to C_{2t+1}$. That is, $g(t) = 4t$.

The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

Prop: $g(t) \leq 10t$ and [Devos] $g(t) \geq 4t$.



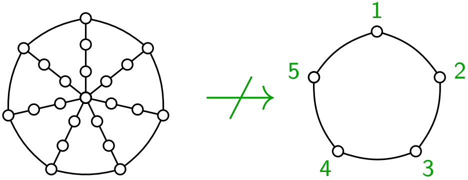
Planar Circular Coloring Conj: If G is planar with girth at least $4t$, then $G \to C_{2t+1}$. That is, $g(t) = 4t$.

Breakthrough [Lovasz–Thomassen–Wu–Zhang '13]**:** $g(t) \leq 6t$.

The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

Prop: $g(t) \leq 10t$ and [Devos] $g(t) \geq 4t$.



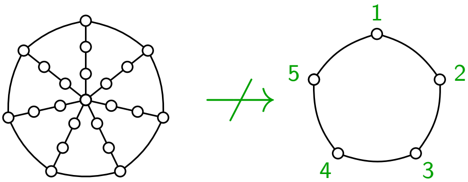
Planar Circular Coloring Conj: If G is planar with girth at least $4t$, then $G \to C_{2t+1}$. That is, $g(t) = 4t$.

Breakthrough [Lovasz–Thomassen–Wu–Zhang '13]**:** $g(t) \leq 6t$.
Our results: $g(2) \leq 10$ and $g(3) \leq 16$.

The Bigger Picture

Big Goal: For each $t \in \mathbb{Z}^+$, find min $g(t)$ s.t. if G is planar with girth at least $g(t)$, then $G \to C_{2t+1}$. Since not all planar graphs are 3-colorable, Grötzsch's Theorem implies $g(1) = 4$.

Prop: $g(t) \leq 10t$ and [Devos] $g(t) \geq 4t$.



Planar Circular Coloring Conj: If G is planar with girth at least $4t$, then $G \to C_{2t+1}$. That is, $g(t) = 4t$.

Breakthrough [Lovasz–Thomassen–Wu–Zhang '13]**:** $g(t) \leq 6t$.

Our results: $g(2) \leq 10$ and $g(3) \leq 16$. That is:

(A) If G is planar with girth ≥ 10, then $G \to C_5$.

(B) If G is planar with girth ≥ 16, then $G \to C_7$.

History

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]
- $g(t) \leq \frac{20t-2}{3}$ [Borodin–Kim–Kostochka–West '04]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]
- $g(t) \leq \frac{20t-2}{3}$ [Borodin–Kim–Kostochka–West '04]
- $g_o(2) \leq 11$ [Dvořák–Postle '17]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]
- $g(t) \leq \frac{20t-2}{3}$ [Borodin–Kim–Kostochka–West '04]
- $g_o(2) \leq 11$ [Dvořák–Postle '17]
- $g_o(3) \leq 17$ [Postle–Smith-Roberge '19+]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]
- $g(t) \leq \frac{20t-2}{3}$ [Borodin–Kim–Kostochka–West '04]
- $g_o(2) \leq 11$ [Dvořák–Postle '17]
- $g_o(3) \leq 17$ [Postle–Smith-Roberge '19+]

- **Conj:** Every $4t$-edge-connected graph has mod $(2t+1)$-orientation. [Jaeger '84]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]
- $g(t) \leq \frac{20t-2}{3}$ [Borodin–Kim–Kostochka–West '04]
- $g_o(2) \leq 11$ [Dvořák–Postle '17]
- $g_o(3) \leq 17$ [Postle–Smith-Roberge '19+]

- **Conj:** Every $4t$-edge-connected graph has mod $(2t+1)$-orientation. [Jaeger '84]
- **Conj:** Every $4t+1$-odd-edge-connected graph has mod $(2t+1)$-orientation. [Zhang '02]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]
- $g(t) \leq \frac{20t-2}{3}$ [Borodin–Kim–Kostochka–West '04]
- $g_o(2) \leq 11$ [Dvořák–Postle '17]
- $g_o(3) \leq 17$ [Postle–Smith-Roberge '19+]

- **Conj:** Every $4t$-edge-connected graph has mod $(2t+1)$-orientation. [Jaeger '84]
- **Conj:** Every $4t+1$-odd-edge-connected graph has mod $(2t+1)$-orientation. [Zhang '02]
- **Thm:** Every $6t+1$-odd-edge-connected graph has mod $(2t+1)$-orientation. [Lovász–Thomassen–Wu–Zhang '13]

