Injective coloring of sparse graphs

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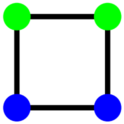

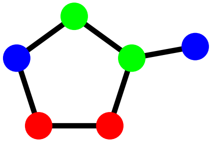

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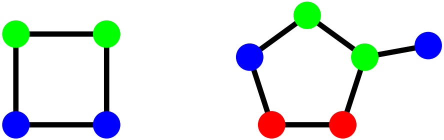

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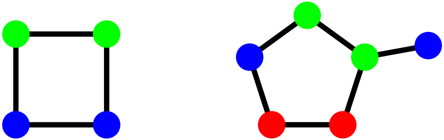

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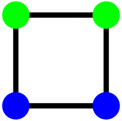

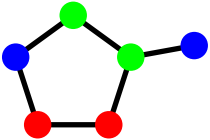

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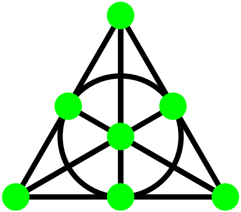

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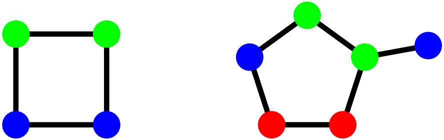

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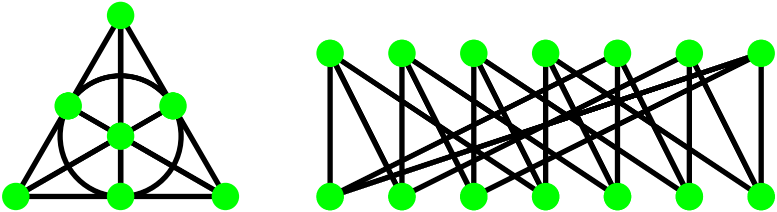

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planar, girth $\geq$ g	$\text{Mad}(G) < \frac{2g}{g-2}$

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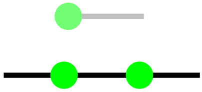

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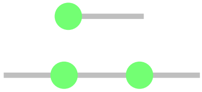

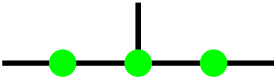

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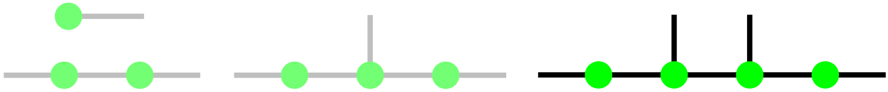

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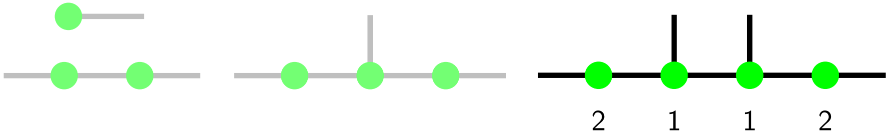

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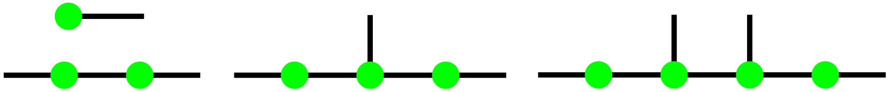

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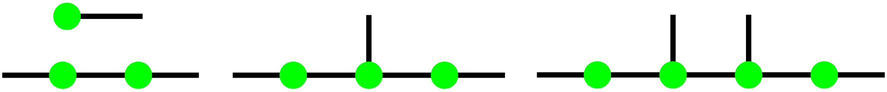

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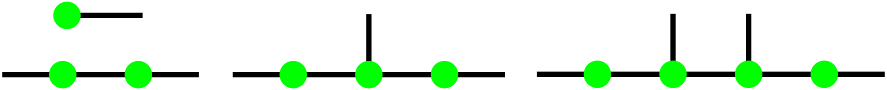

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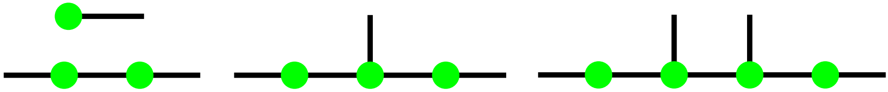

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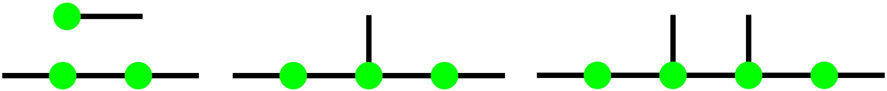

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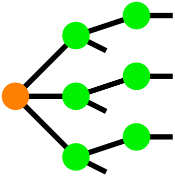

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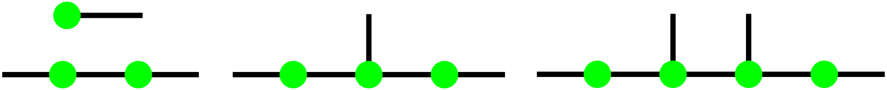

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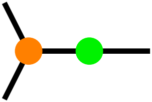

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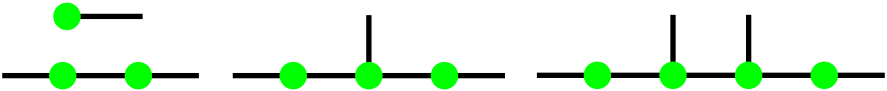

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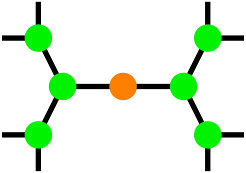

