Painting Squares with Δ^2-1 shades

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Joint with Landon Rabern
Slides available on my webpage

SIAM Discrete Math
19 June 2014

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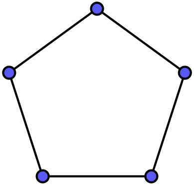

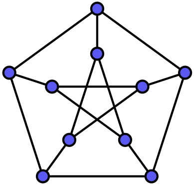

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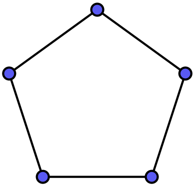

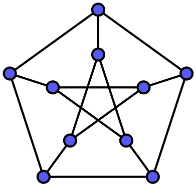

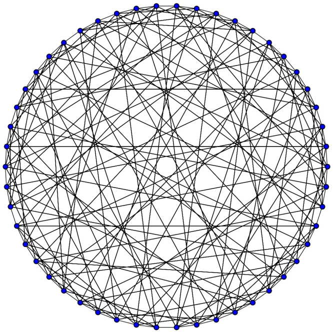

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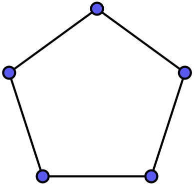

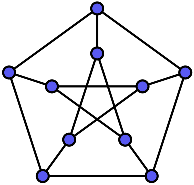

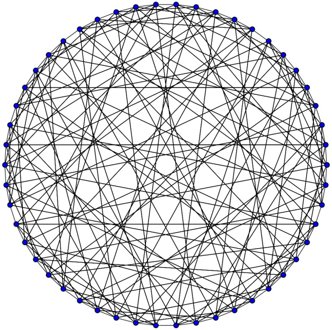

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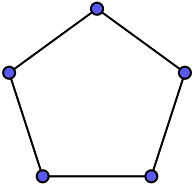

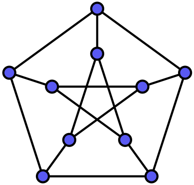

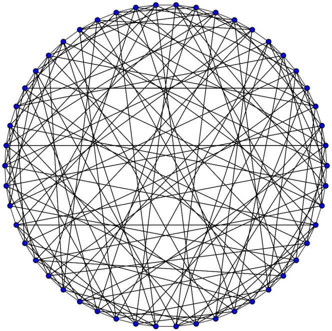

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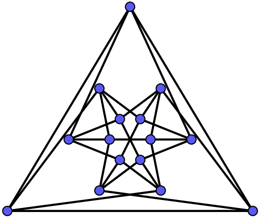

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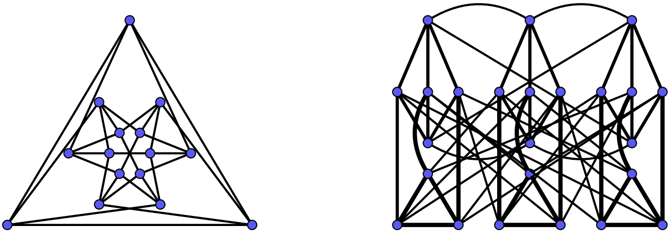

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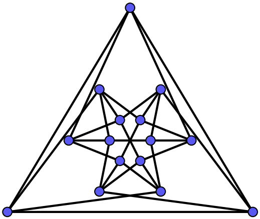 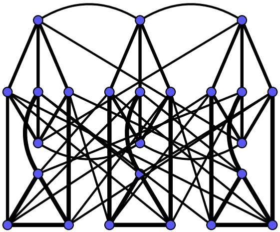

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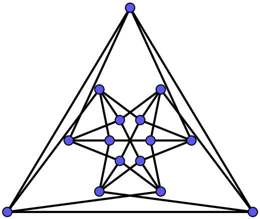

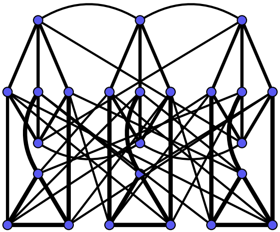

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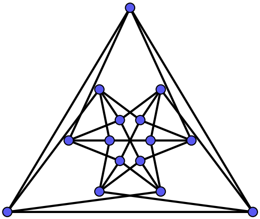

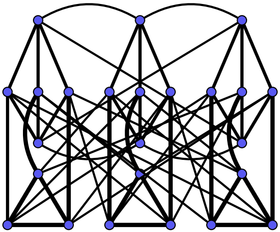

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- Our result implies B–K conj. for G^2 when G has girth ≥ 9.