History

- $g(t) \leq 10t - 4$ [Nesetril–Zhu '96]
- Folding Lemma [Klostermeyer–Zhang '00]
- $g_o(t) \leq 8t - 3$ [Zhu '01]
- $g(t) \leq \frac{20t-2}{3}$ [Borodin–Kim–Kostochka–West '04]
- $g_o(2) \leq 11$ [Dvořák–Postle '17]
- $g_o(3) \leq 17$ [Postle–Smith-Roberge '19+]

- **Conj:** Every $4t$-edge-connected graph has mod $(2t+1)$-orientation. [Jaeger '84]
- **Conj:** Every $4t+1$-odd-edge-connected graph has mod $(2t+1)$-orientation. [Zhang '02]
- **Thm:** Every $6t+1$-odd-edge-connected graph has mod $(2t+1)$-orientation. [Lovász–Thomassen–Wu–Zhang '13]
- **Thm:** Jaeger's conjecture is false for every $t \geq 3$. [Han–Li–Wu–Zhang '18]

Orientations

Def: Let G^* denote planar dual of G.

Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \mod k$ for all $v \in V(G)$.

Orientations

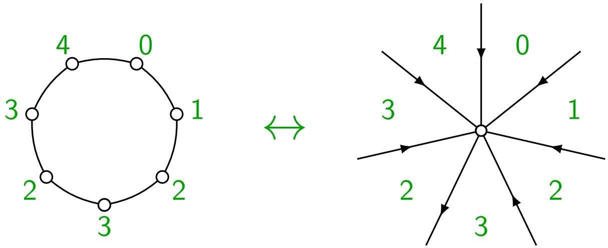
Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \mod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.



Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \mod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$.

Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\bmod\ 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.

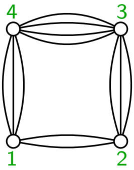
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



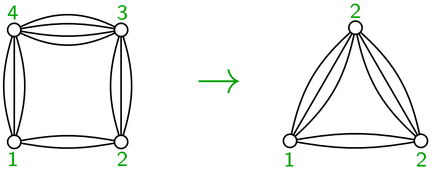
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



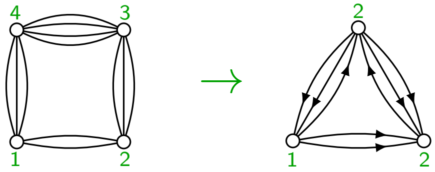
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



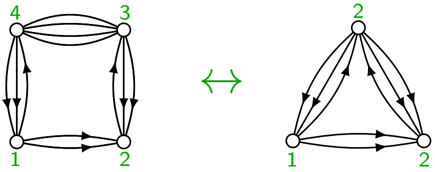
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



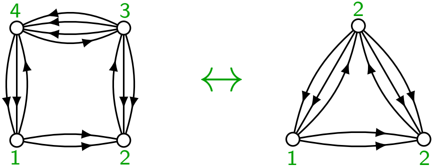
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



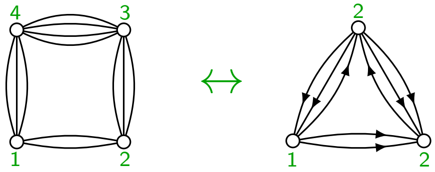
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 5$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



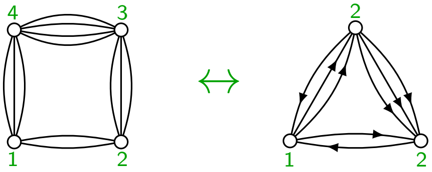
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\bmod\ 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



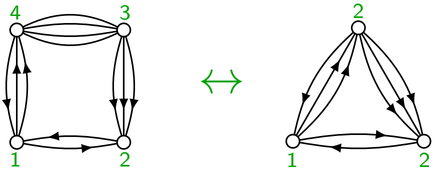
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



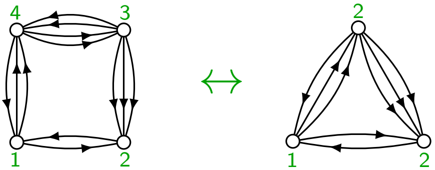
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \mod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



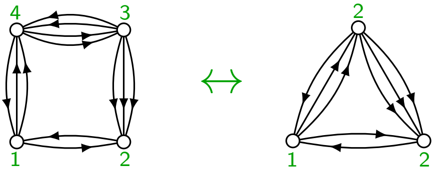
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\bmod\ 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



Def: strongly $\mathbb{Z}_5$-connected: achieves all boundaries.