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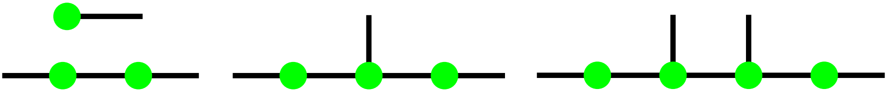

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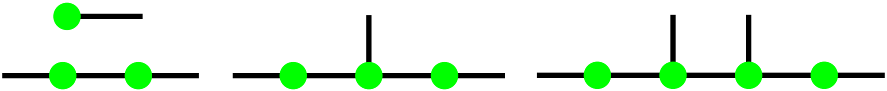

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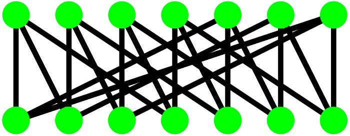

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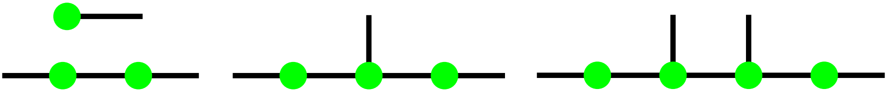

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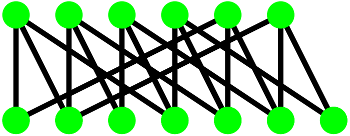

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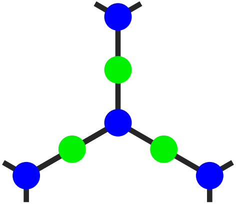
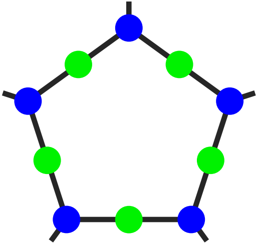
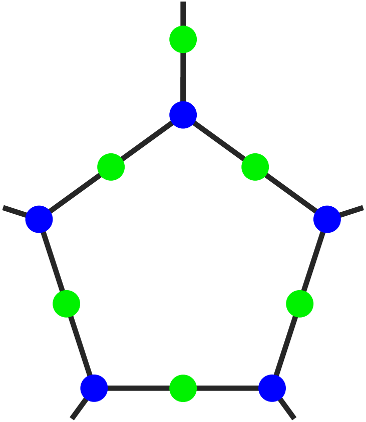




$\Delta = 3$ and $\mathrm{Mad}(G) < \frac{5}{2}$ (again)



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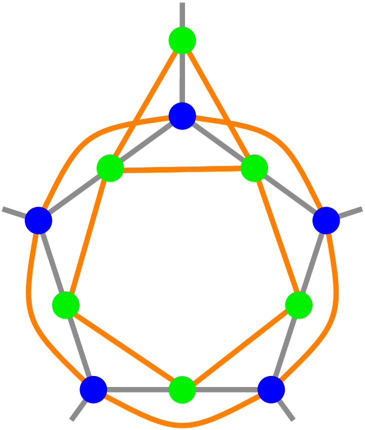

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Lemma (Vizing) For a connected graph G, let L be a list assignment such that $|L(v)| \geq d(v)$ for all v. G is L-colorable if
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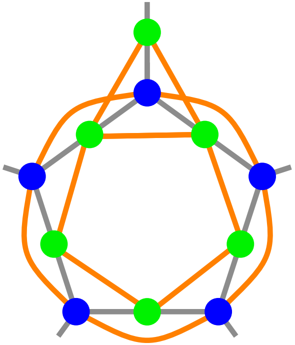

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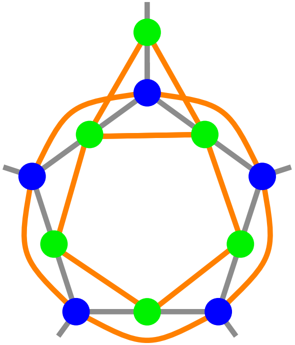

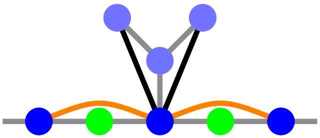

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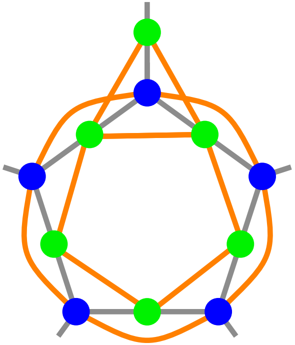

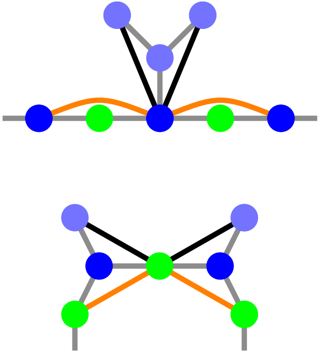

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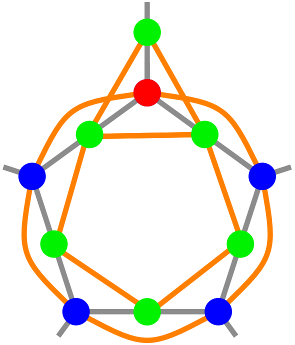

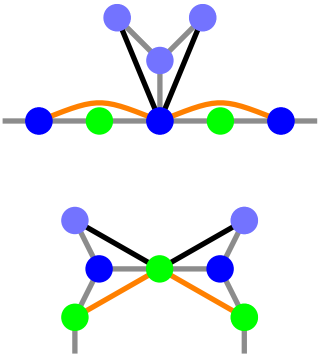

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