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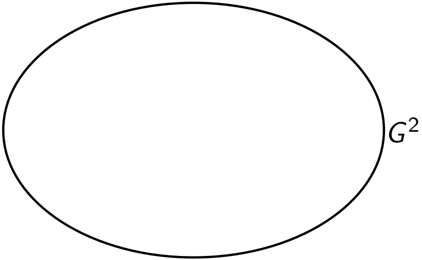

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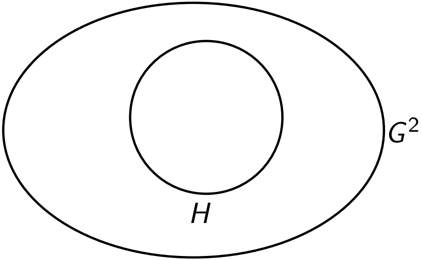

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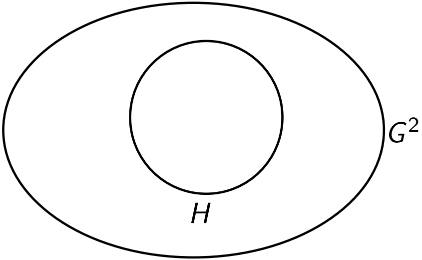

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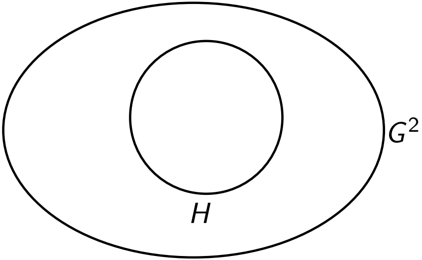

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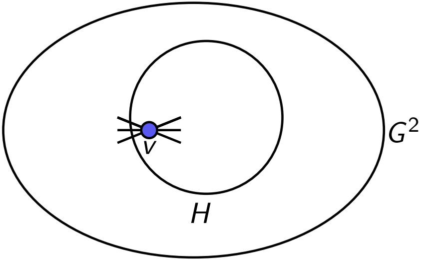

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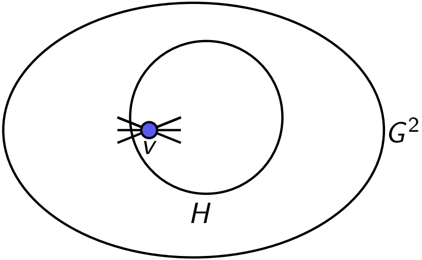

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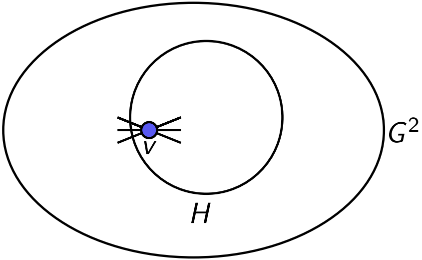

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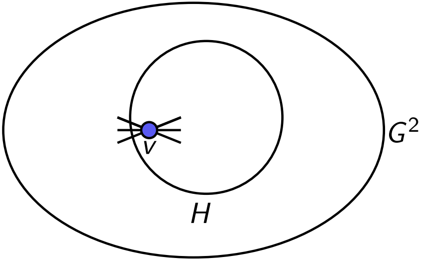

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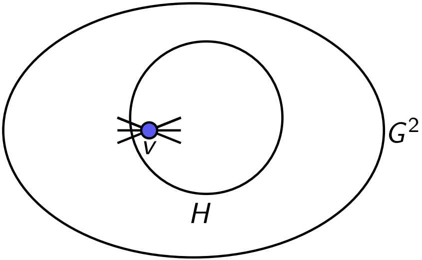

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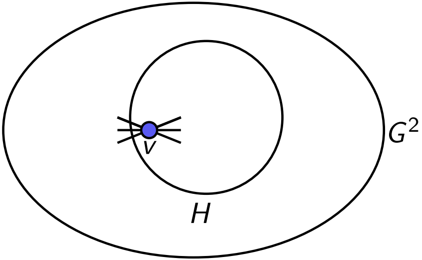

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Where to find d_1-choosable subgraph?