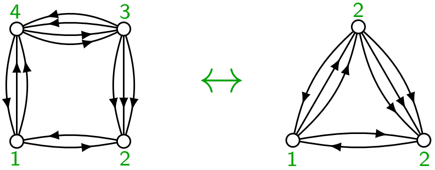
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\bmod\ 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



Def: strongly $\mathbb{Z}_5$-connected: achieves all boundaries.

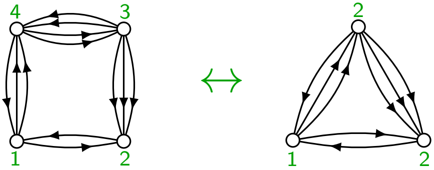
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

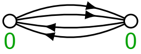
Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



Def: strongly $\mathbb{Z}_5$-connected: achieves all boundaries.



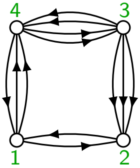
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

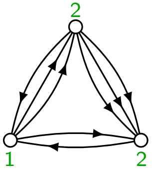
Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

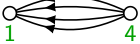
Key Idea: Generality allows induction.



$\leftrightarrow$



Def: strongly $\mathbb{Z}_5$-connected: achieves all boundaries.



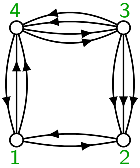
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod k$ for all $v \in V(G)$.

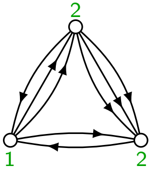
Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\bmod 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

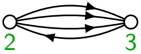
Key Idea: Generality allows induction.



$\leftrightarrow$



Def: strongly $\mathbb{Z}_5$-connected: achieves all boundaries.



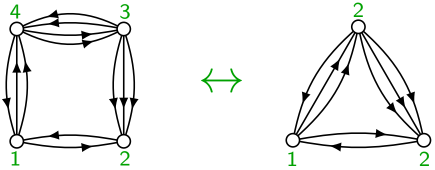
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

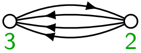
Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\text{mod } 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



Def: strongly $\mathbb{Z}_5$-connected: achieves all boundaries.



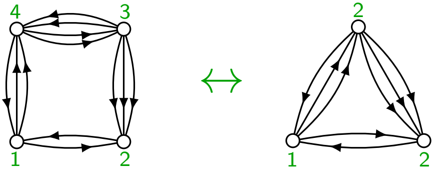
Orientations

Def: Let G^* denote planar dual of G. A (mod k)-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv 0 \bmod\ k$ for all $v \in V(G)$.

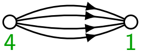
Prop: $G \to C_{2t+1}$ iff G^* has a (mod $2t+1$)-orientation.

Def: A $\mathbb{Z}_5$-boundary β for G is a function $\beta : V(G) \to \mathbb{Z}_5$ s.t. $\sum_{v \in V(G)} \beta(v) \equiv 0 (\bmod\ 5)$. A β-orientation of G is orientation D s.t. $d_D^+(v) - d_D^-(v) \equiv \beta(v)$ for all $v \in V(G)$.

Key Idea: Generality allows induction.



Def: strongly $\mathbb{Z}_5$-connected: achieves all boundaries.

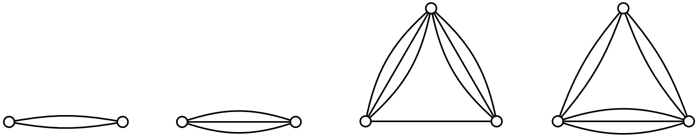


Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.

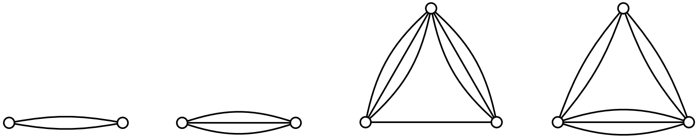
Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.

