Bootstrap Percolation Thresholds in Plane Tilings using Regular Polygons

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Joint with Neal Bushaw
Slides available on my webpage

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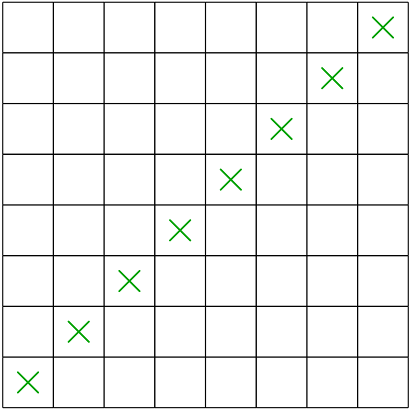

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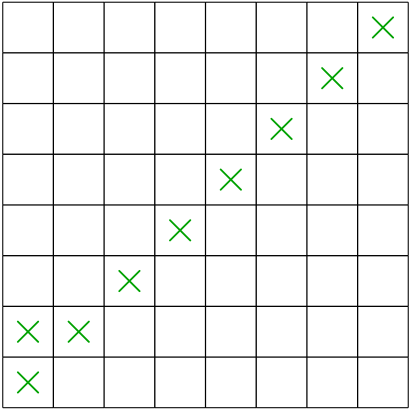

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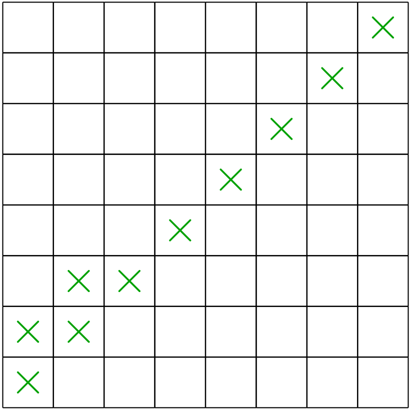

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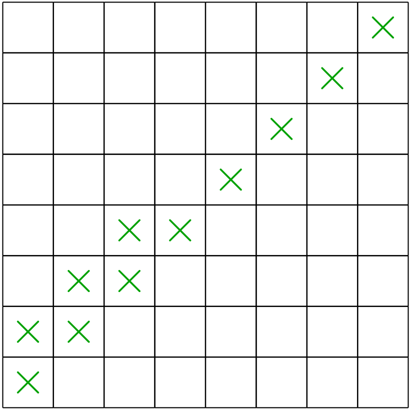

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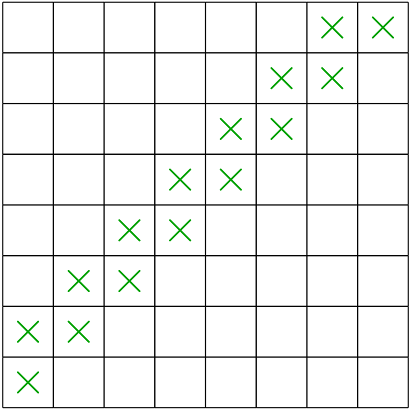

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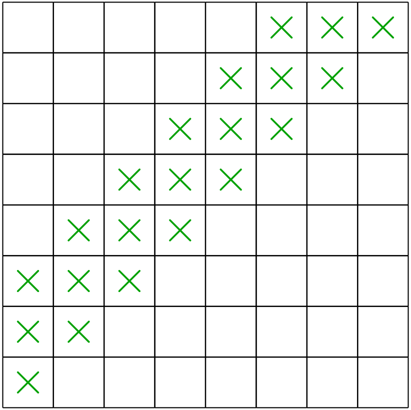

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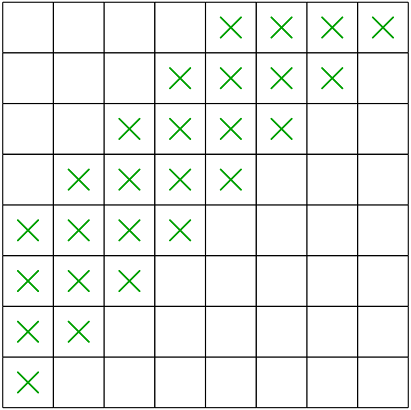

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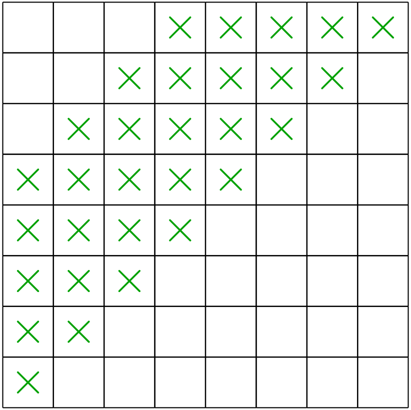

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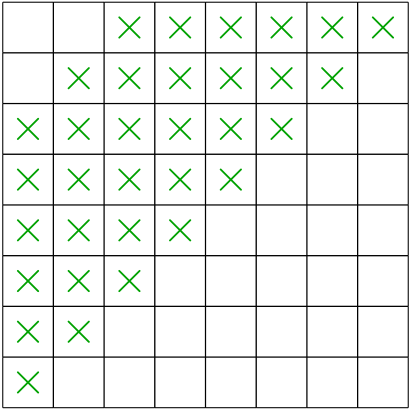

