# A Survey of $k$-critical Graphs

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Albertson's Conjecture and Related Problems (AIM)
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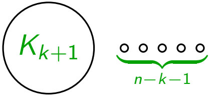

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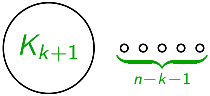


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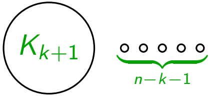


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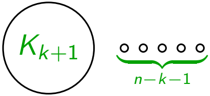


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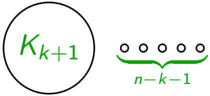


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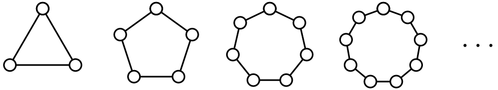

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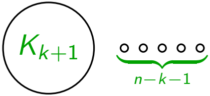


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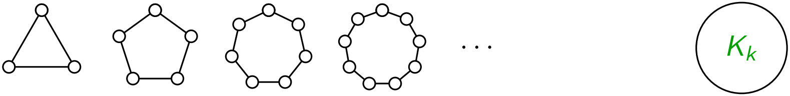

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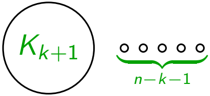


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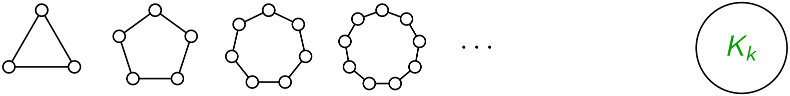


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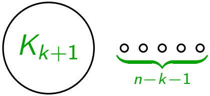


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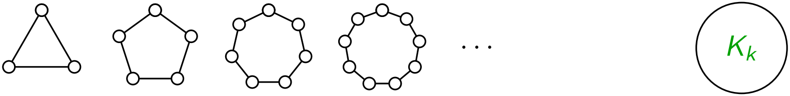


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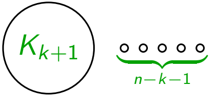


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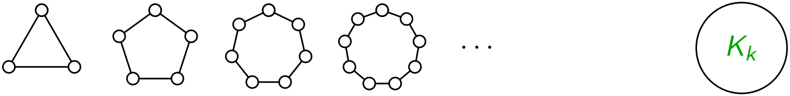


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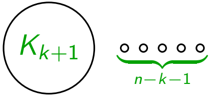


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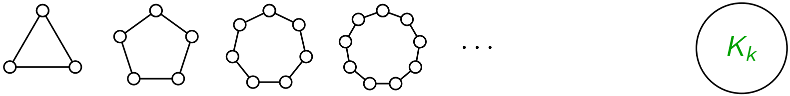


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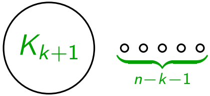


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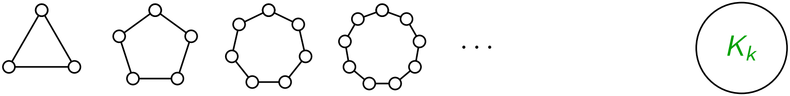


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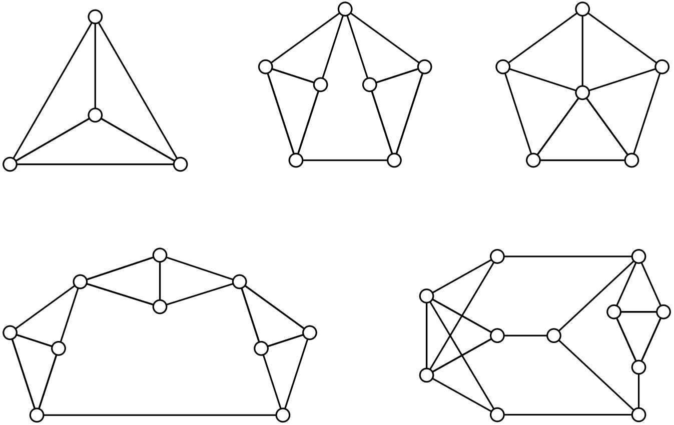

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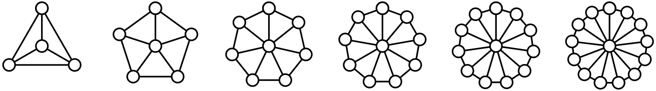

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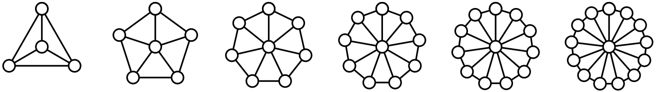

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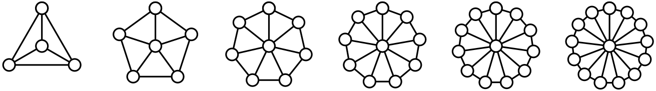

