Boundedness of Max-type Reciprocal Difference Equations

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Joint with Candace Kent
Slides available on my webpage

George Mason
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A brief history

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Bidwell and Franke: If a solution to (1) is bounded, then it is eventually periodic.

Main Result: All Solutions are Bounded

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Proof idea: Assume $\{x_n\}$ is unbounded, and so does not persist. Given ϵ (defined later), find smallest N such that $x_N < \epsilon$. Our lemmas will imply that for some constant C, we get $x_N \geq x_{N-C}$. But now $x_{N-C} < \epsilon$, which contradicts minimality of N.

Determining x_i's . . .

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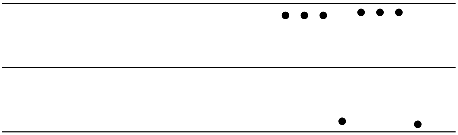

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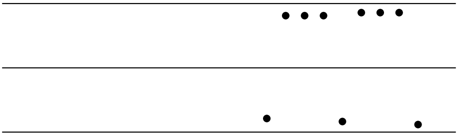

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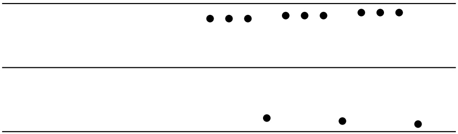

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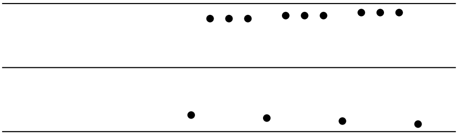

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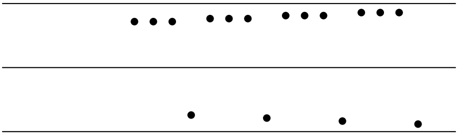

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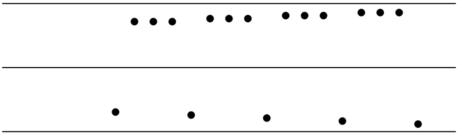

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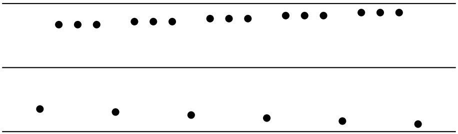

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Example:

Let $\epsilon = 10^{-1000}$, $t = 3$, $\max A_j^i < 10^3$, $\min A_j^i > 10^{-2}$.

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Showing that $x_{P(t+1)} \geq x_0$

Lemma 2: Let the A's be nice, and let $P = P(A)$. If for all $i \in \{1, \ldots, t\}$ and all $k \in \{0, \ldots, P-1\}$ we have

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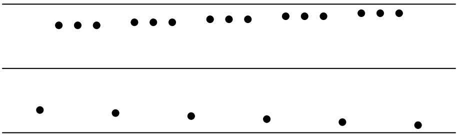

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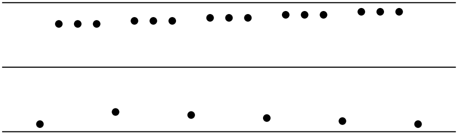

Combining the Lemmas

Lemma 1: There is r s.t. given $P \in \mathbb{Z}^+$ there exists $\epsilon > 0$ s.t. if $x_{P(t+1)} < \epsilon$, then for all $i \in \{1, \ldots, t\}$ and all $k \in \{0, \ldots, P-1\}$

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Corollary: Let the A's be nice. There is $P \in \mathbb{Z}^+$ and $\epsilon > 0$ s.t. if there is $N \geq P(t+1)$ with $x_N < \epsilon$, then $x_N \geq x_{N-P(t+1)}$.

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Main Theorem (Boundedness): If the periodic coefficient A's are "nice", then every positive solution $\{x_n\}$ of (1) is bounded.

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