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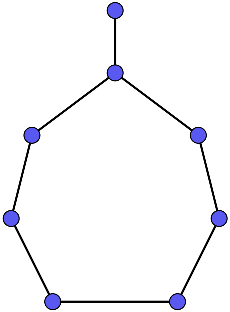

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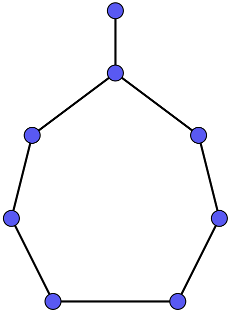

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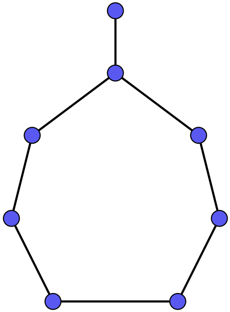

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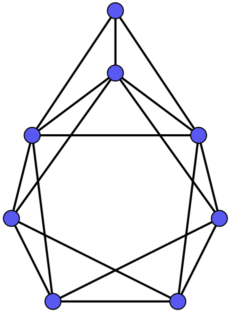

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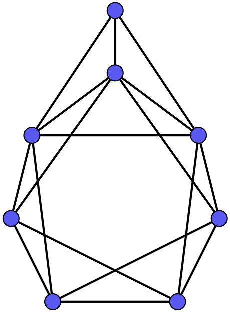

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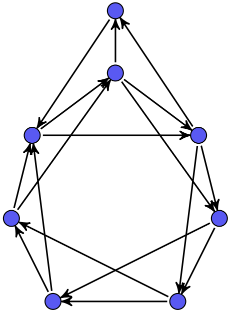

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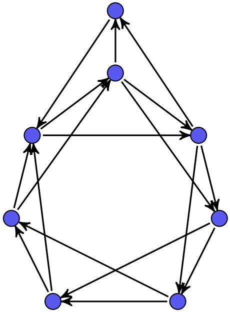

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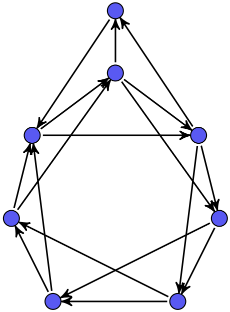

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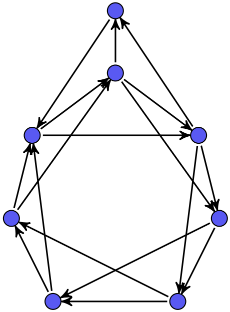

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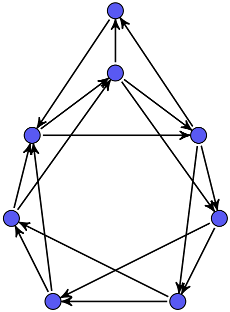

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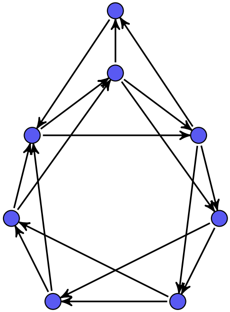

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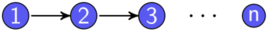

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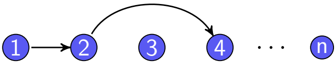

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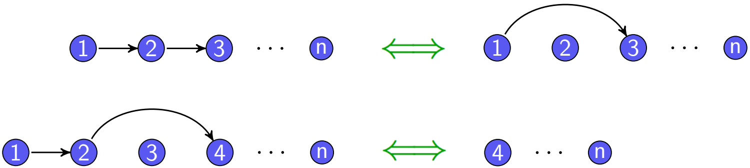

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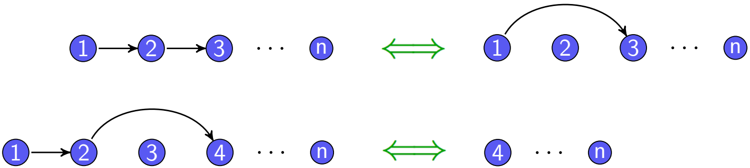

Lemma If $\vec{D}_n$ is the square of C_n, with all edges oriented clockwise, then $|EE(\vec{D}_n)| - |EO(\vec{D}_n)|$ only depends on $n \pmod 3$.

A Gallery of d_1-choosable graphs

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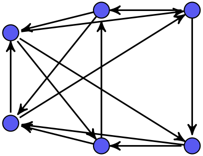
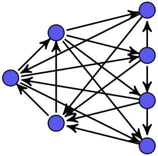


(a) EE=30, EO=28



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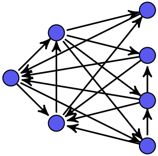
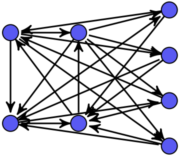
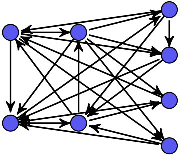


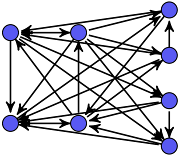
(c) EE=88, EO=87



(d) EE=512, EO=515



(e) EE=751, EO=750



(f) EE=1097, EO=1096

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