Using the Potential Method to Color Near-bipartite Graphs

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Joint with Matthew Yancey

WaterColor

24 September 2019

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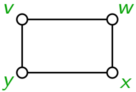

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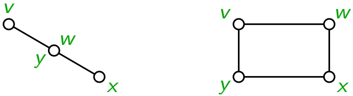

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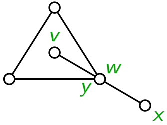

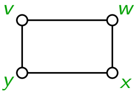

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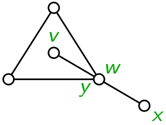

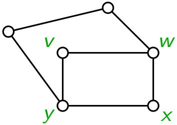

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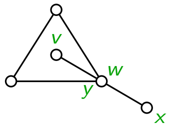

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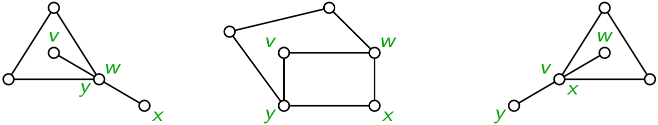

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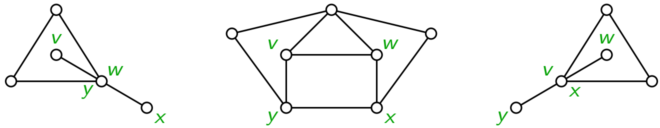

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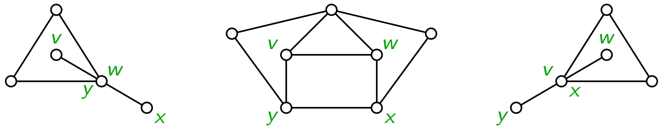

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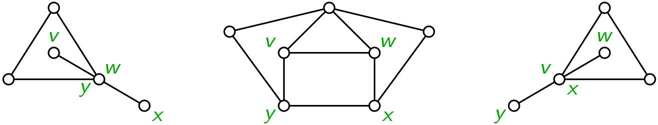

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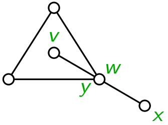

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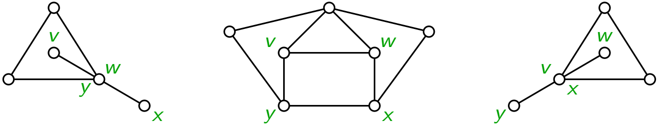

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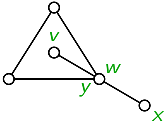

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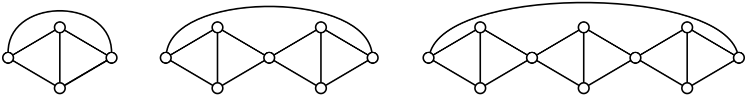

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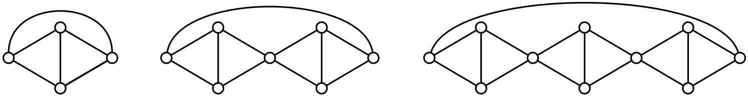

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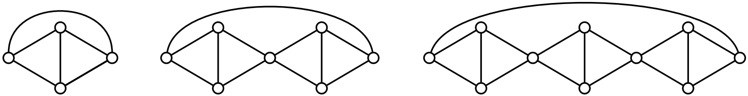

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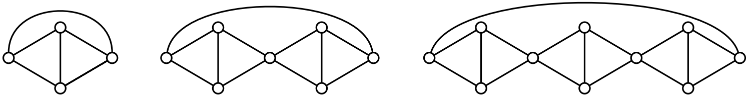

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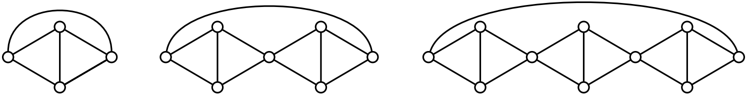

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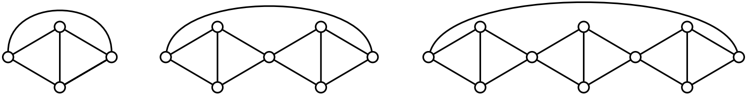

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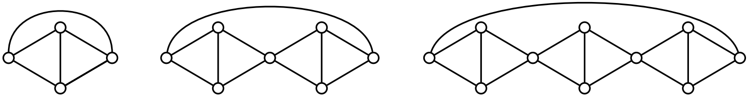

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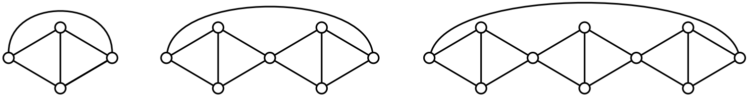

