Overlap Number of Graphs

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Slides available on my preprint page
Joint with Nitish Korula, Tim LeSaulnier, Kevin Milans
Chris Stocker, Jenn Vandenbussche, and Doug West

Atlanta Lecture Series V
26 February 2012

Definitions

Def: A set overlaps another set if they intersect but neither contains the other.

Definitions

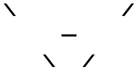
Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap.

Definitions

Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.

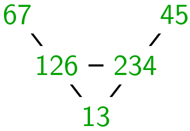
Definitions

Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.



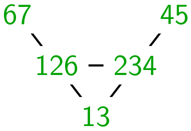
Definitions

Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.



Definitions

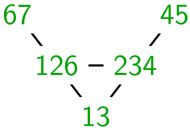
Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.

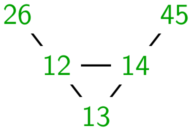


so $\varphi(G) \leq 7$

Definitions

Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.

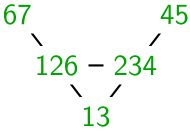




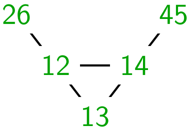
so $\varphi(G) \leq 7$

Definitions

Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.



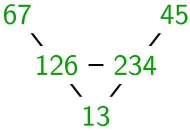
so $\varphi(G) \leq 7$



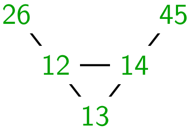
so $\varphi(G) \leq 6$

Definitions

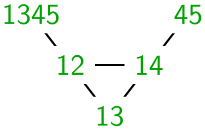
Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.



so $\varphi(G) \leq 7$

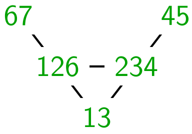


so $\varphi(G) \leq 6$

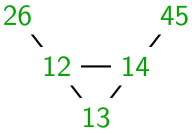


Definitions

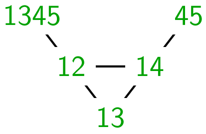
Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.



so $\varphi(G) \le 7$



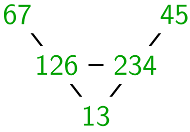
so $\varphi(G) \le 6$



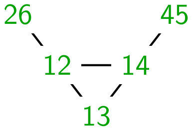
so $\varphi(G) \le 5$

Definitions

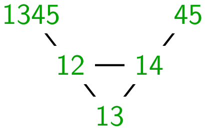
Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.



so $\varphi(G) \leq 7$



so $\varphi(G) \leq 6$

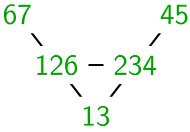


so $\varphi(G) \leq 5$

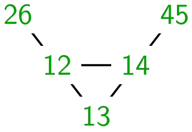
Def: A pure overlap representation f of a graph G is an overlap representation where no set contains another. The pure overlap number $\Phi(G)$ is the minimum size of f.

Definitions

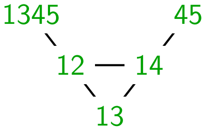
Def: A set overlaps another set if they intersect but neither contains the other. An overlap representation f of a graph G assigns sets to $V(G)$ so that $uv \in E(G)$ iff $f(u)$ and $f(v)$ overlap. The overlap number $\varphi(G)$ is the minimum size of f.



so $\varphi(G) \leq 7$



so $\varphi(G) \leq 6$



so $\varphi(G) \leq 5$

Def: A pure overlap representation f of a graph G is an overlap representation where no set contains another. The pure overlap number $\Phi(G)$ is the minimum size of f.

So $\varphi(G) \leq 5$, but $\Phi(G) \leq 6$.

Main Results

Thm 1: We have a linear-time algorithm to determine $\varphi(T)$ for every tree T. Corollary: $\varphi(T) \leq |T|$.

Main Results

Thm 1: We have a linear-time algorithm to determine $\varphi(T)$ for every tree T. Corollary: $\varphi(T) \leq |T|$.

Thm 2: If G is a planar n-vertex graph and $n \geq 5$, then $\varphi(G) \leq 2n - 5$, which is sharp for $n = 8$ and $n \geq 10$.

