The Search for Moore Graphs: Beauty is Rare

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LSU Student Colloquium
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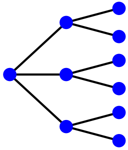

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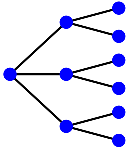

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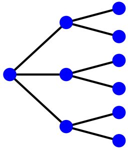

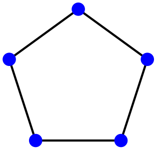

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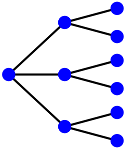

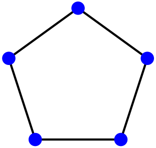

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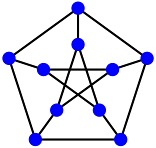

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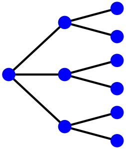

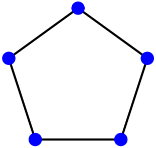

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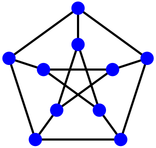

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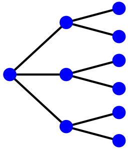

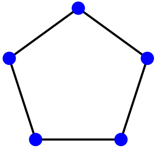

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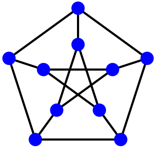

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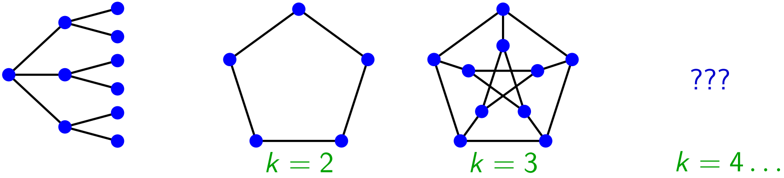

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Main Theorem: [Hoffman-Singleton 1960]
Moore graphs exist only when $k = 2$, 3, 7, and (possibly) 57.
When $k \in \{2, 3, 7\}$, the Moore graph is unique.

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Ex:

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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In fact, $A_5^2 + A_5 - I_5 = J_5$, and more generally:

$$A^2 + A - (k-1)I = J$$

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Fact: Sum of the eigenvalues equals sum of the diagonal entries.

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$0 = k + (r_1 + r_2)\frac{n-1}{2} = k + (-1)\frac{n-1}{2} = k + (-1)\frac{k^2}{2} \Rightarrow k \in \{0, 2\}$

Case 2: r_1 and r_2 are rational, so let $s^2 = 4k - 3$.

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$$s^5 + s^4 + 6s^3 - 2s^2 + (9 - 32m)s - 15 = 0$$

Rational Root Theorem $\Rightarrow s \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$

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3	3	10	$\longleftarrow$ Petersen
5	7	50	$\longleftarrow$?
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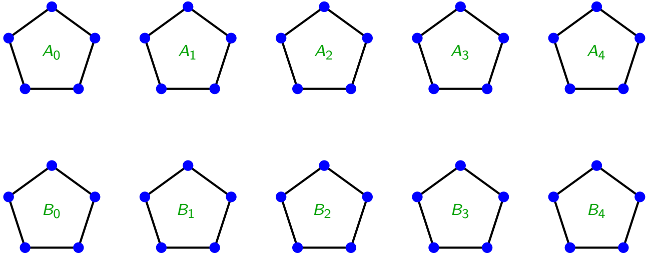

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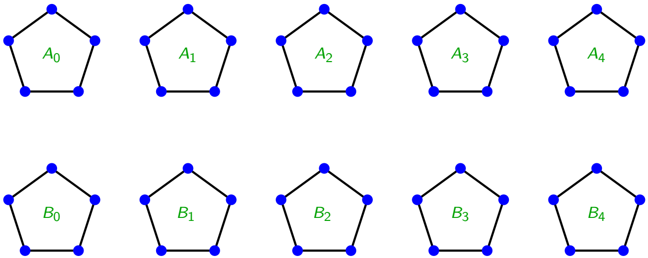