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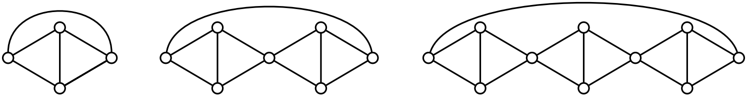

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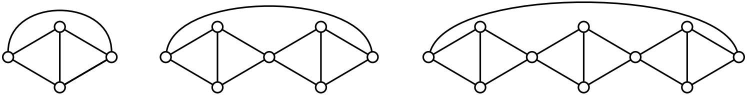

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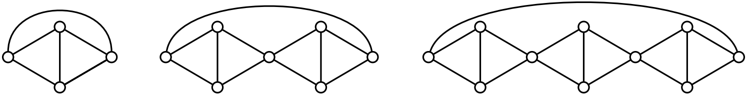

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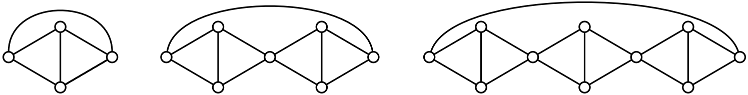

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Problem: Need more power for reducibility.

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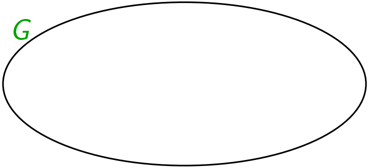

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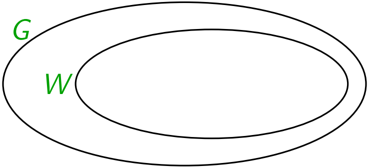

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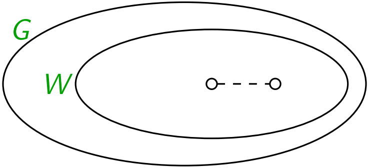

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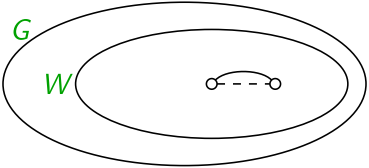

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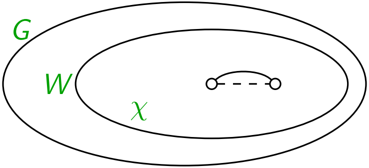

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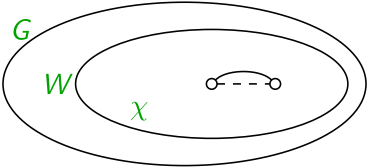

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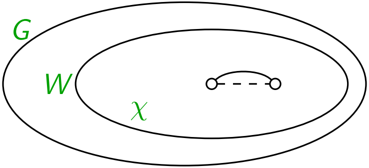

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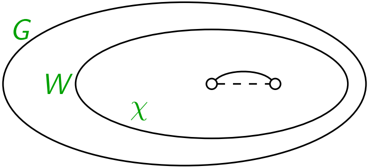

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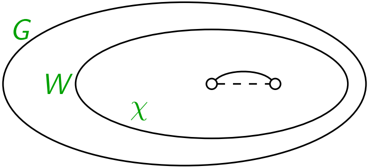

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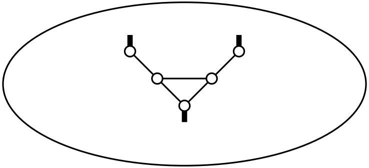

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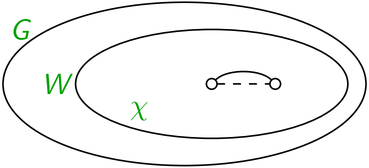

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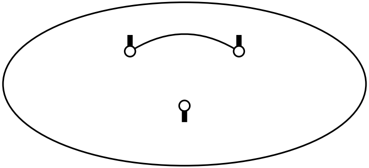

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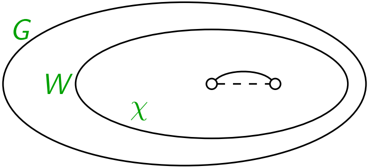

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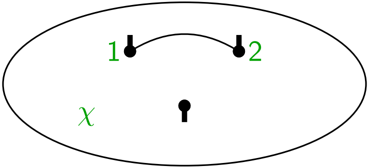

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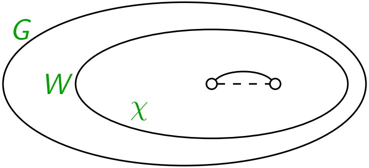

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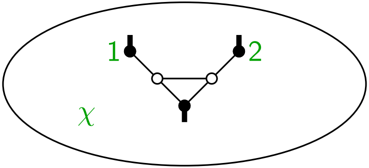

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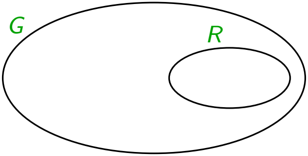

