Reducibility and Discharging: An Introduction by Example

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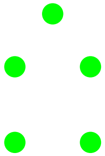

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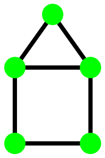

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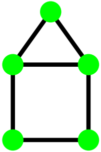

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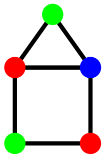

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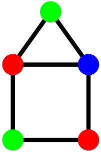

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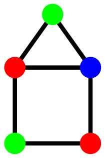

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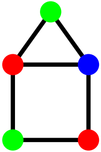

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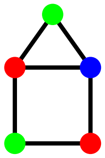

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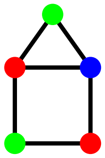

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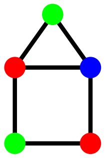

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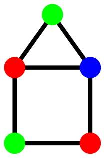

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- Reproved in 1996 by Robertson, Sanders, Seymour, Thomas.

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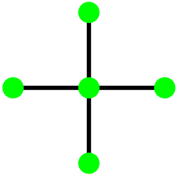

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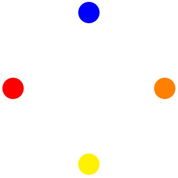

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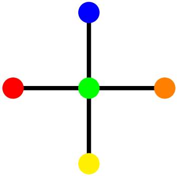

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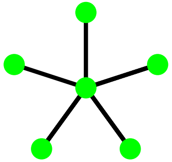

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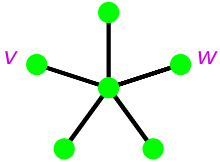

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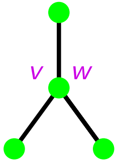

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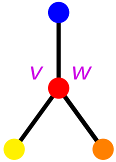

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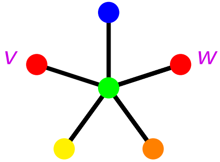

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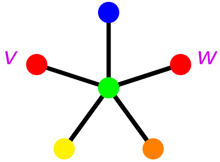

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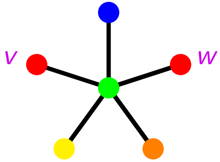

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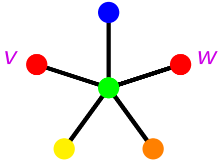

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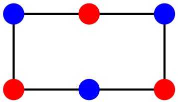

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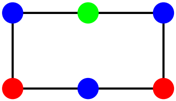

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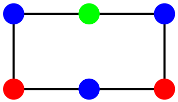

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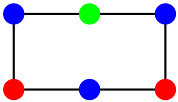

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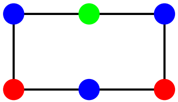

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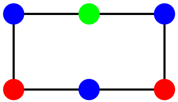

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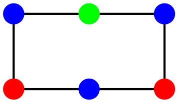

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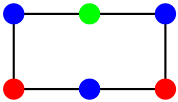

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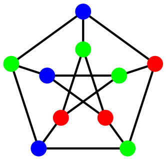

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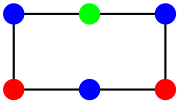

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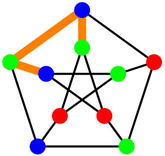

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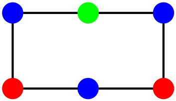

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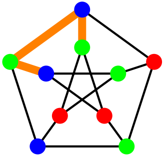

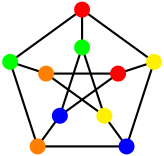

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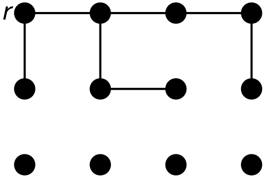

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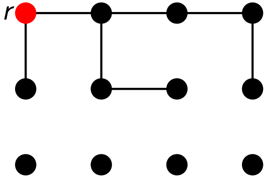

Structural Decomposition

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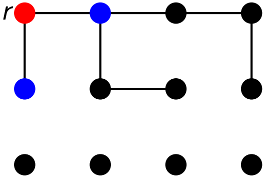

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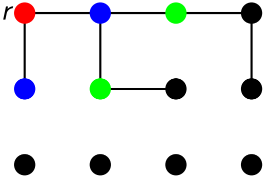

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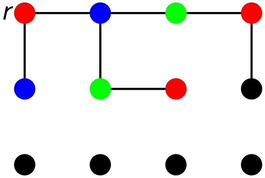

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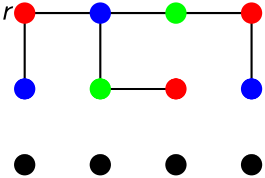

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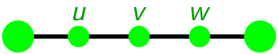

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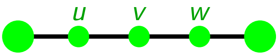

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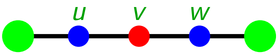

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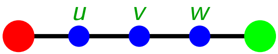

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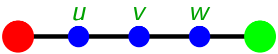

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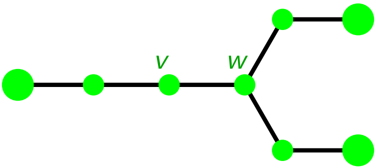

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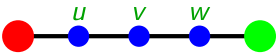

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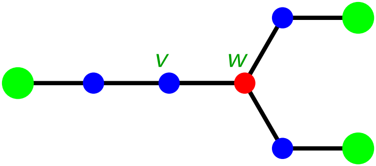

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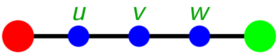

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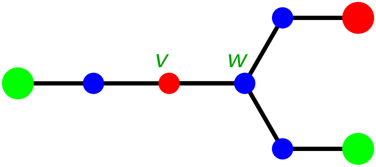

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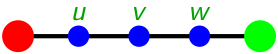

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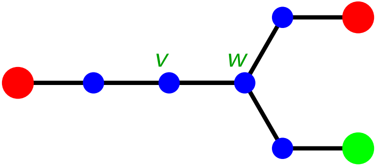

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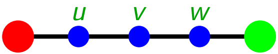

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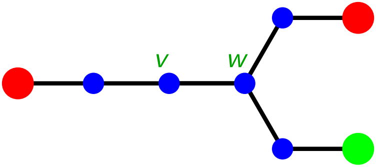

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"nearby" 2-vertices

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Contradiction! So G contains a reducible configuration.

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- For an arbitrary surface S, what is the minimum γ_S s.t. girth $\geq \gamma_S$ and G embedded in S implies an I, F-partition?