Euler's Pentagonal Number Theorem

Dan Cranston

February 22, 2012

Introduction

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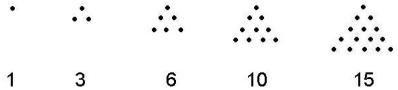
Triangular Numbers: $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$



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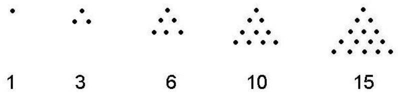
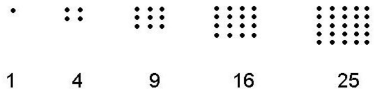


Square Numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...



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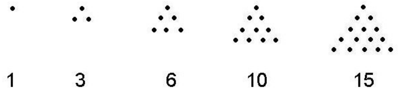

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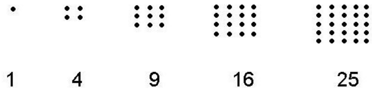
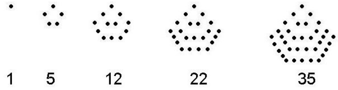
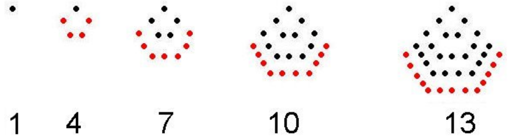


Pentagonal Numbers: 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ...



Generalized Pentagonal Numbers



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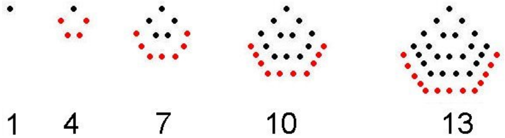

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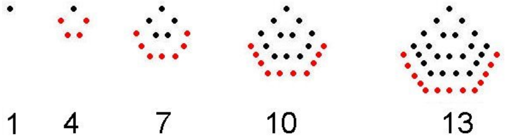

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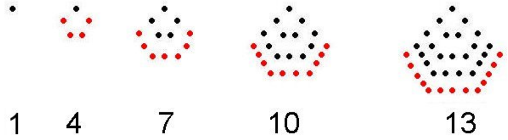

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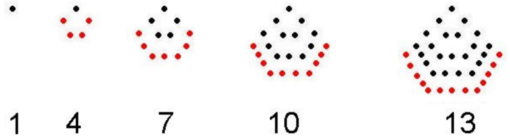

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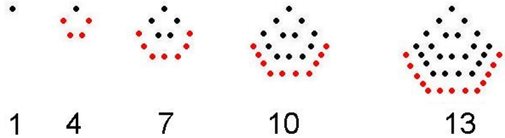

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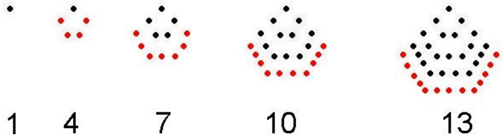

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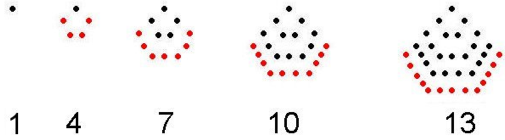

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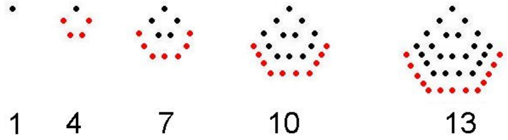

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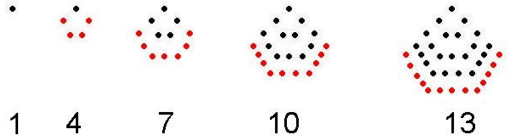

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- $\{P(0), P(1), P(-1), P(2), P(-2), \ldots\} = \{0, 1, 2, 5, 7, \ldots\}$ is an increasing sequence.

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So 3 has 1 part, $2+1$ has 2 parts, and $1+1+1$ has 3 parts.