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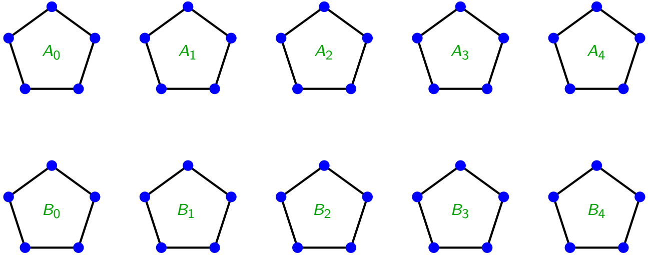

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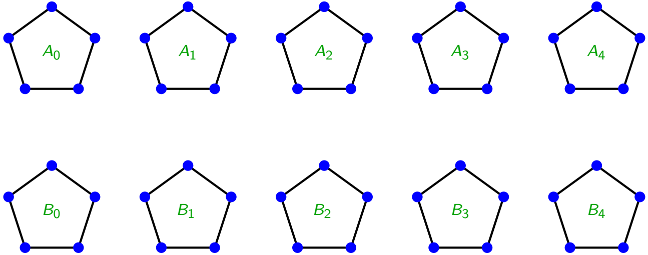

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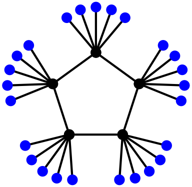

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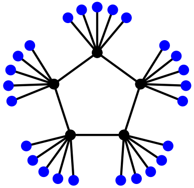

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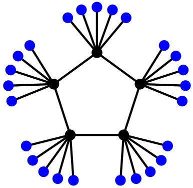

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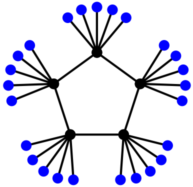

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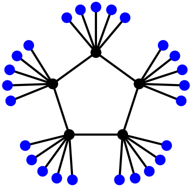

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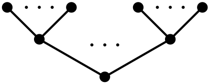

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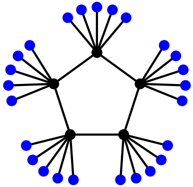

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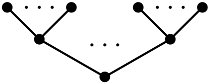

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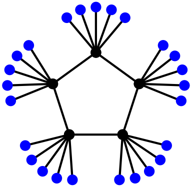

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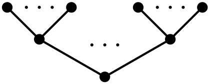

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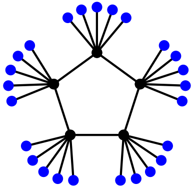

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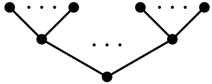

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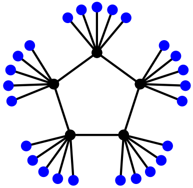

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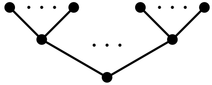

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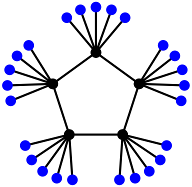

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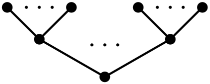

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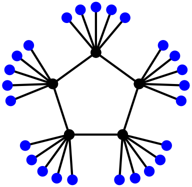

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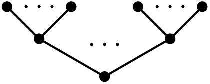

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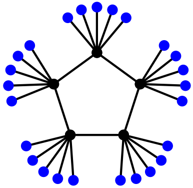

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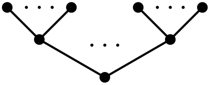

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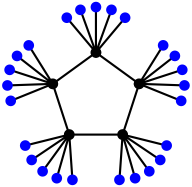

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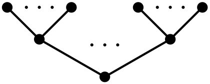

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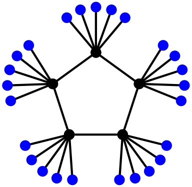

