Rosenfeld Counting: Proper Conflict-free Coloring of Graphs with Large Maximum Degree

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CanaDAM
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Nonrepetitive List-coloring of Paths

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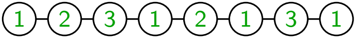

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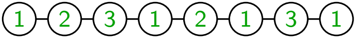

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Since $|\mathcal{C}_1| = 4$, path P_n has more than 2^n nonrepetitive L-colorings.

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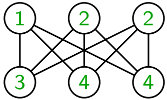

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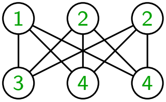

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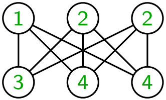

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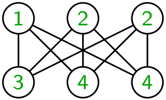

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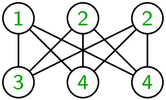

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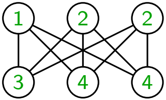

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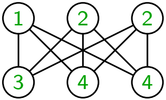

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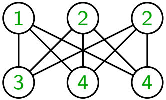

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Rem: Liu and Reed showed that $\chi_{pcf}(G) \leq \Delta(1 + o(1))$. This bound is stronger than ours, but much less general.

Key Rosenfeld Counting Lemma

Defn: For an integer t, a graph G, and a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = V(G)$, a coloring φ is a proper t-conflict-free coloring of $(G, \mathcal{H})$ if φ is a proper coloring of G such that for every $f \in E(\mathcal{H})$, some color is used k times by φ on f for some $k \in \{1, \ldots, t\}$.

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Key Rosenfeld Counting Lemma

Defn: For an integer t, a graph G, and a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = V(G)$, a coloring φ is a proper t-conflict-free coloring of $(G, \mathcal{H})$ if φ is a proper coloring of G such that for every $f \in E(\mathcal{H})$, some color is used k times by φ on f for some $k \in \{1, \dots, t\}$.

Defn: Fix $t, i, d \in \mathbb{Z}^+$. The t-associated Stirling number of second kind, $S_t(d, i)$, is the number of partitions of the set $\{1, \dots, d\}$ into i parts, each of size at least t. E.g. $S_2(2i, i) = (2i)!/(i!2^i)$.

Key Lem: Fix G, $\mathcal{H}$, t as above. Let β be a real number. If a is a real number such that

$$a \geq \Delta(G) + \beta + \sum_{f \in E(\mathcal{H}), f \ni v} \sum_{i=1}^{\lfloor |f|/(t+1) \rfloor} S_{t+1}(|f|, i) \cdot \beta^{i-|f|+1}$$

for every $v \in V(G)$, then for every a-assignment L of G, there are at least $\beta^{|V(G)|}$ proper t-conflict-free L-colorings of $(G, \mathcal{H})$.

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Helper Lem: Fix $i, d \in \mathbb{Z}^+$ with $d \geq 110$. If $0.3d \leq i \leq d/2$, then

$$S_2(d, i) \leq 8i(0.6251d)^{d-i}.$$

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Lem: Fix $d, R \in \mathbb{Z}^+$ with $110 \leq d \leq R$. If $\epsilon, c, \beta \in \mathbb{R}^+$ s.t. $0.6251 \leq \epsilon < 1$, $0.3 \leq c < \epsilon/2$, $\epsilon R \leq \beta \leq R$, and $d \geq f(c, \epsilon, R)$, then

$$\sum_{i=1}^{d/2} S_2(d, i)\beta^{i-d+1} \leq R^{-1/2}.$$

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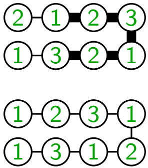

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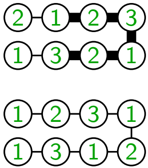

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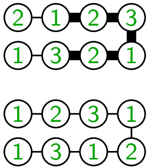

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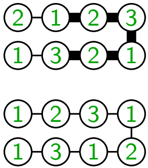

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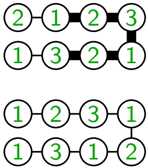

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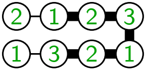

