Fractionally Coloring the Plane

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Joint with Landon Rabern
Slides available on my webpage

VCU Discrete Math Seminar
1 September 2015

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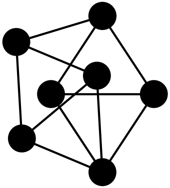

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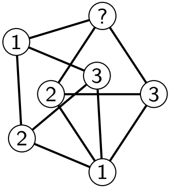

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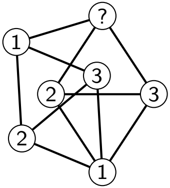

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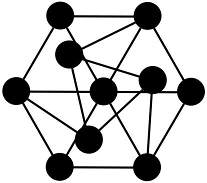

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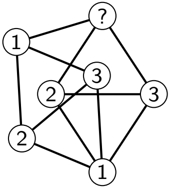

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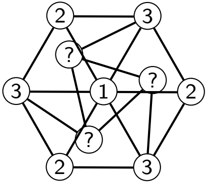

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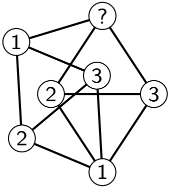

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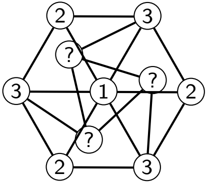

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So $\chi(\mathbb{R}^2) \geq 4$

Coloring the Plane: an Upper Bound

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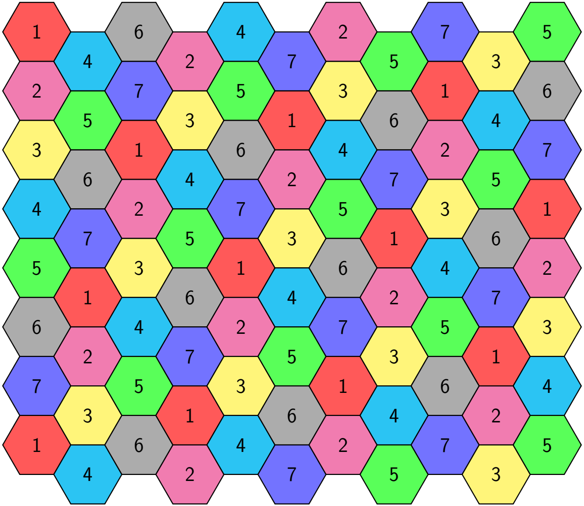
Also, $\chi(\mathbb{R}^2) \leq 7$



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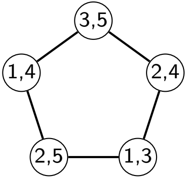

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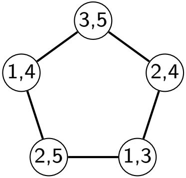

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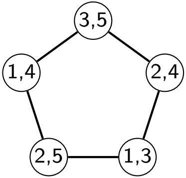

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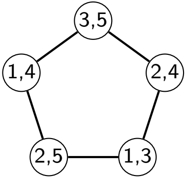

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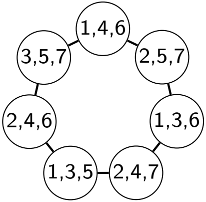

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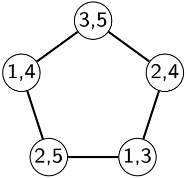

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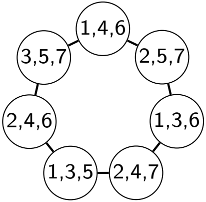

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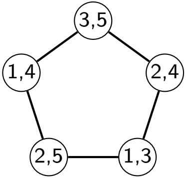

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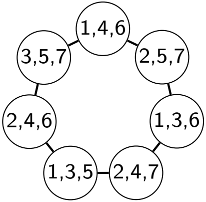

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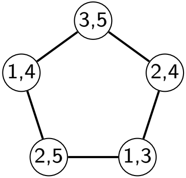

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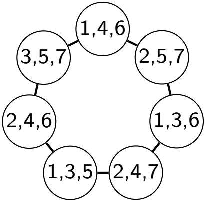

