Crossings, Colorings, and Cliques

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Joint with Mike Albertson and Jacob Fox.

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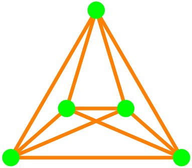

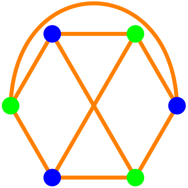

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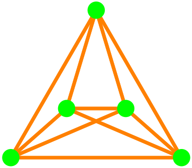

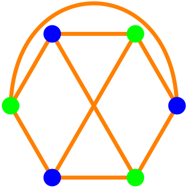

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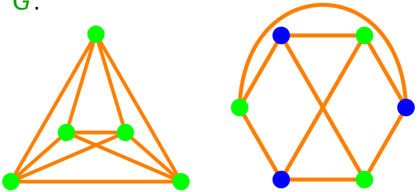

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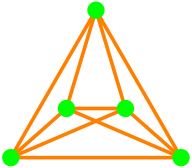

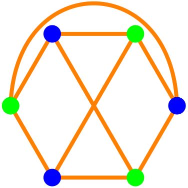

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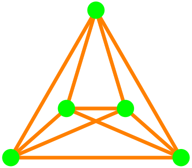

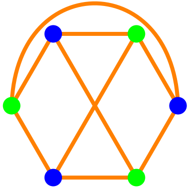

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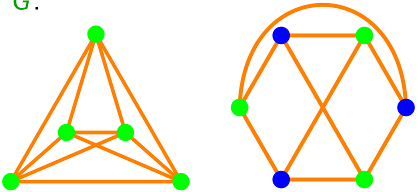

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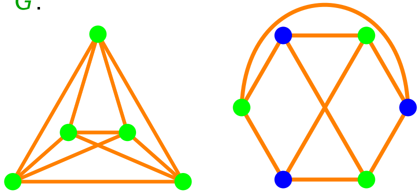

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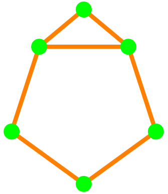

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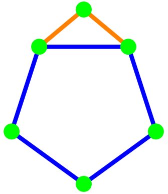

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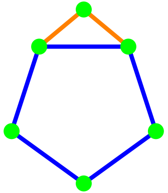

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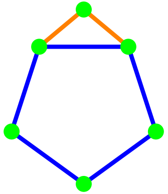

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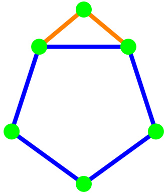

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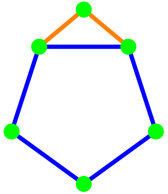

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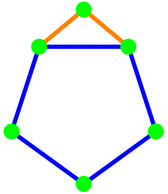

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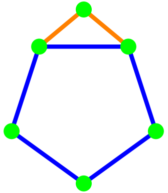

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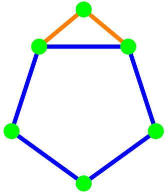

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Hence $m \geq 3n + 1$ and $\mathrm{cr}(G) \geq m - (3n - 6) \geq 7$. ∎

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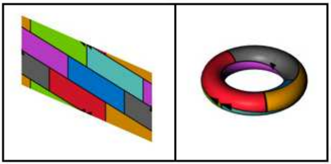

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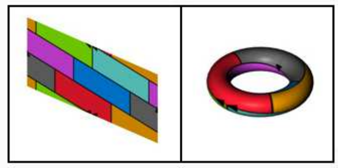

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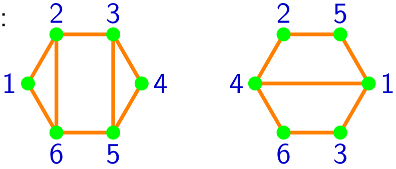

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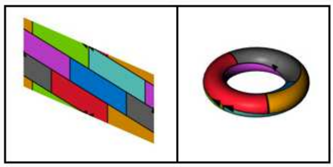

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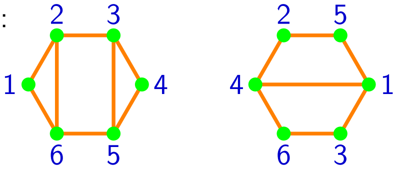