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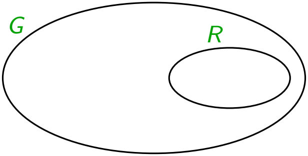

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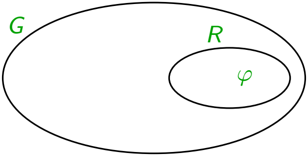

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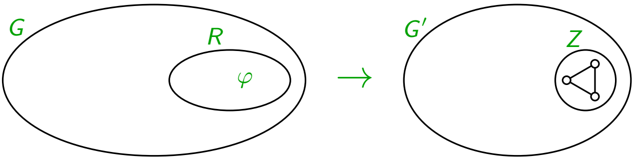

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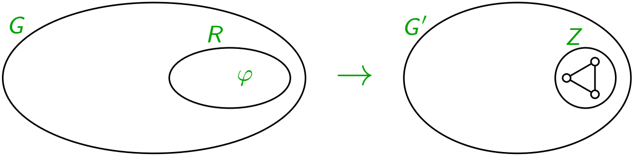

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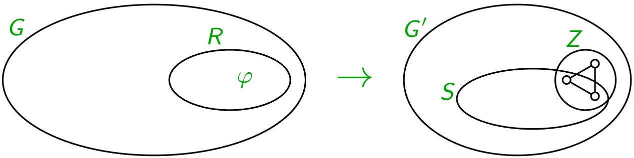

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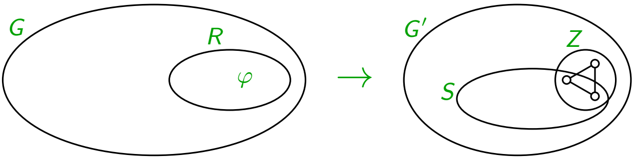

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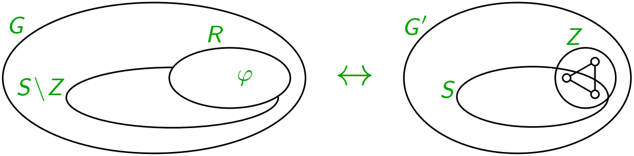

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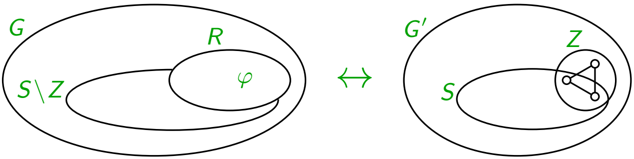

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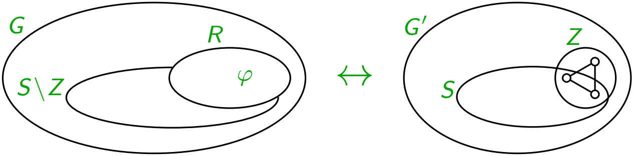

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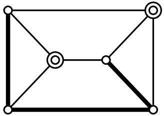

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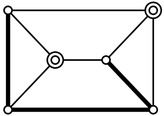

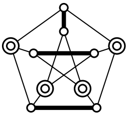

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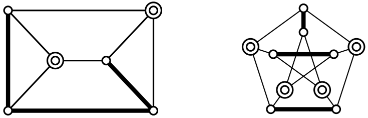

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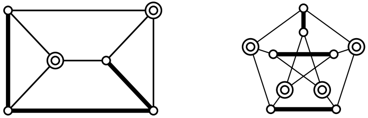

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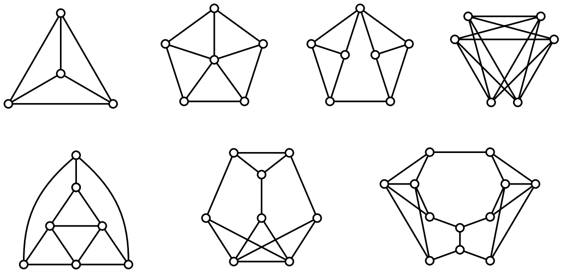

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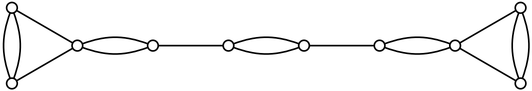

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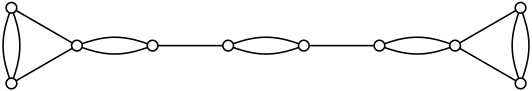

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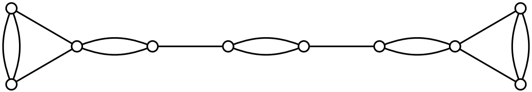