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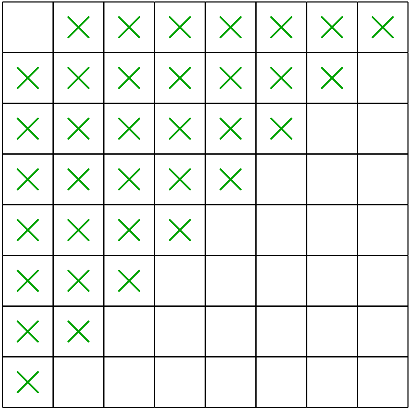

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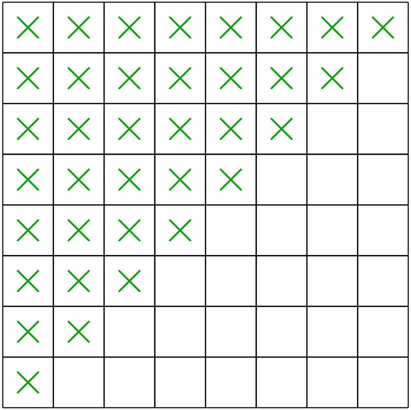

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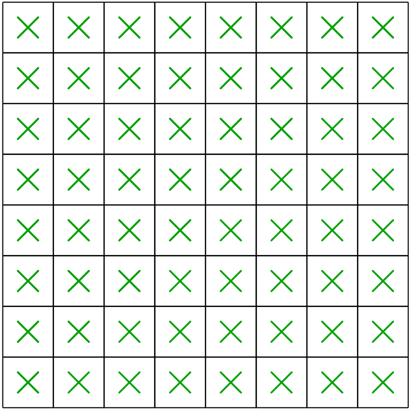

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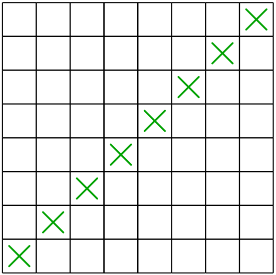

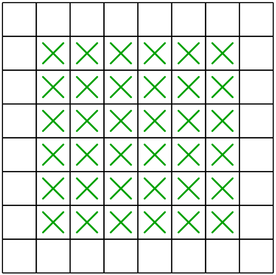

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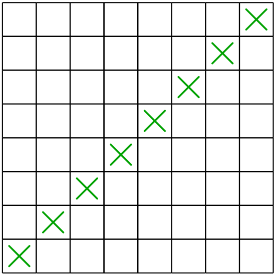

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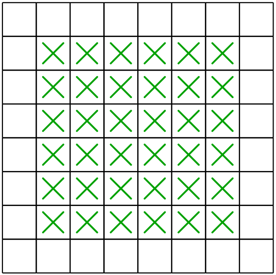

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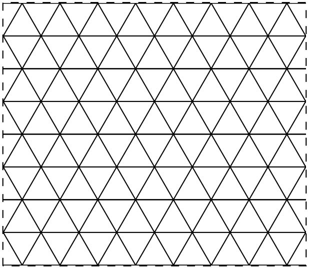

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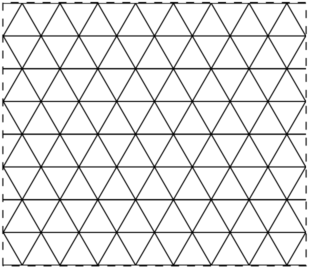

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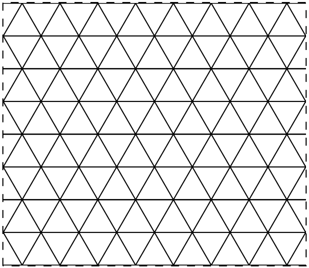

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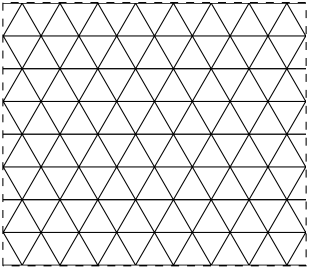

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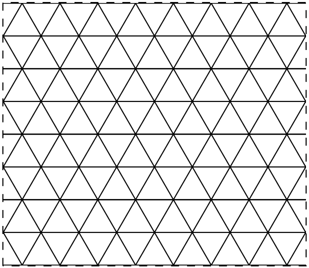

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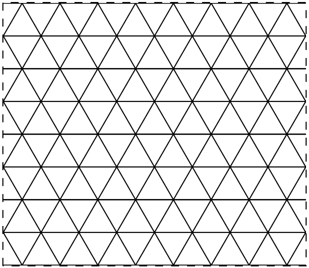

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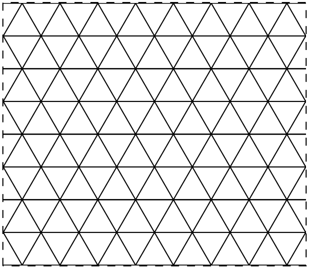

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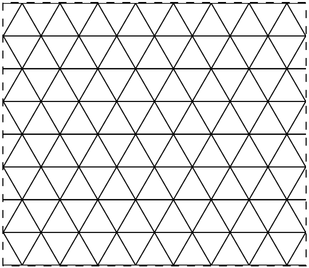