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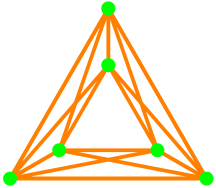

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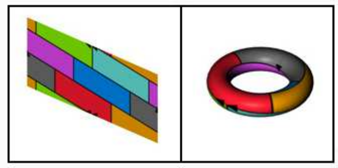

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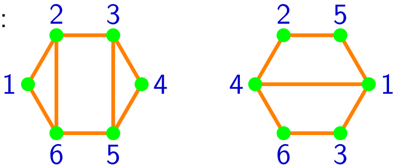

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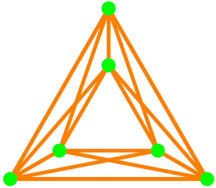

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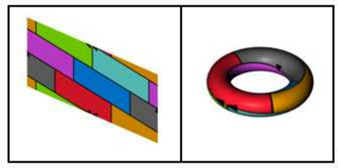

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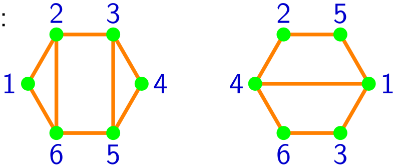

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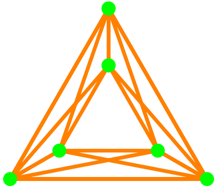

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If so, what are the extremal graphs?

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Thm. [Beineke-Harary '65; Alekseev-Gončakov '76]
$\tau(K_n) = \lceil \frac{n+2}{6} \rceil$ for $n \neq 9, 10$ and $\tau(K_9) = \tau(K_{10}) = 3$.

Chromatic number vs. genus and thickness

Suppose G has genus $g \geq 1$. Recall that $e \leq 3n - 6 + 6g$.
Note that $\chi(G) \leq \min(n, 1 + \frac{2(3n-6+6g)}{n}) \leq \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$.

Thm. [Ringel-Youngs '68]
The max $\chi(G)$ such that G embeds in S_g is $\lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$.

Suppose G has thickness t. Note that $\chi(G) \leq 6t$.
Note that $\tau(K_n) \geq \left\lceil \frac{\binom{n}{2}}{3n-6} \right\rceil = \lceil \frac{n+2}{6} \rceil$.

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Cor. Max $\chi(G)$ such that $\tau(G) = t$ satisfies $6t - 2 \leq \chi(G) \leq 6t$.

Chromatic number vs. crossing number

Thm. (Crossing Lemma) [Leighton; Ajtai et. al. '82]
If $m \geq 4n$, then

$$\mathrm{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

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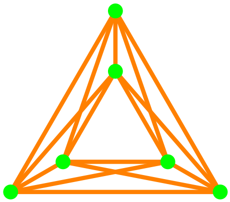

$$\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \approx n^4/64$$

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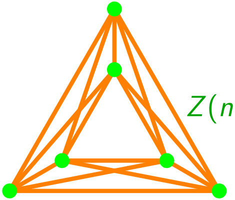

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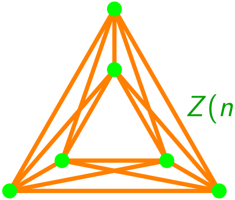

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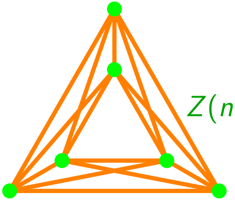

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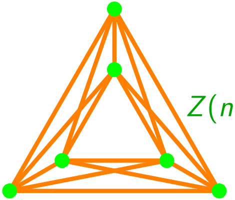

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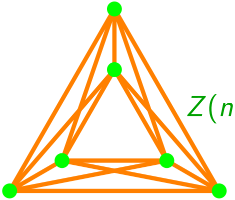

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Thm. [Kostochka-Stiebitz '96] If G is r-critical and $G \neq K_r$ and $n \neq 2r - 1$, then

$$m \geq \frac{r-1}{2}n + r - 3.$$

Proving Albertson's Conjecture (for lots more cases)

Crossing Lemma [Leighton; Ajtai et. al. '82] If $m \geq 4n$, then

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Thm. Albertson's Conjecture is true for $r \leq 12$.