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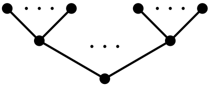

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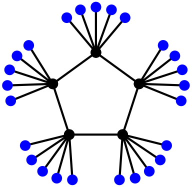

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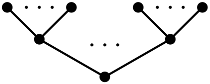

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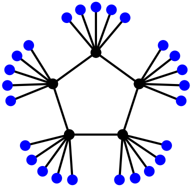

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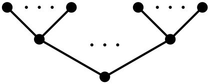

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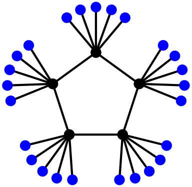

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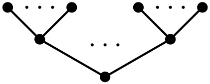

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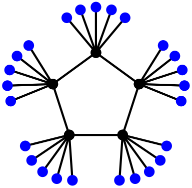

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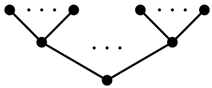

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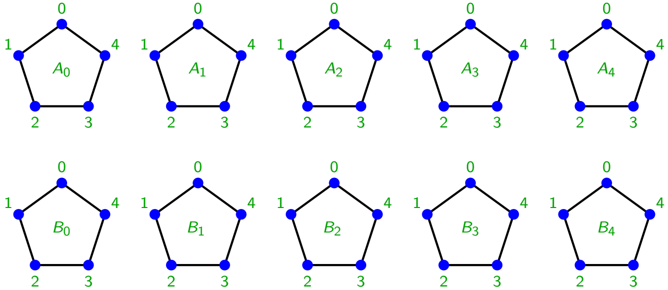
Pf: If not, we have too many $AAAAx$ and $BBBBx$ 5-cycles.

The Finale

Vertex sets A and B each induce 5 disjoint 5-cycles.



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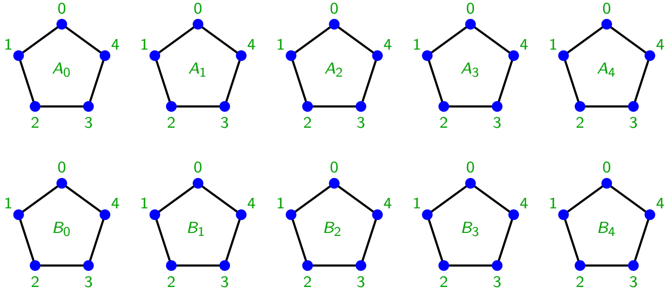

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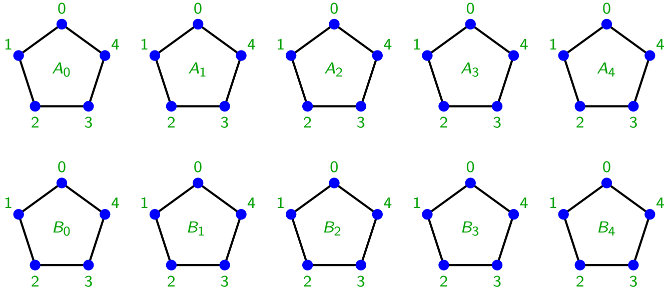

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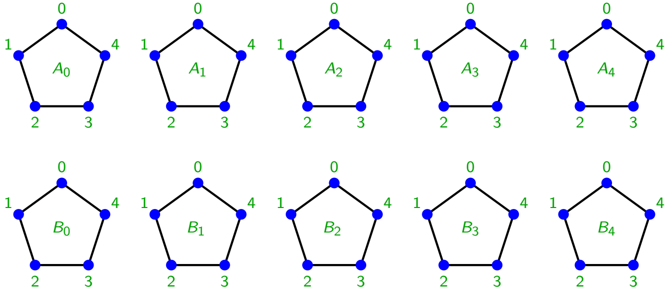