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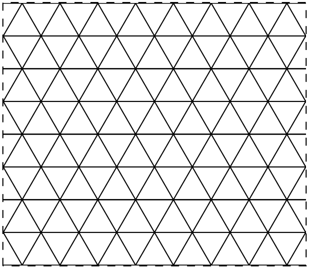

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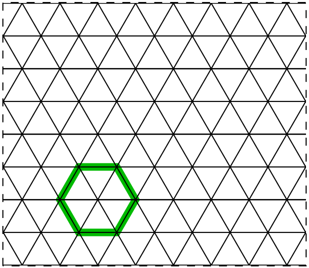

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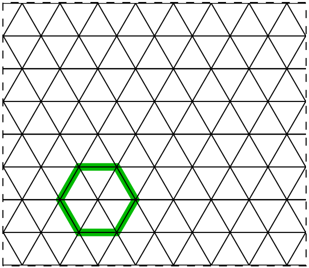

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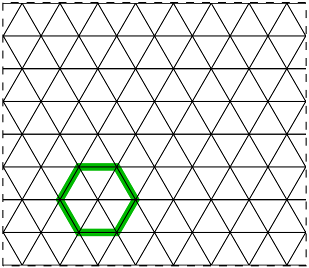

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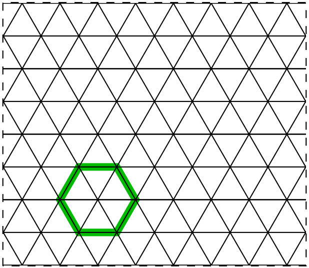

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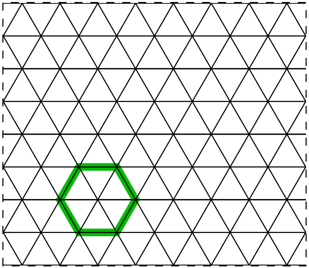

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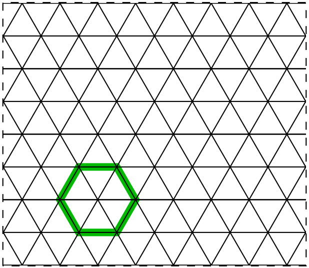

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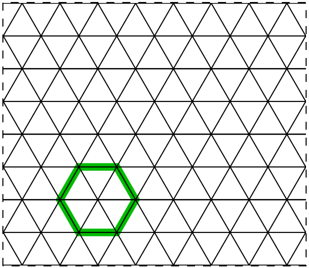

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Lem 2: Let G be the triangular lattice and $\mathcal{I}$ be a p-random set, with $p < 1$. In the 2-bootstrap model, $\mathcal{I}$ percolates with prob. 0.

Cor: The triangular lattice has percolation threshold 1, whenever $0 < p < 1$.

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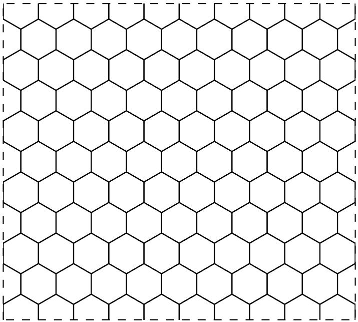

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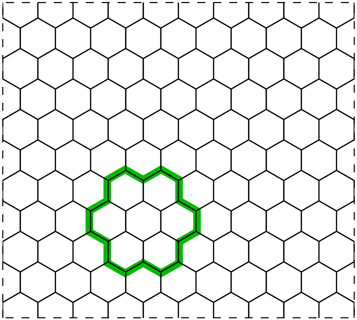

Lem 3: Fix $p < 1$. For the hex lattice, in the 4-bootstrap model, a p-random set $\mathcal{I}$ percolates with prob. 0.

Pf: Same as Lem 2, but with  in place of . ■

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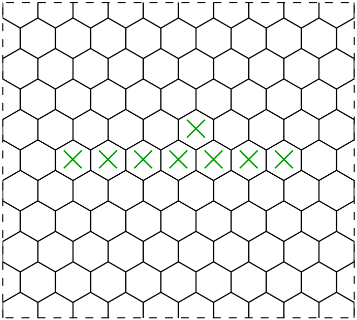

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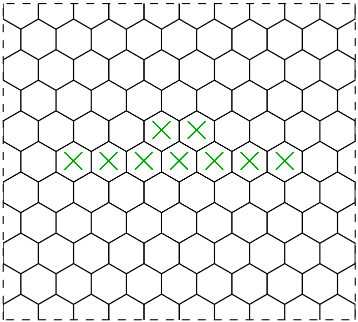

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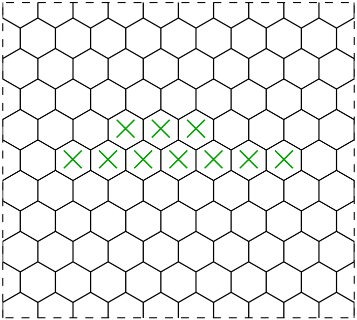

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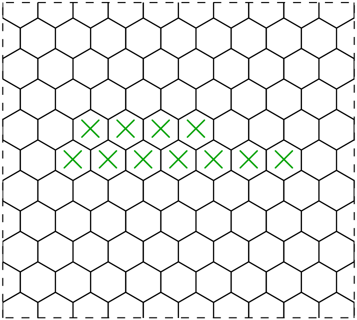