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Three different partitions of 9:

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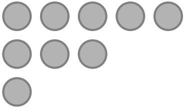
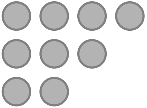


$4 + 3 + 2$



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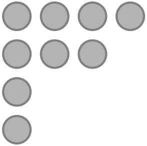

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Pentagonal Number Theorem

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Proof of Lemma 1: The product as a sum

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Pentagonal Number Theorem: Outline of Proof

Lemma 1:

$$\prod_{m=1}^{\infty}(1-x^m) = 1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n$$

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$$1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n = 1 - x - x^2 + x^5 + x^7 + \ldots$$

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Proof Part 2: Cancellation of partition numbers

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- That $p_e(n) - p_o(n) = 0$ unless n is a pentagonal number.

Proof Part 2: Cancellation of partition numbers

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We must show:

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- If n is a pentagonal number $(n = \frac{3k^2-k}{2})$, then $p_e(n) - p_o(n) = (-1)^k$

Proof Part 2: Cancellation of partition numbers

For any partition of n in standard form, we define:
s = number of dots along slope, and
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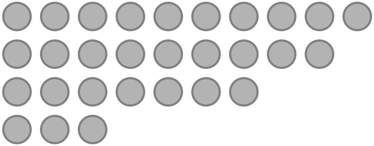

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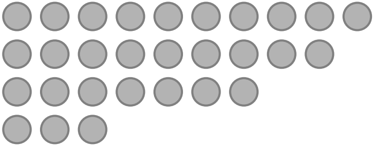

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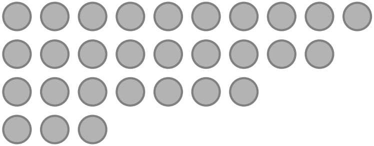

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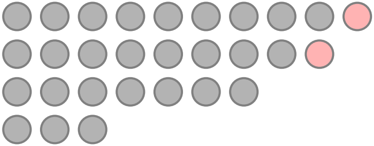

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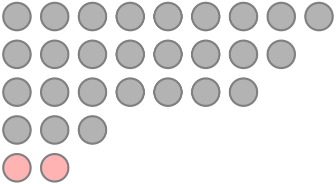

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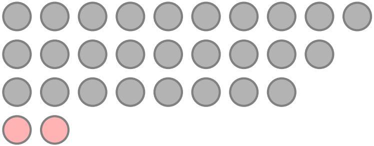

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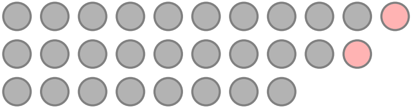

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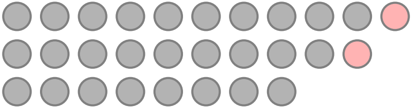

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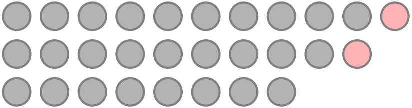

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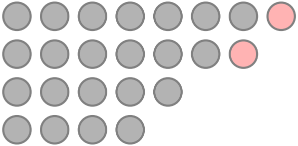

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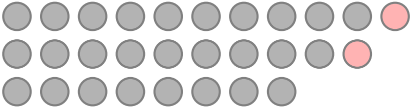

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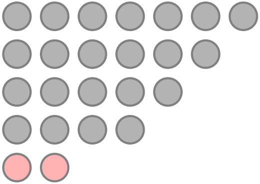

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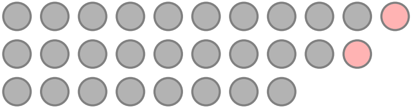

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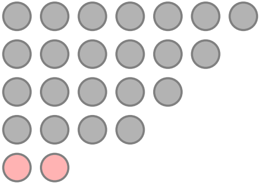
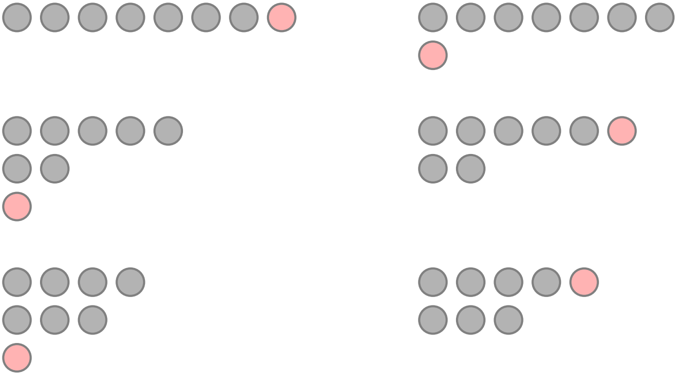


Note: This operation changes the parity of the number of parts.