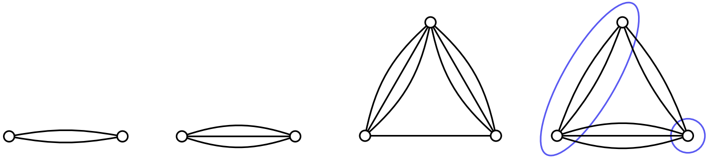


Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



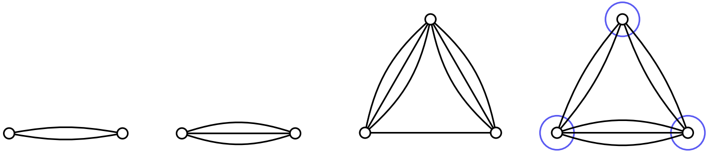
$5 + 5 - 22 + 19 = 7$

Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



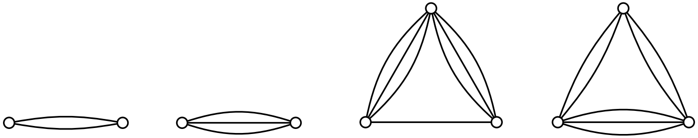
$4 + 5 + 5 - 33 + 19 = 0$

Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



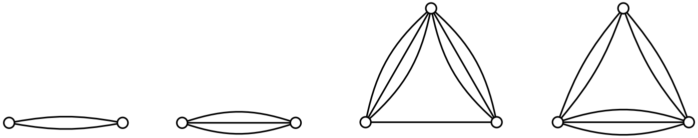
Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



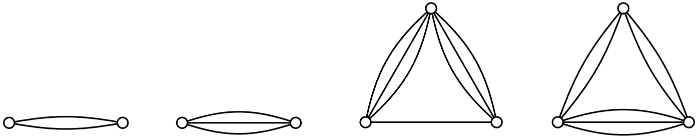
Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$,

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



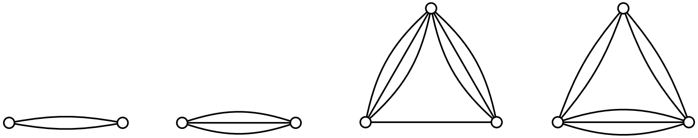
Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$,

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



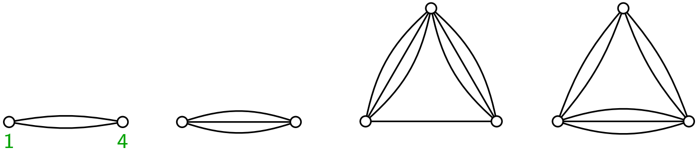
Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



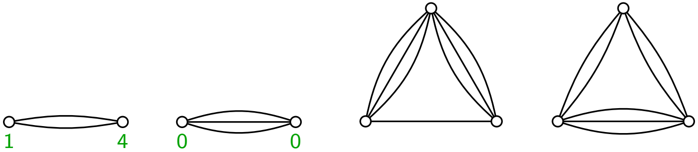
Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



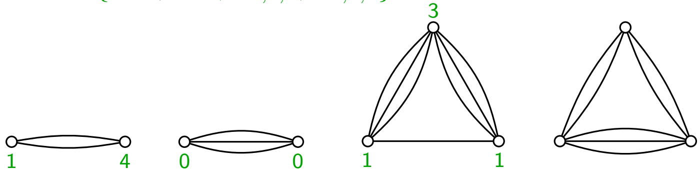
Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



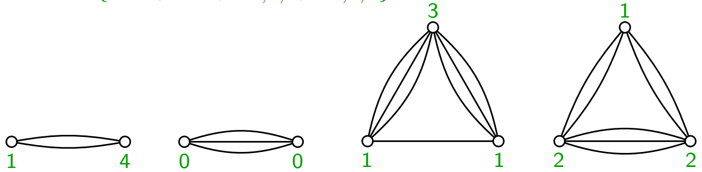
Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



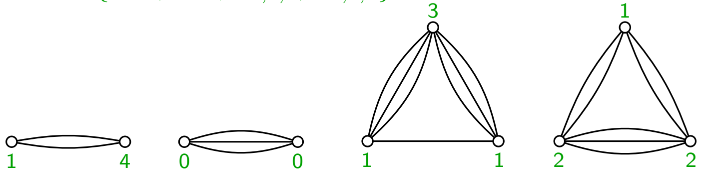
Def: For a partition $\mathcal{P} = \{P_1, \dots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

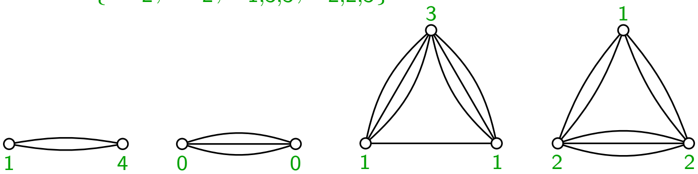
$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Cor: If G^* is an 11-edge-connected planar graph, then G^* is strongly $\mathbb{Z}_5$-connected.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