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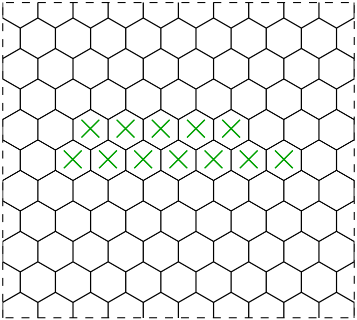

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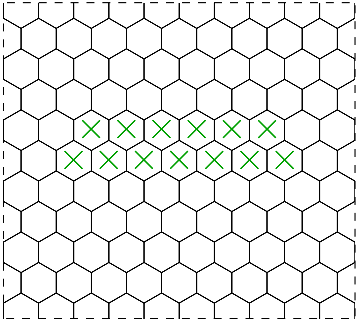

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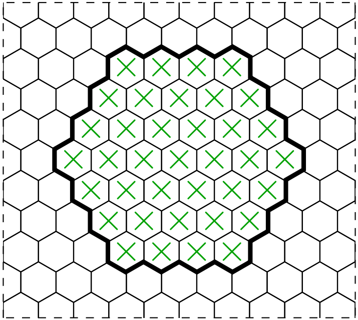

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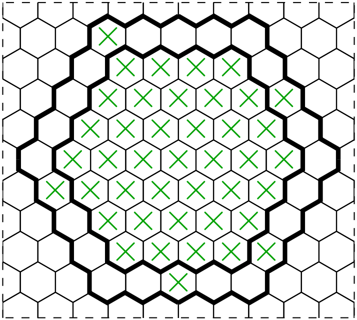

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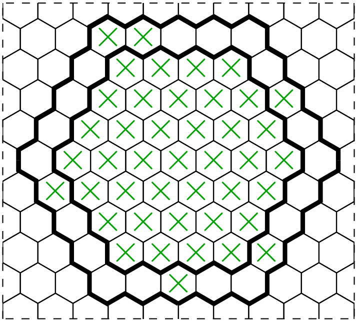

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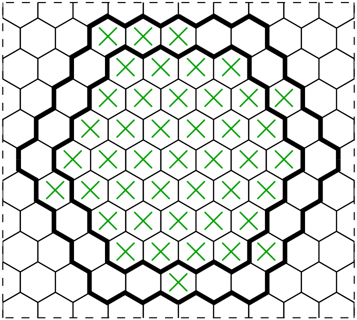

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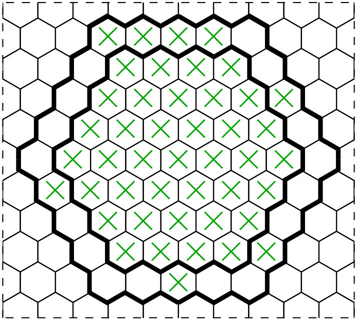

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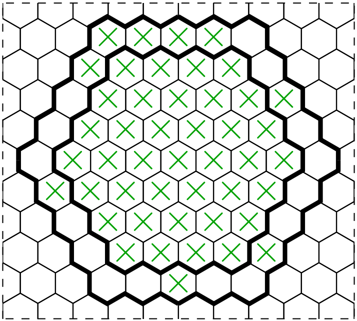

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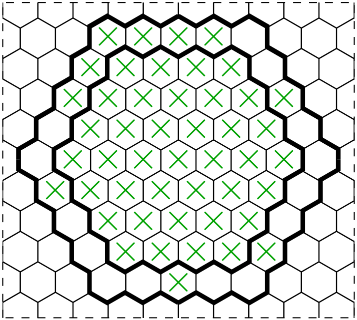

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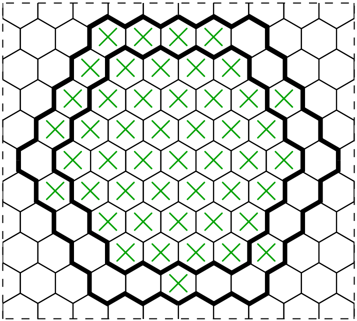

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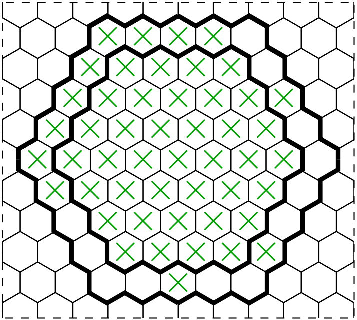

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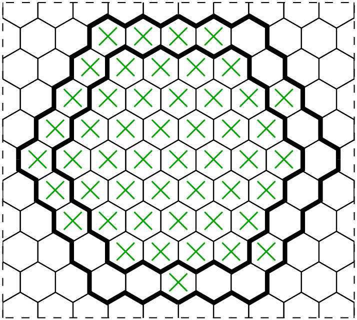

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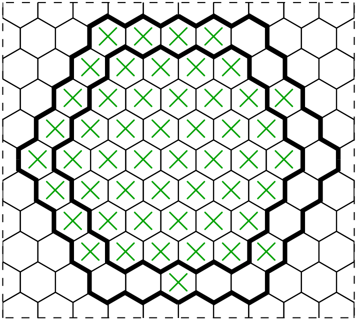