Main Results

Thm 1: We have a linear-time algorithm to determine $\varphi(T)$ for every tree T. Corollary: $\varphi(T) \le |T|$.

Thm 2: If G is a planar n-vertex graph and $n \ge 5$, then $\varphi(G) \le 2n - 5$, which is sharp for $n = 8$ and $n \ge 10$.

Thm 3: If G is an arbitrary n-vertex graph and $n \ge 14$, then $\varphi(G) \le n^2/4 - n/2 - 1$, which is sharp for even n.

Preliminaries

Decomposition Bound: Let $\mathcal{F}$ be a decomposition of graph G into cliques of order at most k, where $k \geq 2$. If $\delta(G) \geq k$, then $\Phi(G) \leq |\mathcal{F}|$. In particular, $\delta(G) \geq 2$ implies $\Phi(G) \leq |E(G)|$.

Preliminaries

Decomposition Bound: Let $\mathcal{F}$ be a decomposition of graph G into cliques of order at most k, where $k \geq 2$. If $\delta(G) \geq k$, then $\Phi(G) \leq |\mathcal{F}|$. In particular, $\delta(G) \geq 2$ implies $\Phi(G) \leq |E(G)|$.

Pf: Give each clique in $\mathcal{F}$ its own label, and give each vertex all the labels of cliques that contain it.

Preliminaries

Decomposition Bound: Let $\mathcal{F}$ be a decomposition of graph G into cliques of order at most k, where $k \geq 2$. If $\delta(G) \geq k$, then $\Phi(G) \leq |\mathcal{F}|$. In particular, $\delta(G) \geq 2$ implies $\Phi(G) \leq |E(G)|$.

Pf: Give each clique in $\mathcal{F}$ its own label, and give each vertex all the labels of cliques that contain it.

Prop: If G is triangle-free, then $\Phi(G) \geq |E(G)|$, and $\Phi(G) = |E(G)|$ when $\delta(G) \geq 2$.

Preliminaries

Decomposition Bound: Let $\mathcal{F}$ be a decomposition of graph G into cliques of order at most k, where $k \geq 2$. If $\delta(G) \geq k$, then $\Phi(G) \leq |\mathcal{F}|$. In particular, $\delta(G) \geq 2$ implies $\Phi(G) \leq |E(G)|$.

Pf: Give each clique in $\mathcal{F}$ its own label, and give each vertex all the labels of cliques that contain it.

Prop: If G is triangle-free, then $\Phi(G) \geq |E(G)|$,
and $\Phi(G) = |E(G)|$ when $\delta(G) \geq 2$.

Pf: We can't do better than one label on each edge.

Preliminaries

Decomposition Bound: Let $\mathcal{F}$ be a decomposition of graph G into cliques of order at most k, where $k \geq 2$. If $\delta(G) \geq k$, then $\Phi(G) \leq |\mathcal{F}|$. In particular, $\delta(G) \geq 2$ implies $\Phi(G) \leq |E(G)|$.

Pf: Give each clique in $\mathcal{F}$ its own label, and give each vertex all the labels of cliques that contain it.

Prop: If G is triangle-free, then $\Phi(G) \geq |E(G)|$, and $\Phi(G) = |E(G)|$ when $\delta(G) \geq 2$.

Pf: We can't do better than one label on each edge.

Deletion Bound: If v is a vertex with $d(v) \leq 2$ in a graph G with at least 3 vertices, then $\Phi(G) \leq \Phi(G - v) + 2$. If $d(v) \leq 1$, then $\varphi(G) \leq \varphi(G - v) + 2$.

Preliminaries

Decomposition Bound: Let $\mathcal{F}$ be a decomposition of graph G into cliques of order at most k, where $k \geq 2$. If $\delta(G) \geq k$, then $\Phi(G) \leq |\mathcal{F}|$. In particular, $\delta(G) \geq 2$ implies $\Phi(G) \leq |E(G)|$.