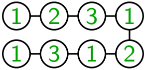

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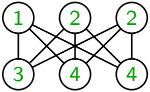

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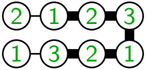

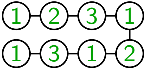

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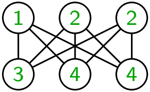

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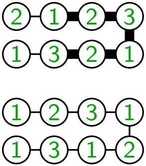

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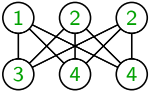

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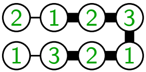

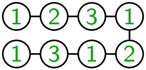

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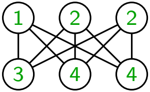

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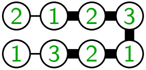

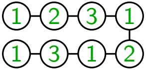

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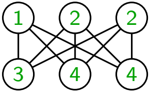

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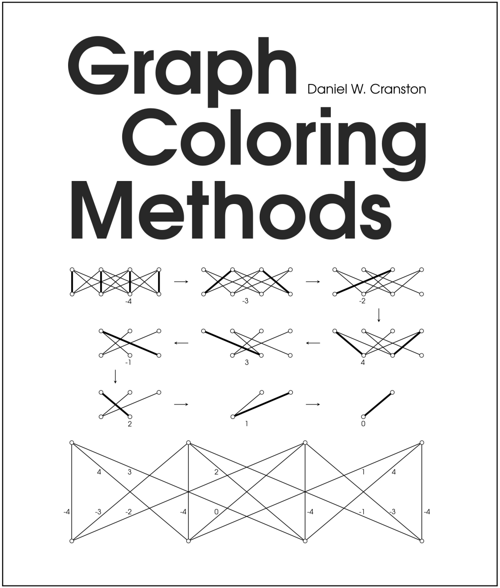

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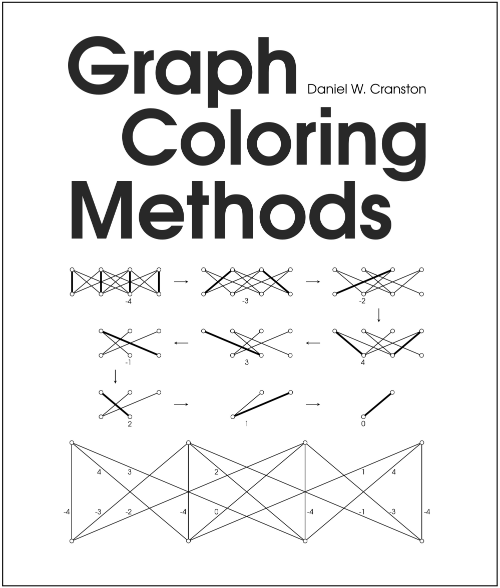

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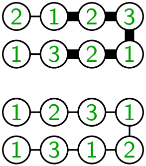

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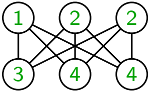

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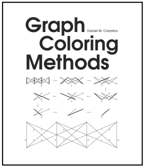

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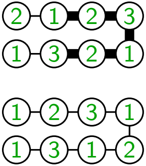

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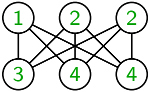

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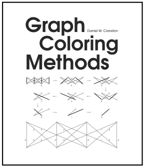

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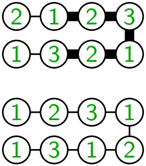

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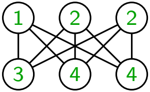

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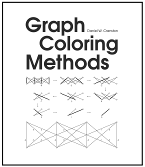

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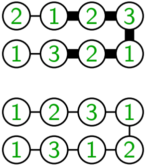

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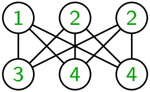

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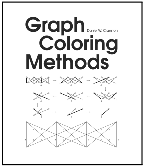

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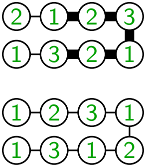

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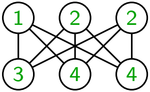

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