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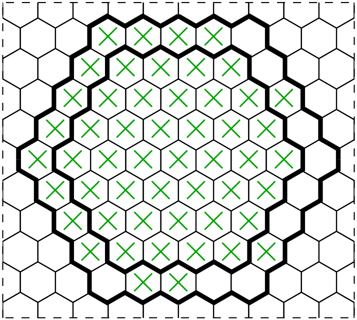

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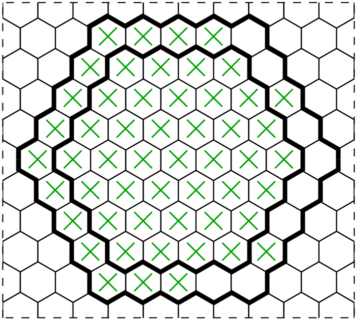

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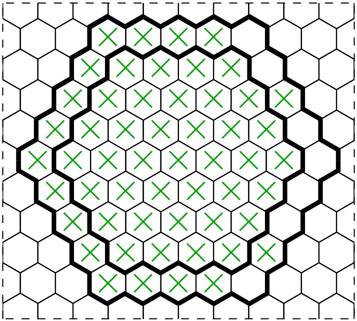

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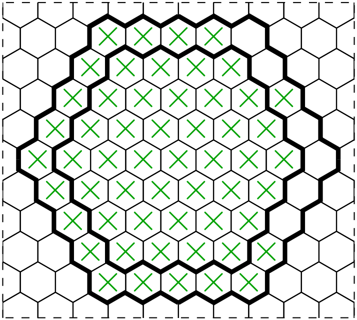

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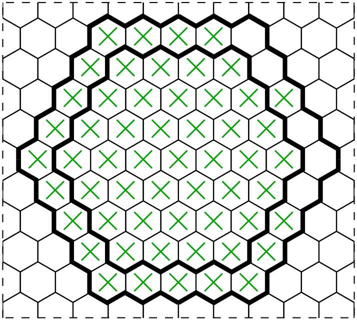

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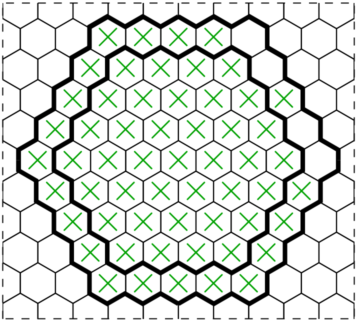

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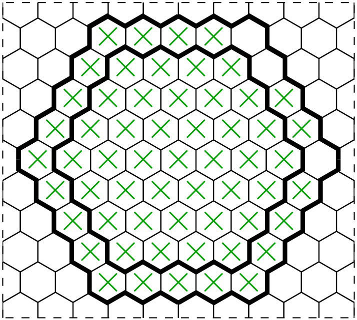

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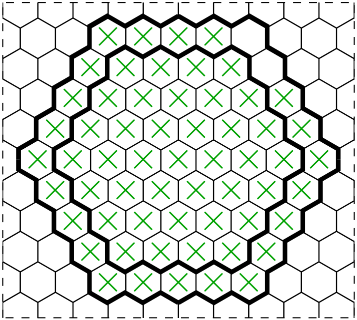

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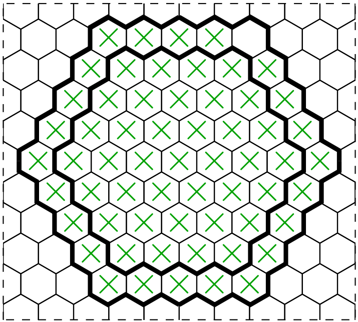

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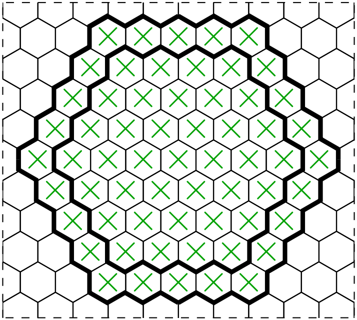

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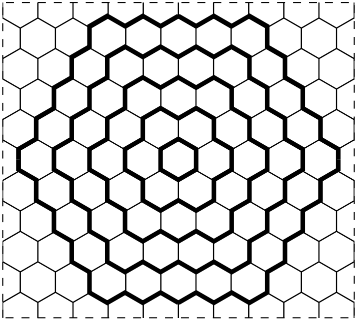

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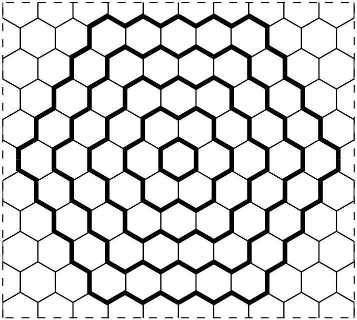

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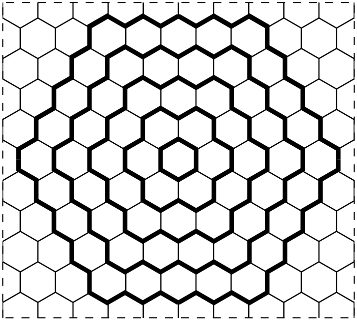