Pf: Give each clique in $\mathcal{F}$ its own label, and give each vertex all the labels of cliques that contain it.

Prop: If G is triangle-free, then $\Phi(G) \geq |E(G)|$, and $\Phi(G) = |E(G)|$ when $\delta(G) \geq 2$.

Pf: We can't do better than one label on each edge.

Deletion Bound: If v is a vertex with $d(v) \leq 2$ in a graph G with at least 3 vertices, then $\Phi(G) \leq \Phi(G - v) + 2$. If $d(v) \leq 1$, then $\varphi(G) \leq \varphi(G - v) + 2$.

Pf: Easy for Φ, and not too hard for φ.

Preliminaries (part 2)

Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.

Preliminaries (part 2)

Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Preliminaries (part 2)

Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Def: A star-cutset in a graph G is a separating set S containing a vertex x adjacent to all of $S - x$.

Preliminaries (part 2)

Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Def: A star-cutset in a graph G is a separating set S containing a vertex x adjacent to all of $S - x$.

Edge Lower Bound: If G is a triangle-free graph with no star-cutset, then $\varphi(G) \geq |E(G)| - 1$.

Preliminaries (part 2)

Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Def: A star-cutset in a graph G is a separating set S containing a vertex x adjacent to all of $S - x$.

Edge Lower Bound: If G is a triangle-free graph with no star-cutset, then $\varphi(G) \geq |E(G)| - 1$.

Pf idea: We can't do anything better than in the Edge Bound.

Preliminaries (part 2)

Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Def: A star-cutset in a graph G is a separating set S containing a vertex x adjacent to all of $S - x$.

Edge Lower Bound: If G is a triangle-free graph with no star-cutset, then $\varphi(G) \geq |E(G)| - 1$.

Pf idea: We can't do anything better than in the Edge Bound.

Cor. 1 If G is a triangle-free plane graph in which every face has length 4, and G has no star-cutset, then $\varphi(G) = 2n - 5$.

Preliminaries (part 2)

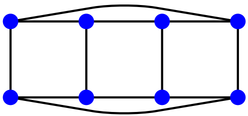
Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Def: A star-cutset in a graph G is a separating set S containing a vertex x adjacent to all of $S - x$.

Edge Lower Bound: If G is a triangle-free graph with no star-cutset, then $\varphi(G) \geq |E(G)| - 1$.

Pf idea: We can't do anything better than in the Edge Bound.

Cor. 1 If G is a triangle-free plane graph in which every face has length 4, and G has no star-cutset, then $\varphi(G) = 2n - 5$.



Preliminaries (part 2)

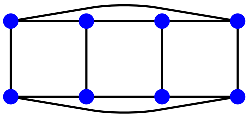
Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Def: A star-cutset in a graph G is a separating set S containing a vertex x adjacent to all of $S - x$.

Edge Lower Bound: If G is a triangle-free graph with no star-cutset, then $\varphi(G) \geq |E(G)| - 1$.

Pf idea: We can't do anything better than in the Edge Bound.

Cor. 1 If G is a triangle-free plane graph in which every face has length 4, and G has no star-cutset, then $\varphi(G) = 2n - 5$.



Cor. 2 For even $n \geq 6$, if we obtain G_n by deleting a matching of size $n/2$ from $K_{n/2,n/2}$, then $\varphi(G_n) = n^2/4 - n/2 - 1$.

Preliminaries (part 2)

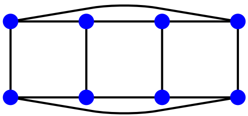
Edge Bound: If $\delta(G) \geq 2$ and $G \neq K_3$, then $\varphi(G) \leq |E(G)| - 1$.
Pf: Slightly modify a pure overlap labeling of size $|E(G)|$.

Def: A star-cutset in a graph G is a separating set S containing a vertex x adjacent to all of $S - x$.

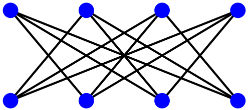
Edge Lower Bound: If G is a triangle-free graph with no star-cutset, then $\varphi(G) \geq |E(G)| - 1$.