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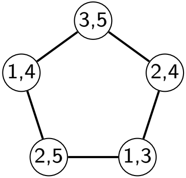

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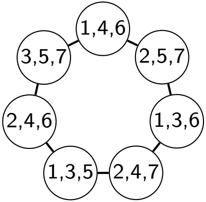

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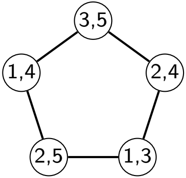

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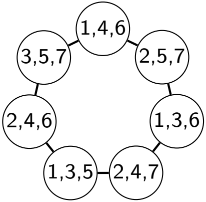

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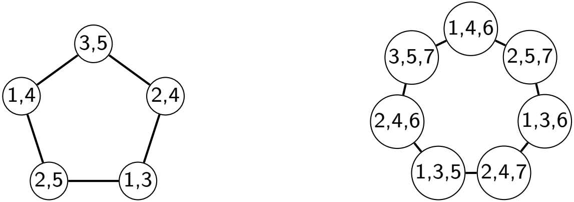

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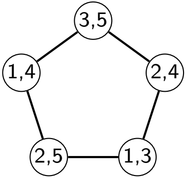

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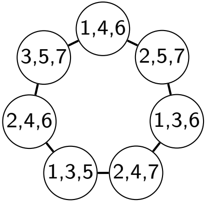

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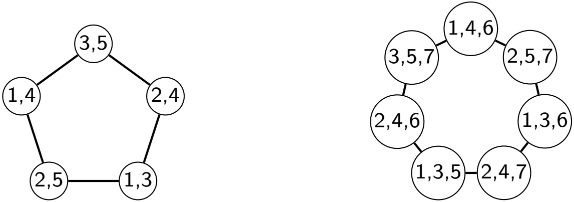

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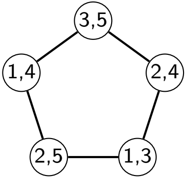

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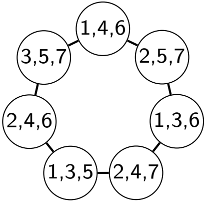

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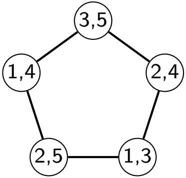

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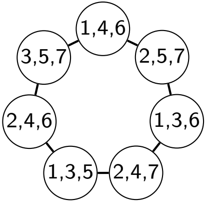

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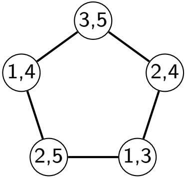

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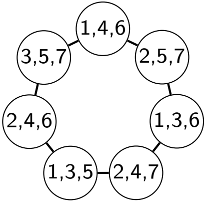

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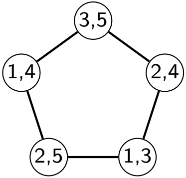

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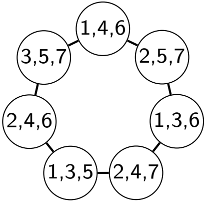

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When G is vertex transitive, $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$.

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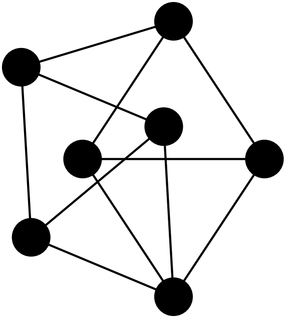

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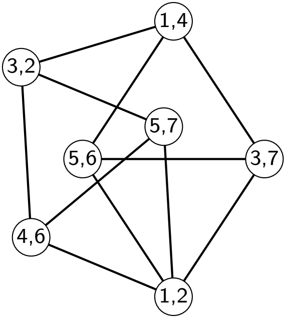

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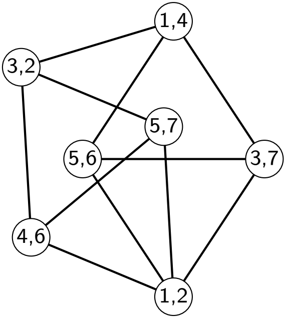