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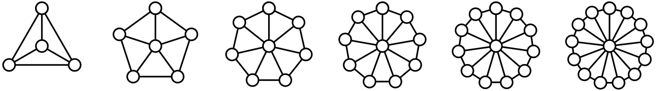

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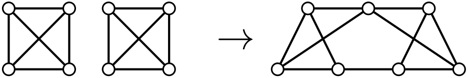

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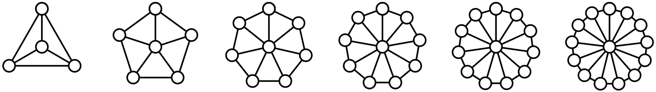

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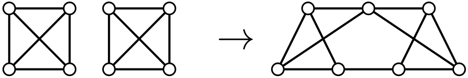

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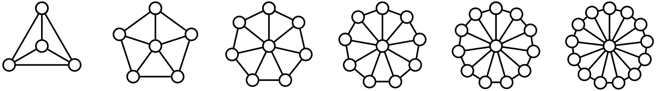

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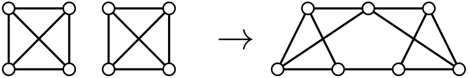

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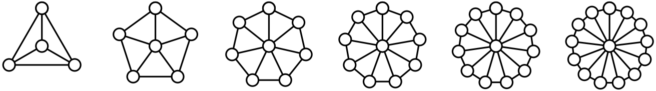

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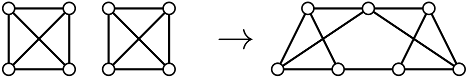

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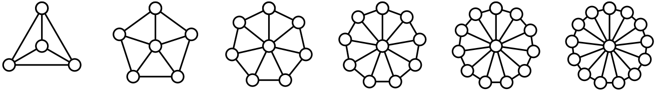

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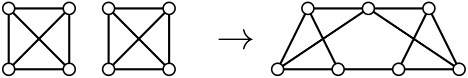

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**Lem:** A graph $H$ appears as induced subgraph of some $k$-critical if and only if $\chi(H/e) \leq k-1$ for all $e \in E(H)$.

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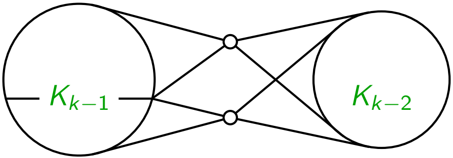

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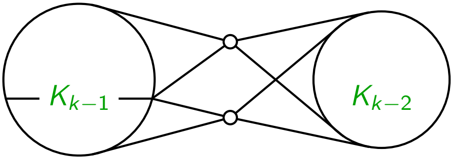


For smaller $n$, join clique to graph above.

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**Pf idea:** Like above; work harder to forbid low degree subgraphs.

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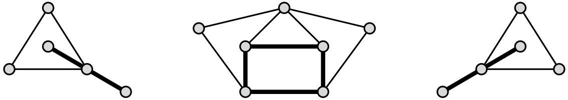

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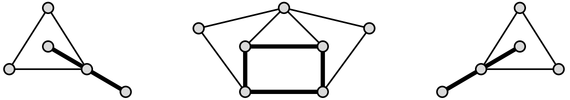

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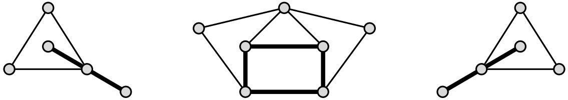

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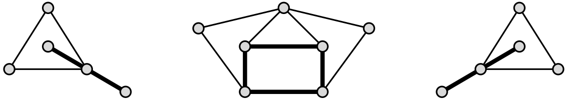

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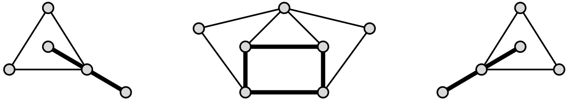

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**Obs:** Proof has some "slack"; allows various strengthenings.

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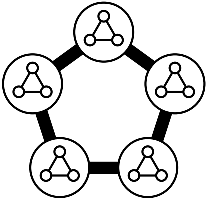

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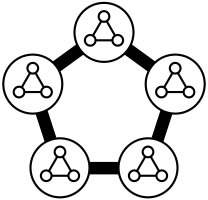

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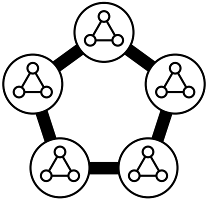

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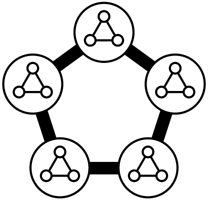

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Hereditary families: $K_{1,3}$-free, $\{P_5, \text{gem}\}$-free, hammer-free, etc.