Pf idea: We can't do anything better than in the Edge Bound.

Cor. 1 If G is a triangle-free plane graph in which every face has length 4, and G has no star-cutset, then $\varphi(G) = 2n - 5$.



Cor. 2 For even $n \geq 6$, if we obtain G_n by deleting a matching of size $n/2$ from $K_{n/2,n/2}$, then $\varphi(G_n) = n^2/4 - n/2 - 1$.



Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles.

Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.

Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles.
We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim.

Planar Graphs

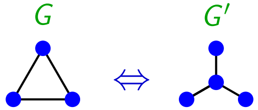
Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles.
We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

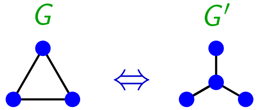


Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles.
We induct on t, the number of facial triangles in G.
If $t=0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1)-5 = 2n-3$,
so $|\mathcal{F}| \leq (2n-3)-3+1 = 2n-5$.

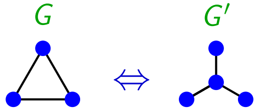


Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1) - 5 = 2n - 3$,
so $|\mathcal{F}| \leq (2n-3) - 3 + 1 = 2n - 5$.



Case 2: G' has only facial 4-cycles.

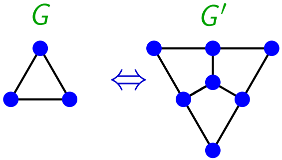
Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1) - 5 = 2n-3$,
so $|\mathcal{F}| \leq (2n-3) - 3 + 1 = 2n-5$.

Case 2: G' has only facial 4-cycles.



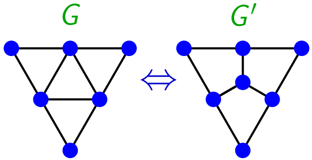
Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1) - 5 = 2n - 3$,
so $|\mathcal{F}| \leq (2n-3) - 3 + 1 = 2n - 5$.

Case 2: G' has only facial 4-cycles.

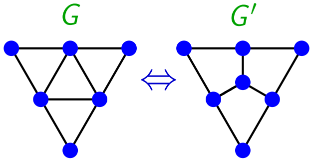


Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t=0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1)-5 = 2n-3$,
so $|\mathcal{F}| \leq (2n-3)-3+1 = 2n-5$.



Case 2: G' has only facial 4-cycles.
Now $|\mathcal{F}'| = 2(n+1)-4 = 2n-2$,

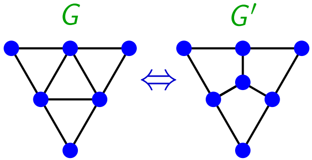
Planar Graphs

Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles.
We induct on t, the number of facial triangles in G.
If $t=0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1)-5 = 2n-3$,
so $|\mathcal{F}| \leq (2n-3)-3+1 = 2n-5$.

Case 2: G' has only facial 4-cycles.
Now $|\mathcal{F}'| = 2(n+1)-4 = 2n-2$,
so $|\mathcal{F}| = (2n-2)-9+3 = 2n-8$.

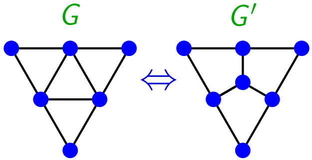


Planar Graphs

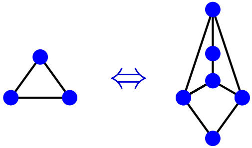
Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1) - 5 = 2n - 3$,
so $|\mathcal{F}| \leq (2n-3) - 3 + 1 = 2n - 5$.



Case 2: G' has only facial 4-cycles.
Now $|\mathcal{F}'| = 2(n+1) - 4 = 2n - 2$,
so $|\mathcal{F}| = (2n-2) - 9 + 3 = 2n - 8$.
Case 3: Or 2 faces share an edge,



Planar Graphs

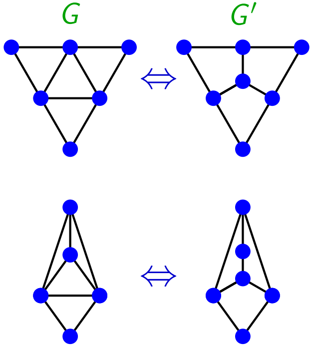
Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1) - 5 = 2n - 3$,
so $|\mathcal{F}| \leq (2n-3) - 3 + 1 = 2n - 5$.