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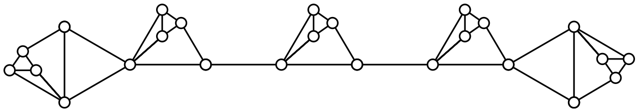

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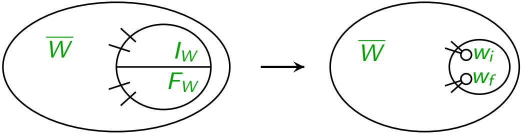

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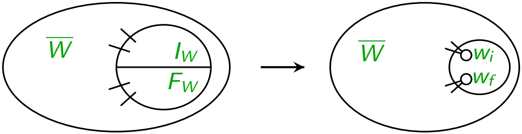

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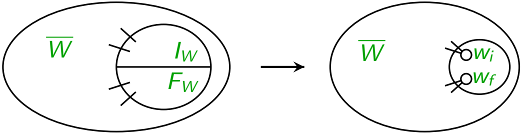

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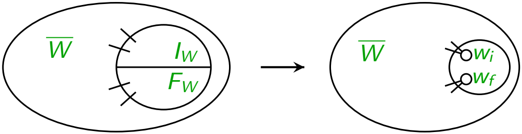

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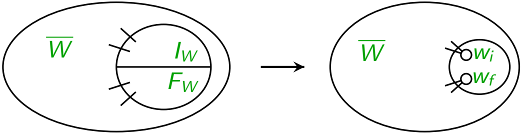

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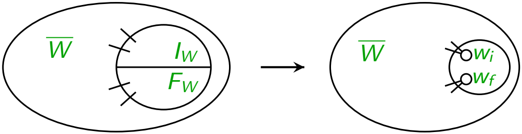

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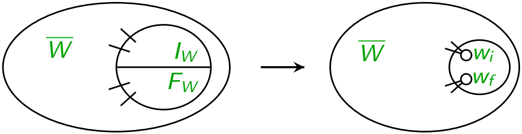

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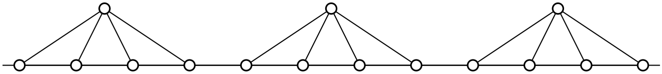

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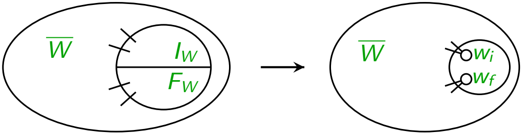

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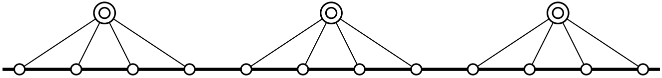

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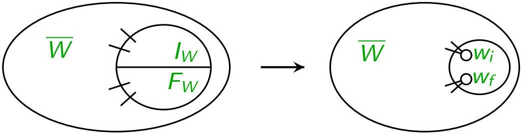

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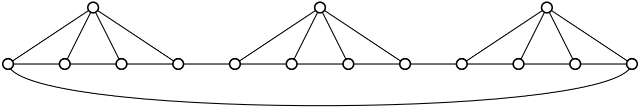

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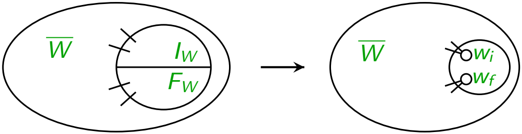

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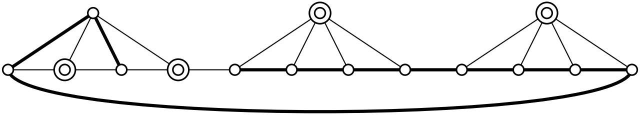

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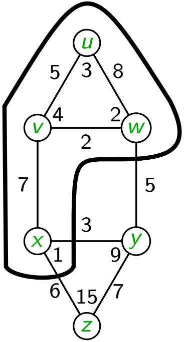
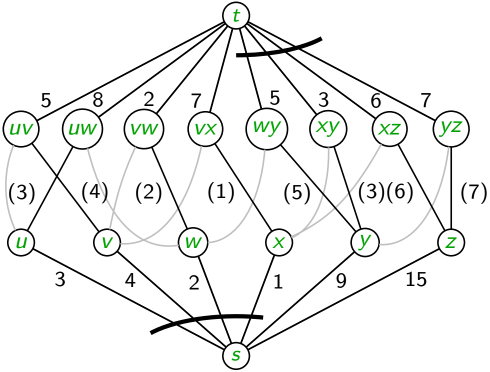
- May have many plausible reductions; need one with no $H \in \mathcal{H}$
  Finding right one takes time $O(n^{21})$; color recursively, extend

Algorithms: Finding Low Potential Sets

Thm [Goldberg '84]**:** Given arbitrary vertex and edge weights, we can find a set of minimum potential in polynomial time.





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