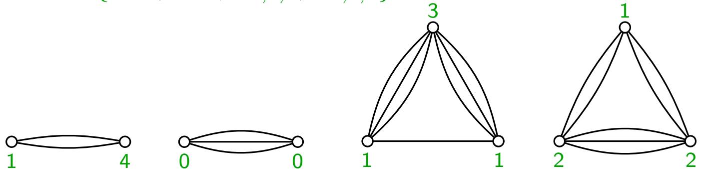
$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Cor: If G^* is an 11-edge-connected planar graph, then G^* is strongly $\mathbb{Z}_5$-connected. So if G has girth ≥ 11, then $G \to C_5$.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

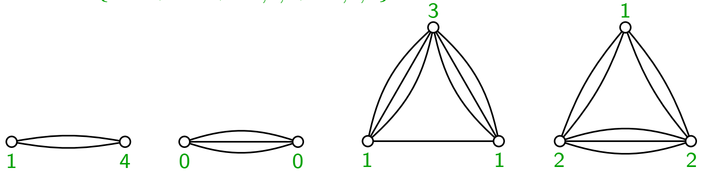
Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Cor: If G^* is an 11-edge-connected planar graph, then G^* is strongly $\mathbb{Z}_5$-connected. So if G has girth ≥ 11, then $G \to C_5$.

Pf: Fix a partition $\mathcal{P}$. Since G^* is 11-edge-connected, $d(P_i) \geq 11$ for each i.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

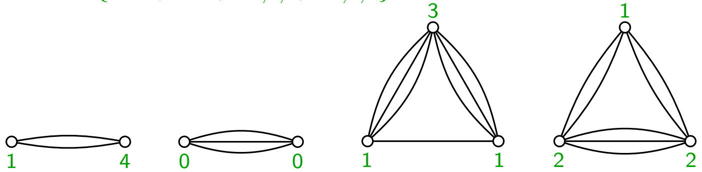
Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Cor: If G^* is an 11-edge-connected planar graph, then G^* is strongly $\mathbb{Z}_5$-connected. So if G has girth ≥ 11, then $G \to C_5$.

Pf: Fix a partition $\mathcal{P}$. Since G^* is 11-edge-connected, $d(P_i) \geq 11$ for each i. So $\sum_{i=1}^{t} d(P_i) - 11t + 19 \geq 19$.

Main Theorem

Def: $\mathcal{T} = \{2K_2, 3K_2, T_{1,3,3}, T_{2,2,3}\}$.



Def: For a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let

$$w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19.$$

Main Thm: Let G be planar. If $w(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ of $V(G)$, and no $\mathcal{P}$ has $G/\mathcal{P} \in \mathcal{T}$, then G is strongly $\mathbb{Z}_5$-connected.

Cor: If G^* is an 11-edge-connected planar graph, then G^* is strongly $\mathbb{Z}_5$-connected. So if G has girth ≥ 11, then $G \to C_5$.

Pf: Fix a partition $\mathcal{P}$. Since G^* is 11-edge-connected, $d(P_i) \geq 11$ for each i. So $\sum_{i=1}^{t} d(P_i) - 11t + 19 \geq 19$. Since each $H \in \mathcal{T}$ is not 11-edge-connected, we are done by the Main Theorem.

Strongly $\mathbb{Z}_5$-connected Graphs

Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?

Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?

Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.

Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?

Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow$ 4 edge-disjoint spanning trees.

Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?
Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.
Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Obs: Why are graphs not strongly $\mathbb{Z}_5$-connected?

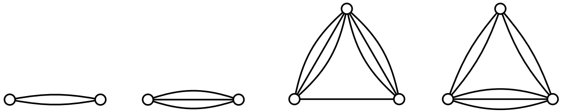
Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?

Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.

Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Obs: Why are graphs not strongly $\mathbb{Z}_5$-connected?



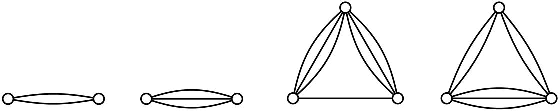
Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?

Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.

Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Obs: Why are graphs not strongly $\mathbb{Z}_5$-connected? Too few edges.