Case 2: G' has only facial 4-cycles.
Now $|\mathcal{F}'| = 2(n+1) - 4 = 2n - 2$,
so $|\mathcal{F}| = (2n-2) - 9 + 3 = 2n - 8$.

Case 3: Or 2 faces share an edge,



Planar Graphs

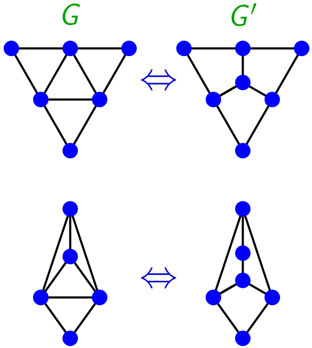
Lemma 1: If G is planar with $n \geq 5$ vertices, then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Pf: Let $\mathcal{F}$ denote our decomposition of G into edges and triangles. We induct on t, the number of facial triangles in G.
If $t = 0$, then Euler's formula implies the claim. So suppose $t \geq 1$.

Case 1: G' has a facial (non-4)-cycle.
Now $|\mathcal{F}'| \leq 2(n+1) - 5 = 2n - 3$,
so $|\mathcal{F}| \leq (2n-3) - 3 + 1 = 2n - 5$.

Case 2: G' has only facial 4-cycles.
Now $|\mathcal{F}'| = 2(n+1) - 4 = 2n - 2$,
so $|\mathcal{F}| = (2n-2) - 9 + 3 = 2n - 8$.

Case 3: Or 2 faces share an edge,
so $|\mathcal{F}| \leq |\mathcal{F}'| - 8 + 4 = 2n - 6$.



Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Cor: If G is planar, $n \geq 5$, and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n-5$, unless G has $2n-4$ edges and every face is a 4-cycle.

Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Cor: If G is planar, $n \geq 5$, and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n - 5$, unless G has $2n - 4$ edges and every face is a 4-cycle.

Pf: This follows from Lemma 1 and the Decomposition Bound.

Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Cor: If G is planar, $n \geq 5$, and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n-5$, unless G has $2n-4$ edges and every face is a 4-cycle.

Pf: This follows from Lemma 1 and the Decomposition Bound.

Thm 2: If G is a planar n-vertex graph and $n \geq 5$, then $\varphi(G) \leq 2n-5$, which is sharp for $n = 8$ and $n \geq 10$.

Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n - 5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n - 4$ edges).

Cor: If G is planar, $n \geq 5$, and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n - 5$, unless G has $2n - 4$ edges and every face is a 4-cycle.

Pf: This follows from Lemma 1 and the Decomposition Bound.

Thm 2: If G is a planar n-vertex graph and $n \geq 5$, then $\varphi(G) \leq 2n - 5$, which is sharp for $n = 8$ and $n \geq 10$.

Pf sketch: Use the Deletion Bound ($\Phi(G) \leq \Phi(G - v) + 2$ if $d(v) \leq 2$) to reduce to $\delta(G) \geq 3$, then invoke the corollary above.

Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Cor: If G is planar, $n \geq 5$, and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n-5$, unless G has $2n-4$ edges and every face is a 4-cycle.

Pf: This follows from Lemma 1 and the Decomposition Bound.

Thm 2: If G is a planar n-vertex graph and $n \geq 5$, then $\varphi(G) \leq 2n-5$, which is sharp for $n = 8$ and $n \geq 10$.

Pf sketch: Use the Deletion Bound ($\Phi(G) \leq \Phi(G-v)+2$ if $d(v) \leq 2$) to reduce to $\delta(G) \geq 3$, then invoke the corollary above. If G consists of $2n-4$ edges, then $\varphi(G) \leq |E(G)| - 1 = 2n-5$.

Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Cor: If G is planar, $n \geq 5$, and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n-5$, unless G has $2n-4$ edges and every face is a 4-cycle.

Pf: This follows from Lemma 1 and the Decomposition Bound.

Thm 2: If G is a planar n-vertex graph and $n \geq 5$, then $\varphi(G) \leq 2n-5$, which is sharp for $n = 8$ and $n \geq 10$.

Pf sketch: Use the Deletion Bound ($\Phi(G) \leq \Phi(G-v)+2$ if $d(v) \leq 2$) to reduce to $\delta(G) \geq 3$, then invoke the corollary above. If G consists of $2n-4$ edges, then $\varphi(G) \leq |E(G)|-1 = 2n-5$. What's missing?

Planar Graphs

Lemma 1: If G is planar with n vertices and $n \geq 5$ then G decomposes into at most $2n-5$ edges and facial triangles unless every face is a 4-cycle (then G consists of $2n-4$ edges).

Cor: If G is planar, $n \geq 5$, and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n-5$, unless G has $2n-4$ edges and every face is a 4-cycle.

Pf: This follows from Lemma 1 and the Decomposition Bound.

Thm 2: If G is a planar n-vertex graph and $n \geq 5$, then $\varphi(G) \leq 2n-5$, which is sharp for $n=8$ and $n \geq 10$.

Pf sketch: Use the Deletion Bound ($\Phi(G) \leq \Phi(G-v)+2$ if $d(v) \leq 2$) to reduce to $\delta(G) \geq 3$, then invoke the corollary above. If G consists of $2n-4$ edges, then $\varphi(G) \leq |E(G)|-1 = 2n-5$. What's missing? Lot's of messy base cases.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

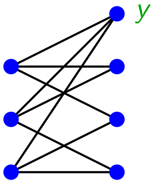
Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,
and some $y \in Y$ has degree k
and all others have degree $k - 1$.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,
and some $y \in Y$ has degree k
and all others have degree $k - 1$.

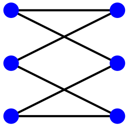


Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,
and some $y \in Y$ has degree k
and all others have degree $k - 1$.



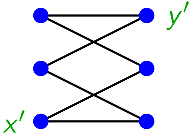
Delete y to form G'. Now
$\Phi(G') \leq |E(G')| = \lfloor n^2/4 - n/2 + 1 \rfloor - \lfloor n/2 \rfloor = \lfloor n^2/4 - n + 1 \rfloor$.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,
and some $y \in Y$ has degree k
and all others have degree $k-1$.



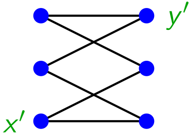
Delete y to form G'. Now
$\Phi(G') \leq |E(G')| = \left\lfloor n^2/4 - n/2 + 1 \right\rfloor - \lfloor n/2 \rfloor = \left\lfloor n^2/4 - n + 1 \right\rfloor$.
Let f be a pure overlap labeling of G' using one label per edge.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,
and some $y \in Y$ has degree k
and all others have degree $k - 1$.



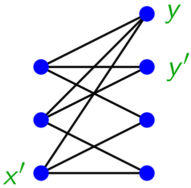
Delete y to form G'. Now
$\Phi(G') \leq |E(G')| = \lfloor n^2/4 - n/2 + 1 \rfloor - \lfloor n/2 \rfloor = \lfloor n^2/4 - n + 1 \rfloor$.
Let f be a pure overlap labeling of G' using one label per edge.
Let y' be a vertex of Y in G' and let x' be its non-neighbor in X.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,
and some $y \in Y$ has degree k
and all others have degree $k-1$.



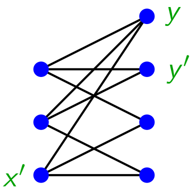
Delete y to form G'. Now
$\Phi(G') \leq |E(G')| = \lfloor n^2/4 - n/2 + 1 \rfloor - \lfloor n/2 \rfloor = \lfloor n^2/4 - n + 1 \rfloor$.
Let f be a pure overlap labeling of G' using one label per edge.
Let y' be a vertex of Y in G' and let x' be its non-neighbor in X.
Extend f to G as follows: let $f(y) = f(y') \cup a$ (where a is a new label) and add a to $f(x')$.