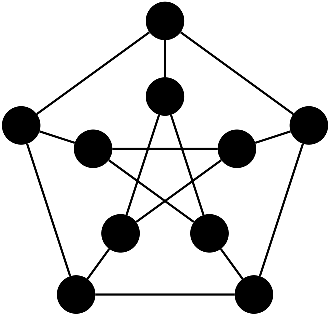

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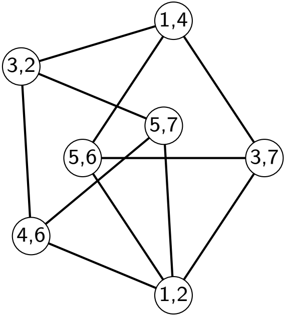

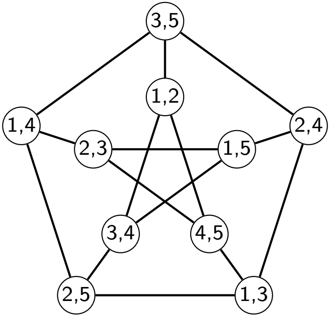

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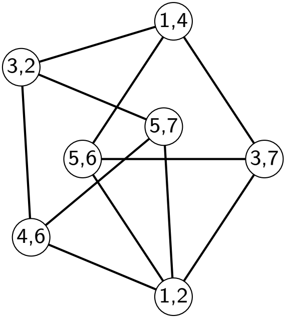

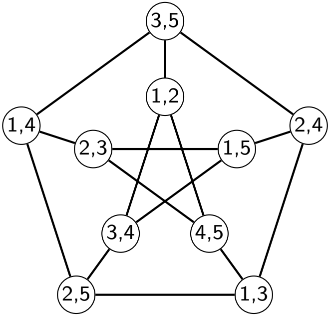

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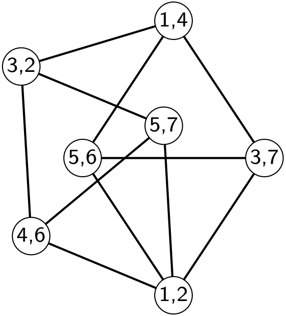

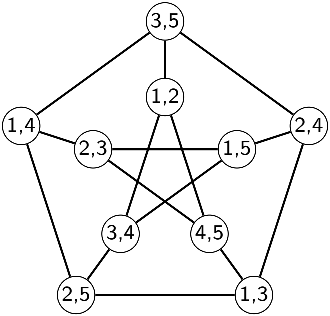

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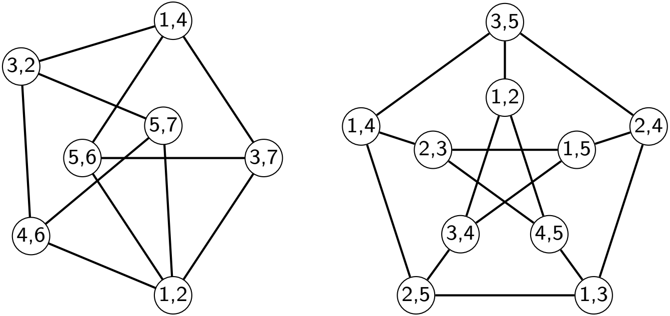

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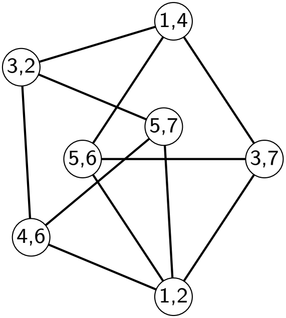
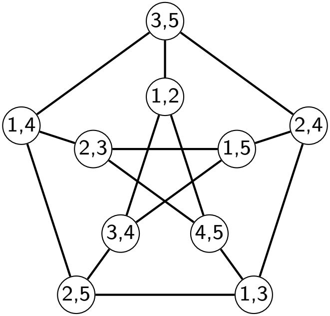
- $|V_\mu(G)| := \sum_{v \in V} \mu(v)$ and $\alpha_\mu(G) := \max_{I \in \mathcal{I}} \sum_{v \in I} \mu(v)$

Fractional Coloring, II

Recall $\chi_f(G) \geq |V(G)|/\alpha(G)$.