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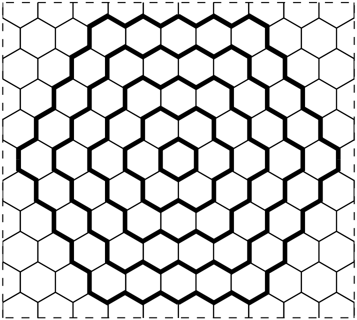

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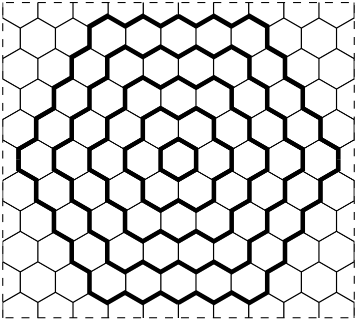

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$\Pr[\mathcal{I} \text{ percolates}] \geq (1-S) \Pr[\text{all small rings good}] > 0$. ■

0–1 Laws

Goal: For hex lattice, in 3-bootstrap model, want to show that p-random set percolates with prob. 1.

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Thm 5: For the hex lattice, in the 3-bootstrap model, a p-random set percolates with prob. 1.
Cor 6: The hex lattice has threshold 3.

Regular Lattices and Beyond

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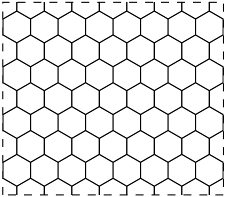
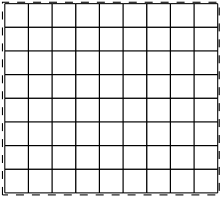
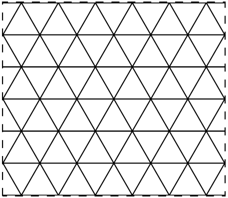

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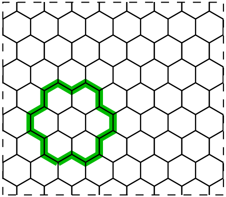

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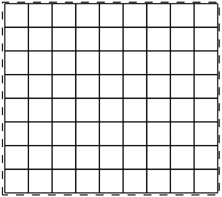

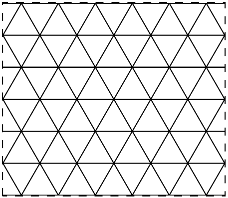

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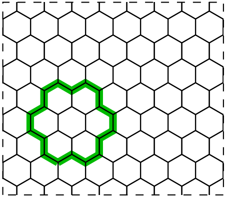

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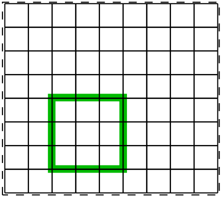

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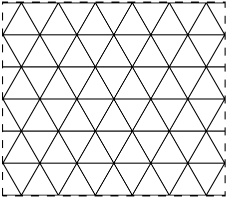

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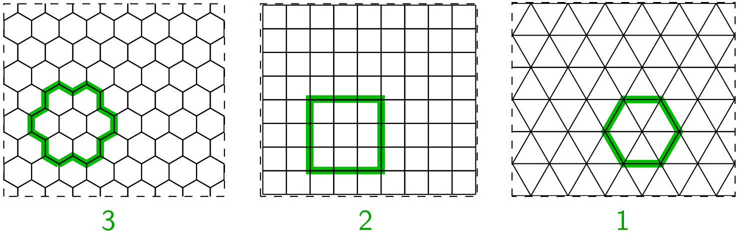

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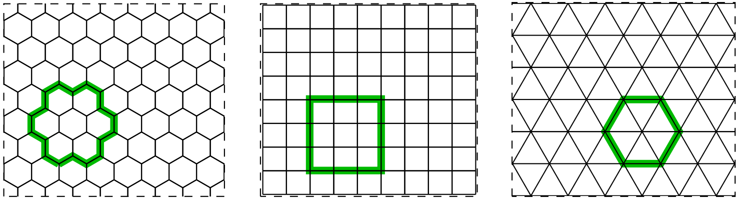

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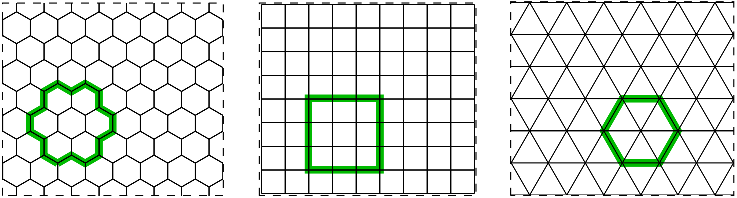
What other graphs to consider?

Regular Lattices and Beyond



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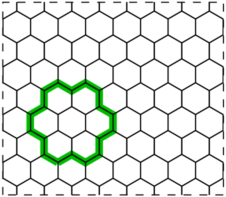
What other graphs to consider?

- Allow more face lengths in same graph

Regular Lattices and Beyond



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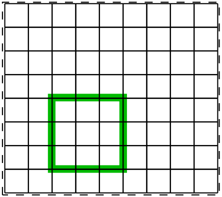

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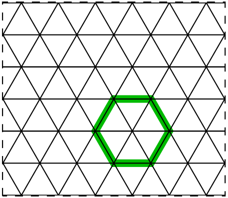

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What other graphs to consider?