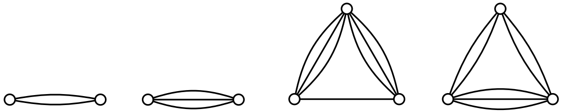
Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?

Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.

Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Obs: Why are graphs not strongly $\mathbb{Z}_5$-connected? Too few edges.

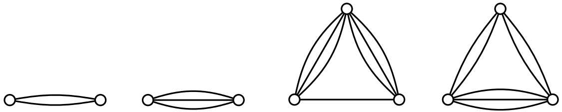


Lem: Small strongly $\mathbb{Z}_5$-connected graphs.

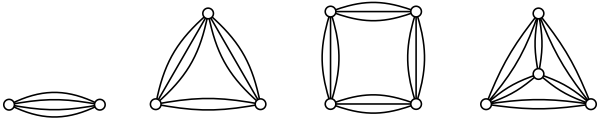
Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?
Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.
Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Obs: Why are graphs not strongly $\mathbb{Z}_5$-connected? Too few edges.



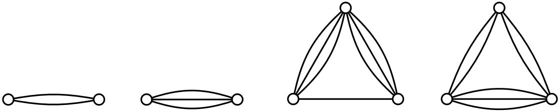
Lem: Small strongly $\mathbb{Z}_5$-connected graphs.



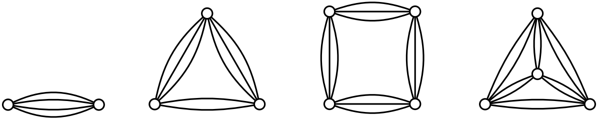
Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?
Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.
Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Obs: Why are graphs not strongly $\mathbb{Z}_5$-connected? Too few edges.



Lem: Small strongly $\mathbb{Z}_5$-connected graphs.



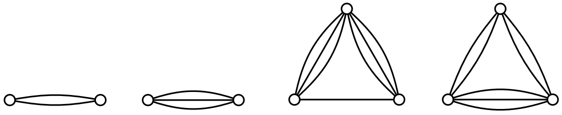
Strongly $\mathbb{Z}_5$-connected Graphs

Ques: Why $w(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$?

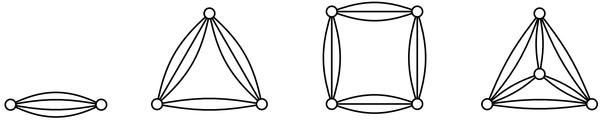
Lem: Strongly $\mathbb{Z}_5$-connected $\Rightarrow 4$ edge-disjoint spanning trees.

Thm [Tutte; Nash-Williams '60s]**:** G has k edge-disjoint spanning tree iff $\sum_{i=1}^{t} d(P_i) - 2tk + 2k \geq 0$ for all $\{P_1, \ldots, P_t\}$ of $V(G)$.

Obs: Why are graphs not strongly $\mathbb{Z}_5$-connected? Too few edges.

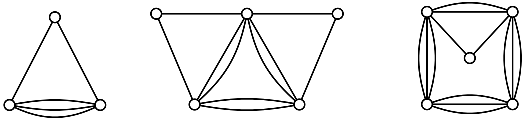


Lem: Small strongly $\mathbb{Z}_5$-connected graphs. All forbidden in G.



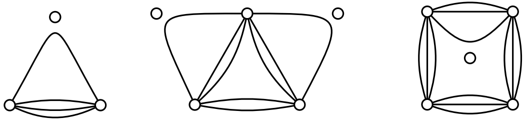
Simulating Edges and the Gap Lemma

Key Idea: Show the following subgraphs are forbidden in G.



Simulating Edges and the Gap Lemma

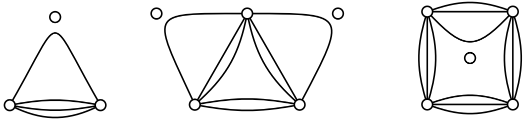
Key Idea: Show the following subgraphs are forbidden in G.



Pf Idea: Simulate one or two edges of a strongly $\mathbb{Z}_5$-connected graph H by a pair of edges with a common endpt.

Simulating Edges and the Gap Lemma

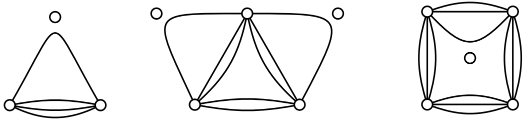
Key Idea: Show the following subgraphs are forbidden in G.