More generally:

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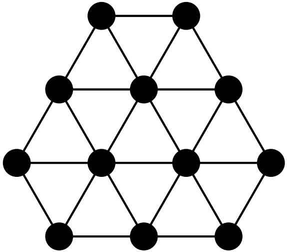

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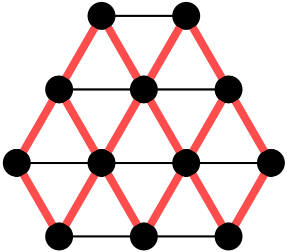

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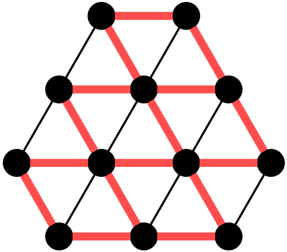

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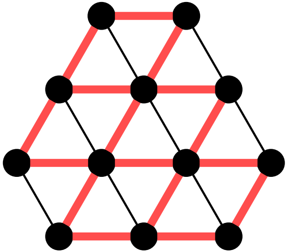

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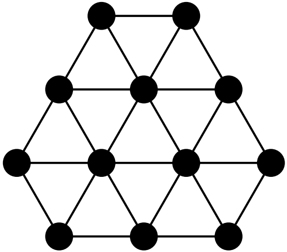

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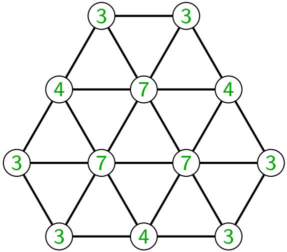

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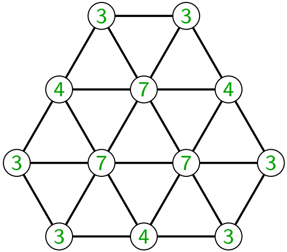

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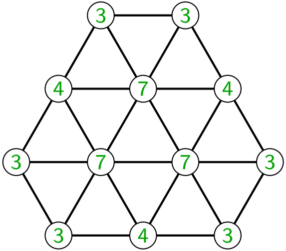

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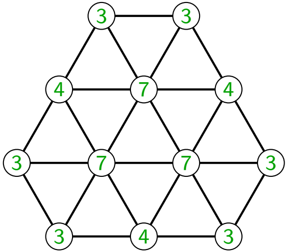

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$$\chi_f(H) \geq 96/27 = 32/9 = 3.5555\ldots$$

Bigger Cores

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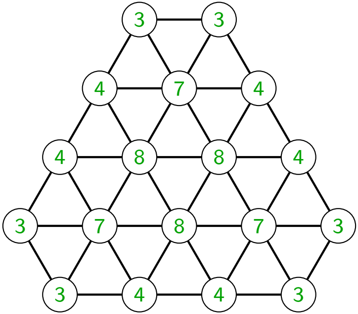

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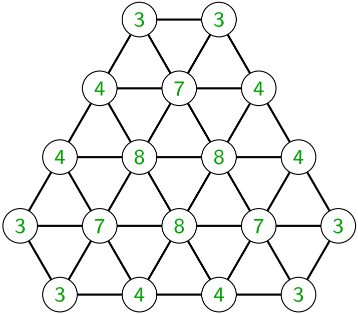

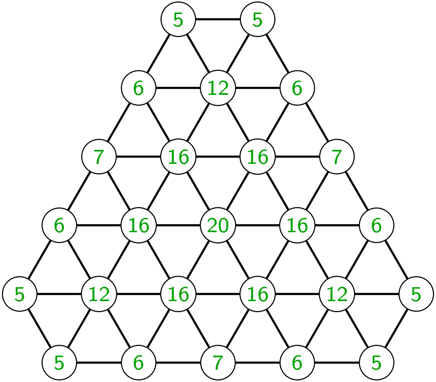

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Spindle weight 2 gives
$\chi_f \geq \frac{491}{137} \approx 3.5839$