- Allow more face lengths in same graph
- All faces are still regular polygons

How Could Vertices Look?

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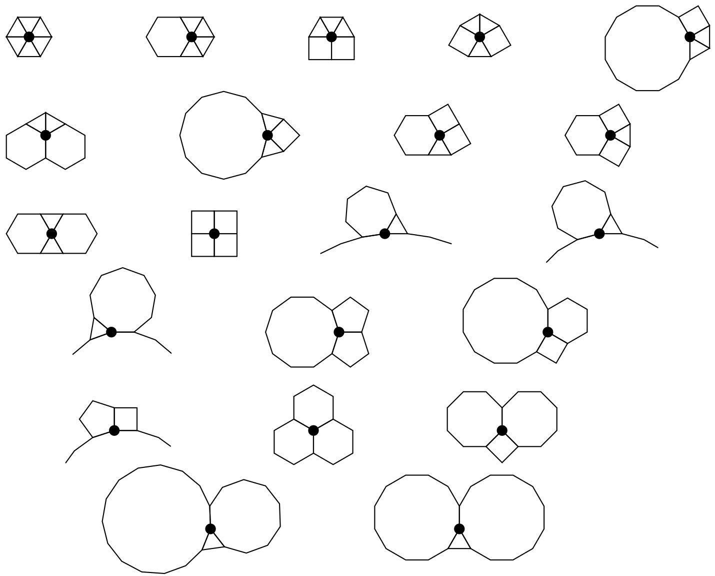
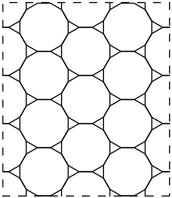
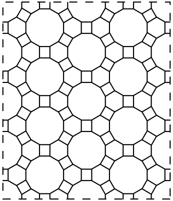


Archimedean Lattices



(3.12.12)



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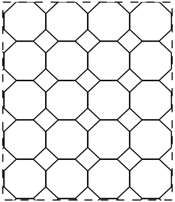
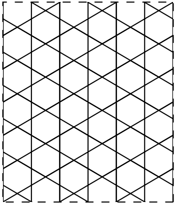
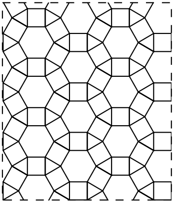


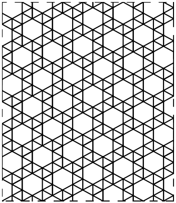
(4.8.8)



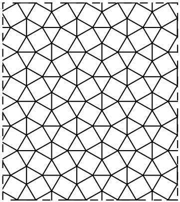
(3.6.3.6)



(3.4.6.4)



(3.3.3.3.6)



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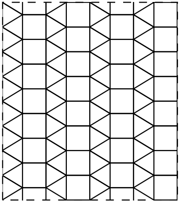
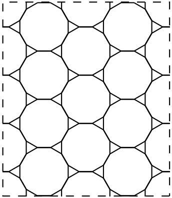
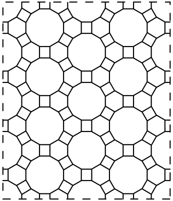


(3.3.3.4.4)

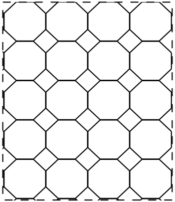
Archimedean Lattices



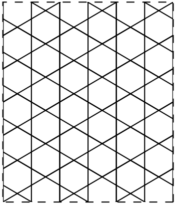
(3.12.12) 3



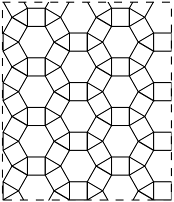
(4.6.12) 3



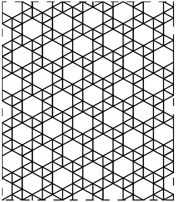
(4.8.8) 3



(3.6.3.6) 2



(3.4.6.4) 1



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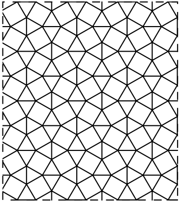
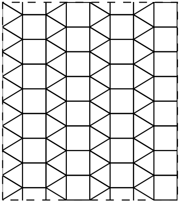
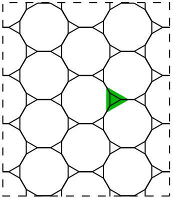


(3.3.4.3.4) 1



(3.3.3.4.4) 1

Archimedean Lattices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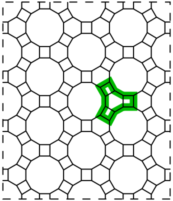
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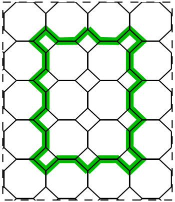
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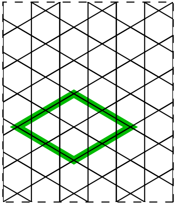
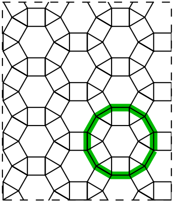
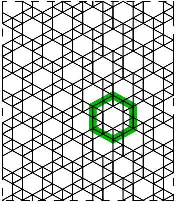
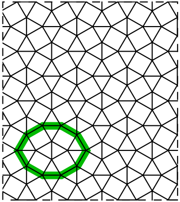