Pf Idea: Simulate one or two edges of a strongly $\mathbb{Z}_5$-connected graph H by a pair of edges with a common endpt. Contract H, find good orienation by minimality, extend to G.

Simulating Edges and the Gap Lemma

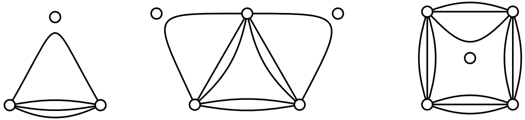
Key Idea: Show the following subgraphs are forbidden in G.



Pf Idea: Simulate one or two edges of a strongly $\mathbb{Z}_5$-connected graph H by a pair of edges with a common endpt. Contract H, find good orienation by minimality, extend to G.
Problem: Simulating edges of H reduces weights in G/H.

Simulating Edges and the Gap Lemma

Key Idea: Show the following subgraphs are forbidden in G.



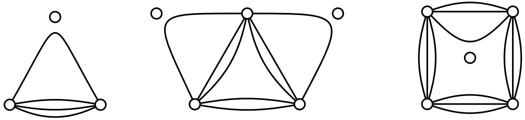
Pf Idea: Simulate one or two edges of a strongly $\mathbb{Z}_5$-connected graph H by a pair of edges with a common endpt. Contract H, find good orienation by minimality, extend to G.
Problem: Simulating edges of H reduces weights in G/H.

Gap Lem: Fix a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$.

- If $|\mathcal{P}| < |G|$, then $w(\mathcal{P}) \geq 5$.
- If $1 < |\mathcal{P}| < |G| - 1$, then $w(\mathcal{P}) \geq 8$.

Simulating Edges and the Gap Lemma

Key Idea: Show the following subgraphs are forbidden in G.



Pf Idea: Simulate one or two edges of a strongly $\mathbb{Z}_5$-connected graph H by a pair of edges with a common endpt. Contract H, find good orienation by minimality, extend to G.
Problem: Simulating edges of H reduces weights in G/H.

Gap Lem: Fix a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$.

- If $|\mathcal{P}| < |G|$, then $w(\mathcal{P}) \geq 5$.
- If $1 < |\mathcal{P}| < |G| - 1$, then $w(\mathcal{P}) \geq 8$.

Solution: Each simulated edge drops weight by at most 4. So if $w(\mathcal{P}) \geq 8$ before simulating two edges, then $w(\mathcal{P}) \geq 0$ afterward.

Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph.

Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph. Each face f gets charge $\ell(f)$.

Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph. Each face f gets charge $\ell(f)$.
Since $0 \leq w(G) = 2|E(G)| - 11|G| + 19$, Euler's gives

Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph. Each face f gets charge $\ell(f)$.
Since $0 \leq w(G) = 2|E(G)| - 11|G| + 19$, Euler's gives

$$\sum_{f \in F(G)} \ell(f) = 2|E(G)| \leq \frac{22}{9}|F(G)| - \frac{2}{3}.$$

Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph. Each face f gets charge $\ell(f)$.
Since $0 \leq w(G) = 2|E(G)| - 11|G| + 19$, Euler's gives

$$\sum_{f \in F(G)} \ell(f) = 2|E(G)| \leq \frac{22}{9}|F(G)| - \frac{2}{3}.$$

Redistribute charge, so each f ends with $\geq 22/9$

Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph. Each face f gets charge $\ell(f)$.
Since $0 \leq w(G) = 2|E(G)| - 11|G| + 19$, Euler's gives

$$\sum_{f \in F(G)} \ell(f) = 2|E(G)| \leq \frac{22}{9}|F(G)| - \frac{2}{3}.$$

Redistribute charge, so each f ends with $\geq 22/9$, contradiction!

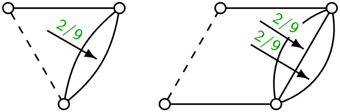
Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph. Each face f gets charge $\ell(f)$.
Since $0 \leq w(G) = 2|E(G)| - 11|G| + 19$, Euler's gives

$$\sum_{f \in F(G)} \ell(f) = 2|E(G)| \leq \frac{22}{9}|F(G)| - \frac{2}{3}.$$

Redistribute charge, so each f ends with $\geq 22/9$, contradiction!