Our Biggest Core

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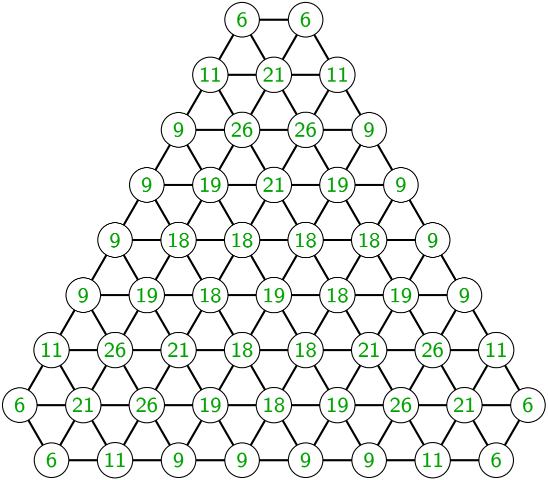

Spindle weight 3 gives $\chi_f \geq \frac{1732}{481} \approx 3.6008$

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Big Idea: Extend same approach to entire plane.

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$$\chi_f \geq 21M/(6M) = 7/2 = 3.5$$

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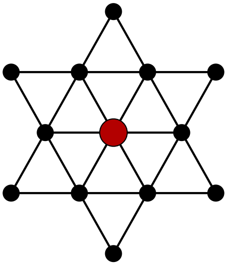

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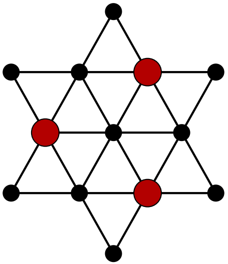

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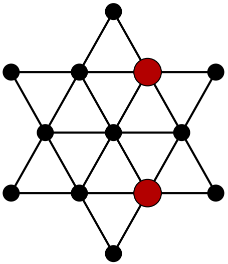

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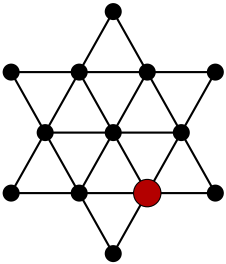

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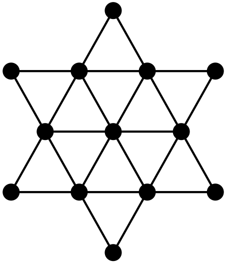

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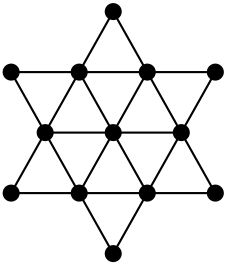

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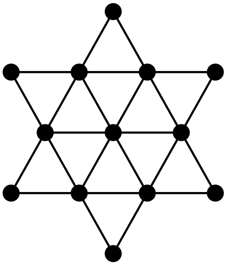

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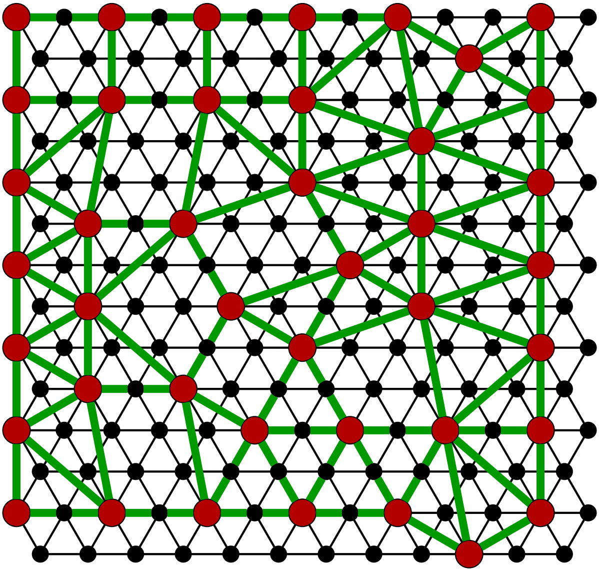
$$\chi_f(\mathbb{R}^2) \geq \frac{105}{29} \approx 3.6207$$

A Tiling for a Better Bound



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