(3.6.3.6) 2


(3.4.6.4) 1


(3.3.3.3.6) 1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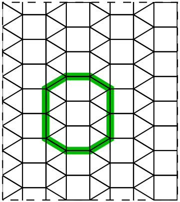
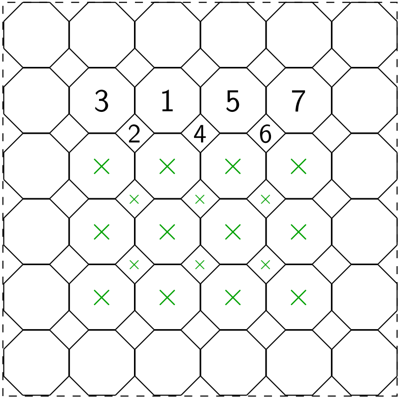

(3.3.3.4.4) 1

Archimedean Lattices: Lower Bounds



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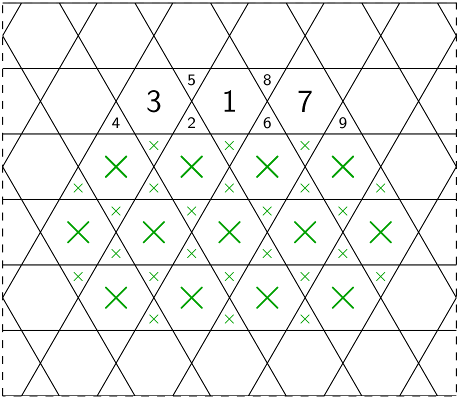

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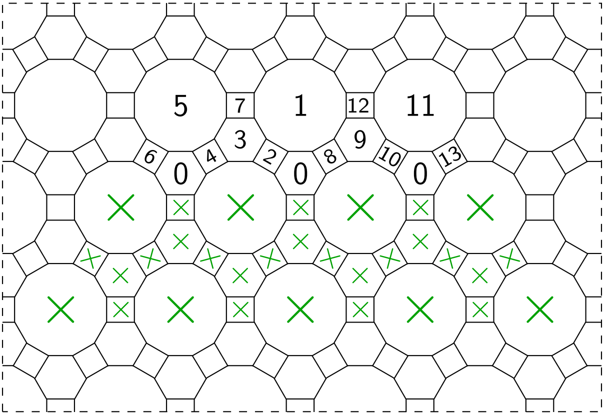

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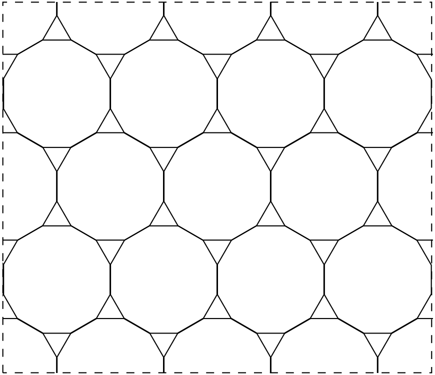
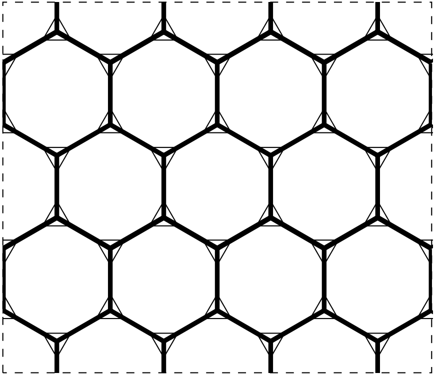


Archimedean Lattices: Lower Bounds



More General Tilings

Defn: Let $\mathcal{T}$ be set of all plane tilings such that if $T \in \mathcal{T}$ and T has one copy of some vertex type, then T has infinitely many copies of that type.

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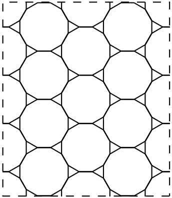
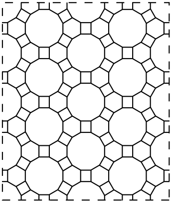
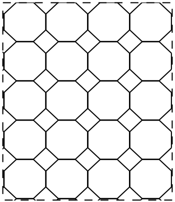


(3.12.12)



(4.6.12)



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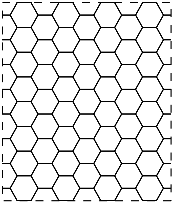

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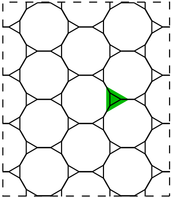
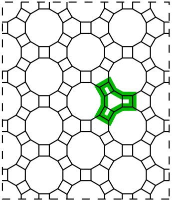
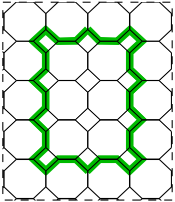


(3.12.12) 3



(4.6.12) 3



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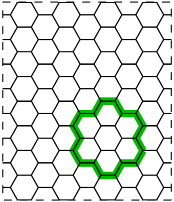

(6.6.6) 3