Proof Part 2: Cancellation of partition numbers

Example: $n = 8$



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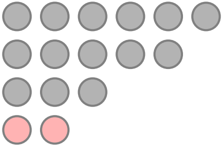

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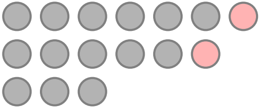

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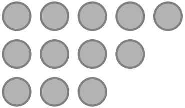

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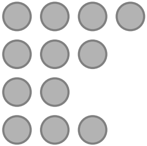

not in standard form!

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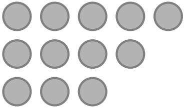

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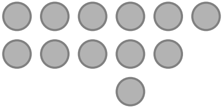

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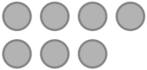

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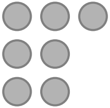

not a valid partition!

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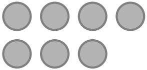

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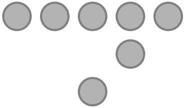

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Proof Part 2: Cancellation of partition numbers

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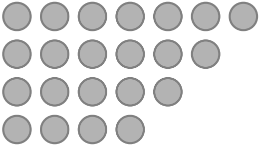
Case 1: $b = s$

Note: The "parity" of this partition is the parity of b.



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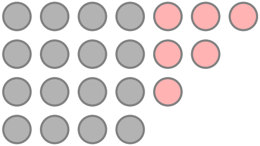

$n = b^2 + \sum_{i=1}^{b-1} i = \frac{2b^2+b(b-1)}{2} = \frac{3b^2-b}{2} = P(b)$

For such n, $p_e(n) - p_o(n) = (-1)^b$.

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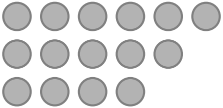

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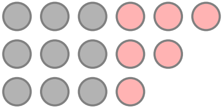

$n = (b-1)^2 + \sum_{i=1}^{b-1} i = \frac{2(b-1)^2 + b(b-1)}{2} = \frac{2(b-1)^2 + b^2 - b}{2} = \frac{2(b-1)^2 + b^2 - 2b - 1 + b - 1}{2} = \frac{3(b-1)^2 + (b-1)}{2} = P(-(b-1))$

For such n, $p_e(n) - p_o(n) = (-1)^{b-1}$.

Proof Part 2: Cancellation of partition numbers

Summary: When n is a pentagonal number, n has exactly one problem partition. We can tell whether the problem partition is even or odd by examining k, where $n = \frac{3k^2-k}{2}$. Otherwise, n has no problem partitions, so we have a bijection between P_e and P_o.

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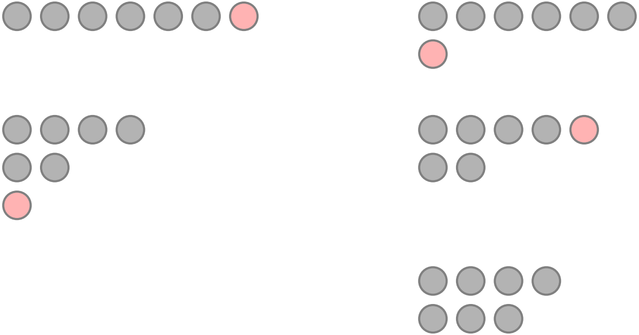
Example: $n = 7$



Pentagonal Number Theorem: Outline of Proof

Lemma 1:

$$\prod_{m=1}^{\infty}(1 - x^m) = 1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n$$

Lemma 2:

$$1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n = 1 - x - x^2 + x^5 + x^7 + \dots$$

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We may now conclude that indeed,

$$\prod_{m=1}^{\infty}(1-x^m) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} + \ldots$$

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