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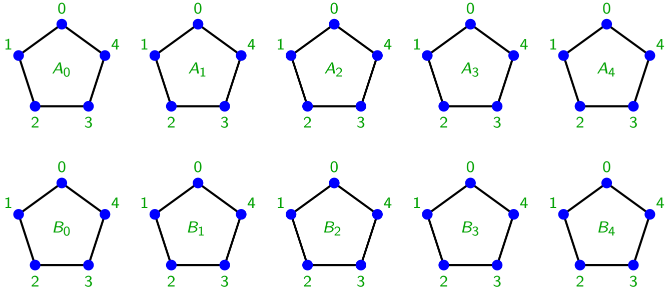

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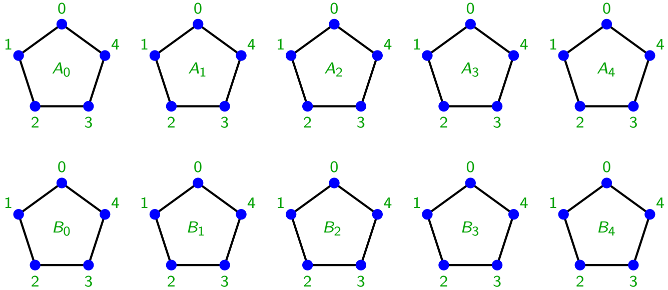

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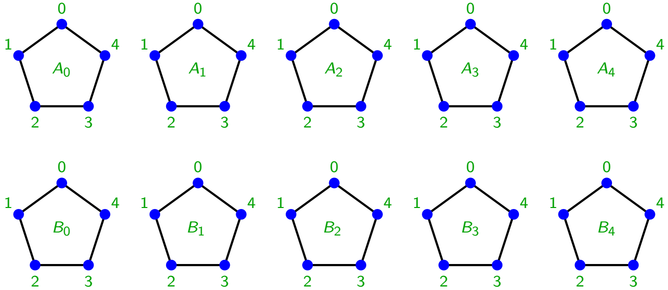

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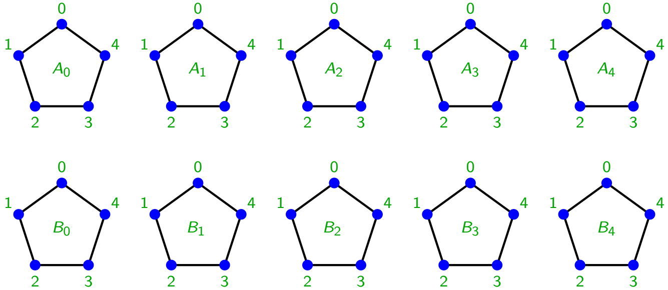

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So for $k = 7$ our desired Moore graph exists and is unique!

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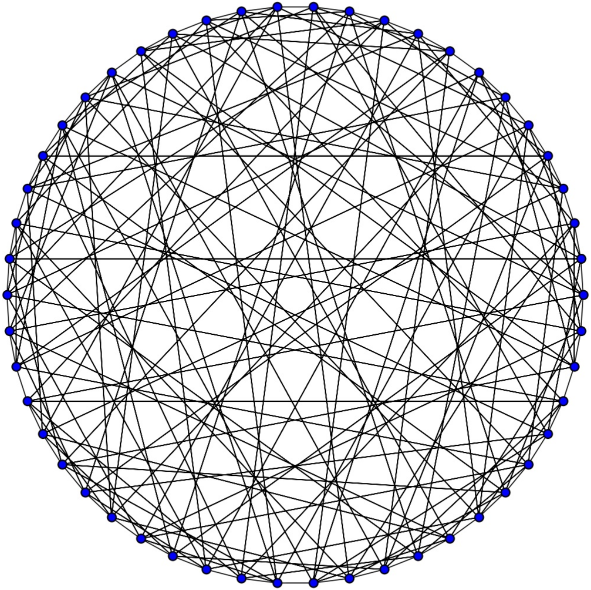

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