Bipartite Graphs

Lemma: Let G be an n-vertex bipartite graph.
If $n \geq 7$ and $\delta(G) \geq 2$, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

Pf: Since $\varphi(G) \leq |E(G)| - 1$, we have $|E(G)| > n^2/4 - n/2$.
Let X and Y be the parts, with $k = |X| \leq |Y|$. If G has a clone, we can delete it. So at most one vertex of Y has degree k.

Thus $|E(G)| \leq (k-1)(n-k) + 1$,
and $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$,
and some $y \in Y$ has degree k
and all others have degree $k - 1$.



Delete y to form G'. Now
$\Phi(G') \leq |E(G')| = \lfloor n^2/4 - n/2 + 1 \rfloor - \lfloor n/2 \rfloor = \lfloor n^2/4 - n + 1 \rfloor$.
Let f be a pure overlap labeling of G' using one label per edge.
Let y' be a vertex of Y in G' and let x' be its non-neighbor in X.
Extend f to G as follows: let $f(y) = f(y') \cup a$ (where a is a new label) and add a to $f(x')$. So $\varphi(G) \leq \Phi(G') + 1 \leq \lfloor n^2/4 - n + 2 \rfloor$.

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \le n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \le \Phi(G - T) + n$.
Lemma: If $n \ge 7$, then $\Phi(G) \le \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k + 1$

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k+1$
  $|E(G)| \leq (2k+1) + k(n-(2k+1)) + (n-(2k+1))^2/4$

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k+1$
  $|E(G)| \leq (2k+1) + k(n-(2k+1)) + (n-(2k+1))^2/4$
  Edge bound is good enough unless $k = 2$, ...

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k+1$
  $|E(G)| \leq (2k+1) + k(n-(2k+1)) + (n-(2k+1))^2/4$
  Edge bound is good enough unless $k = 2, \ldots$
- G has a triangle T

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k+1$
  $|E(G)| \leq (2k+1) + k(n-(2k+1)) + (n-(2k+1))^2/4$
  Edge bound is good enough unless $k = 2, \ldots$
- G has a triangle T
  - $G - T$ is bipartite

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k+1$
  $|E(G)| \leq (2k+1) + k(n-(2k+1)) + (n-(2k+1))^2/4$
  Edge bound is good enough unless $k = 2, \ldots$
- G has a triangle T
  - $G - T$ is bipartite
  - $G - T$ is triangle-free, but not bipartite

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k + 1$
  $|E(G)| \leq (2k + 1) + k(n - (2k + 1)) + (n - (2k + 1))^2/4$
  Edge bound is good enough unless $k = 2, \ldots$
- G has a triangle T
  - $G - T$ is bipartite
  - $G - T$ is triangle-free, but not bipartite
  - $G - T$ has a triangle T'

General n-vertex graphs

Theorem: If G is an n-vertex graph, then $\varphi(G) \leq n^2/4 - n/2 - 1$.
Lemma: If G has a triangle T, then $\Phi(G) \leq \Phi(G - T) + n$.
Lemma: If $n \geq 7$, then $\Phi(G) \leq \lfloor n^2/4 \rfloor$.

Pf sketch of theorem:

- G is bipartite
- G is triangle-free, but not bipartite
  Consider shortest odd cycle C, with length $2k+1$
  $|E(G)| \leq (2k+1) + k(n-(2k+1)) + (n-(2k+1))^2/4$
  Edge bound is good enough unless $k = 2, \ldots$
- G has a triangle T
  - $G - T$ is bipartite
  - $G - T$ is triangle-free, but not bipartite
  - $G - T$ has a triangle T'
    Now $\Phi(G - T - T') \leq \lfloor (n-6)^2/4 \rfloor$, so
    $\Phi(G) \leq \lfloor (n-6)^2/4 \rfloor + 2n - 3 \leq n^2/4 - n/2 - 1$