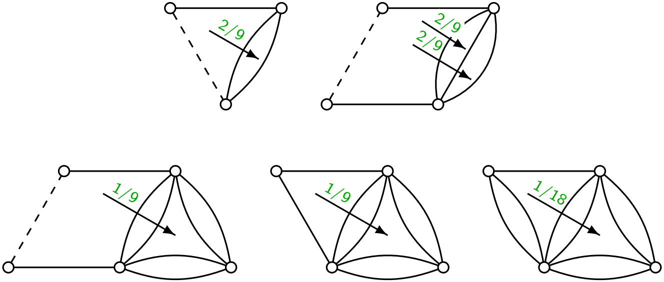
Discharging

Big Idea: Counting argument shows G has forbidden subgraph.

Assume no forbidden subgraph. Each face f gets charge $\ell(f)$.
Since $0 \leq w(G) = 2|E(G)| - 11|G| + 19$, Euler's gives

$$\sum_{f \in F(G)} \ell(f) = 2|E(G)| \leq \frac{22}{9}|F(G)| - \frac{2}{3}.$$

Redistribute charge, so each f ends with $\geq 22/9$, contradiction!



Summary

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).
Best known [LTWZ '13]**:** Planar and girth $\geq 6t \Rightarrow G \to C_{2t+1}$

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).
Best known [LTWZ '13]**:** Planar and girth $\geq 6t \Rightarrow G \to C_{2t+1}$

Our results:
Planar and girth $\geq 10 \Rightarrow G \to C_5$.
Planar and girth $\geq 16 \Rightarrow G \to C_7$.

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).
Best known [LTWZ '13]**:** Planar and girth $\geq 6t \Rightarrow G \to C_{2t+1}$

Our results:
Planar and girth $\geq 10 \Rightarrow G \to C_5$. Also [Dvořák–Postle '17].
Planar and girth $\geq 16 \Rightarrow G \to C_7$. Also [Postle–Smith-Roberge '19+].

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).
Best known [LTWZ '13]**:** Planar and girth $\geq 6t \Rightarrow G \to C_{2t+1}$

Our results:
Planar and girth $\geq 10 \Rightarrow G \to C_5$. Also [Dvořák–Postle '17].
Planar and girth $\geq 16 \Rightarrow G \to C_7$. Also [Postle–Smith-Roberge '19+].

Outline: For planar dual, want mod $(2t+1)$-orientation.

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).
Best known [LTWZ '13]**:** Planar and girth $\geq 6t \Rightarrow G \to C_{2t+1}$

Our results:
Planar and girth $\geq 10 \Rightarrow G \to C_5$. Also [Dvořák–Postle '17].
Planar and girth $\geq 16 \Rightarrow G \to C_7$. Also [Postle–Smith-Roberge '19+].

Outline: For planar dual, want mod $(2t+1)$-orientation.
Generalize to $\mathbb{Z}_{2t+1}$-boundary; allows induction by contraction.

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).
Best known [LTWZ '13]**:** Planar and girth $\geq 6t \Rightarrow G \to C_{2t+1}$

Our results:
Planar and girth $\geq 10 \Rightarrow G \to C_5$. Also [Dvořák–Postle '17].
Planar and girth $\geq 16 \Rightarrow G \to C_7$. Also [Postle–Smith-Roberge '19+].

Outline: For planar dual, want mod $(2t+1)$-orientation.
Generalize to $\mathbb{Z}_{2t+1}$-boundary; allows induction by contraction.
Key Tools: strongly $\mathbb{Z}_{2t+1}$-connected graphs, simulating, gap lem

Summary

Recall: Grötzsch's Thm: Triangle-free planar $\Rightarrow$ 3-colorable
Goal: Refine this for planar graphs with high girth.
Want to map $V(G)$ into $V(C_{2t+1})$ to preserve edges ($G \to C_{2t+1}$).

Conj: Planar and girth $\geq 4t \Rightarrow G \to C_{2t+1}$ (best possible).
Best known [LTWZ '13]**:** Planar and girth $\geq 6t \Rightarrow G \to C_{2t+1}$

Our results:
Planar and girth $\geq 10 \Rightarrow G \to C_5$. Also [Dvořák–Postle '17].
Planar and girth $\geq 16 \Rightarrow G \to C_7$. Also [Postle–Smith-Roberge '19+].

Outline: For planar dual, want mod $(2t+1)$-orientation.
Generalize to $\mathbb{Z}_{2t+1}$-boundary; allows induction by contraction.
Key Tools: strongly $\mathbb{Z}_{2t+1}$-connected graphs, simulating, gap lem

Read More: `https://arxiv.org/abs/1812.09833`