Nowhere-zero Flows: An Introduction

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Slides available on my webpage

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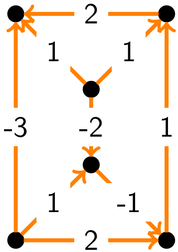

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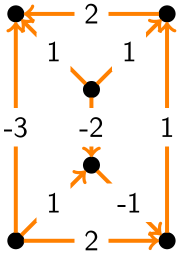

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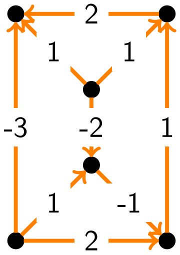

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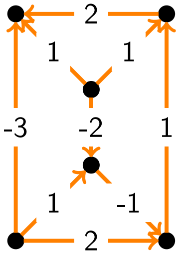

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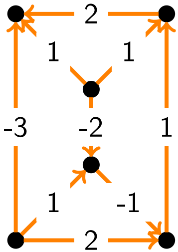

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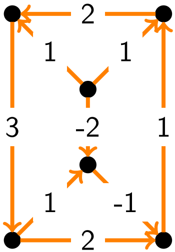

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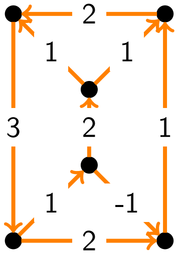

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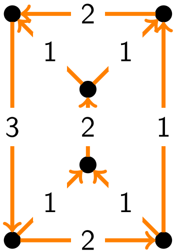

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Cor: Each bridgeless G has nowhere-zero k-flow for some k.

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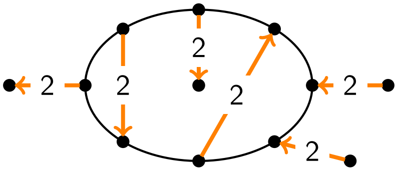

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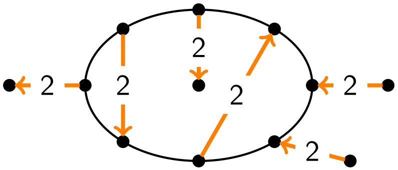

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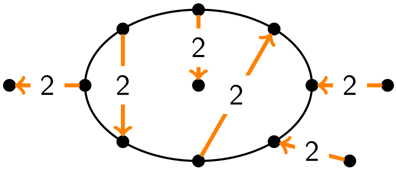

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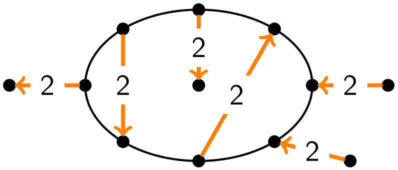

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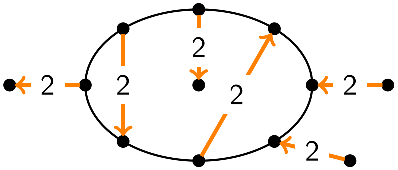

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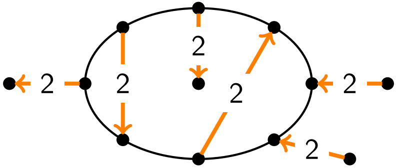

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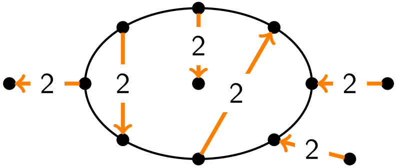

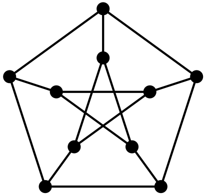

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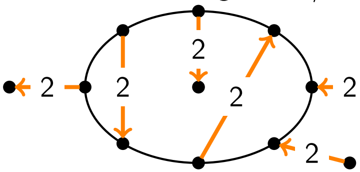

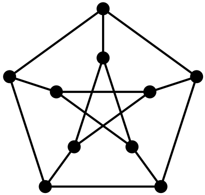

Cor: Petersen has no 3-edge-coloring, so no nowhere-zero 4-flow.

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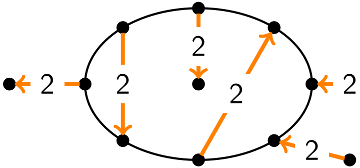

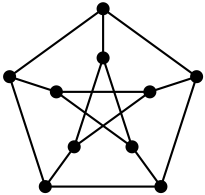

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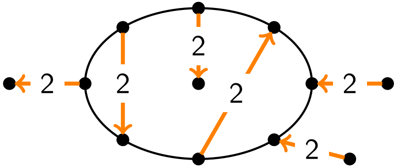

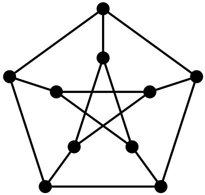

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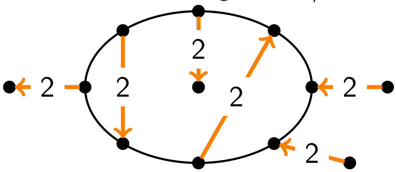

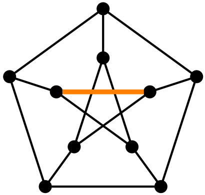

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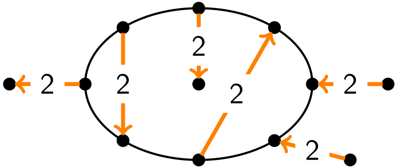

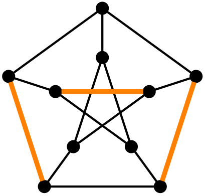

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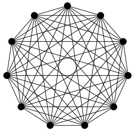

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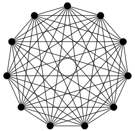

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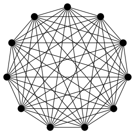

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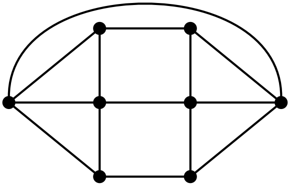

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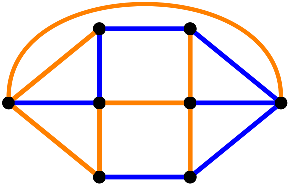

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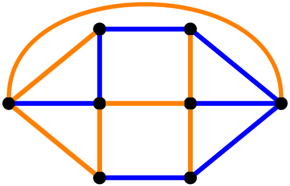

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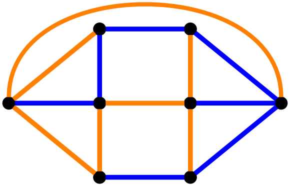

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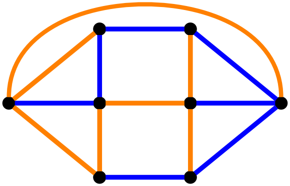

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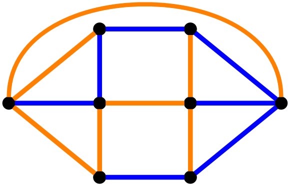

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Obs: The complement of a parity subgraph is an even graph.

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