Fractional Coloring of Planar Graphs and the Plane

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Joint with Landon Rabern
Slides available on my webpage

Cycles & Colourings
High Tatras
9 September 2015

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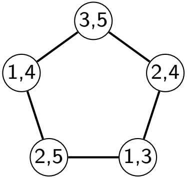

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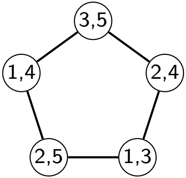

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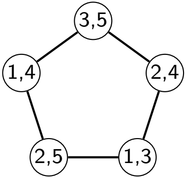

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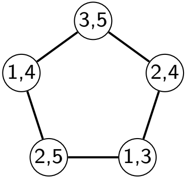

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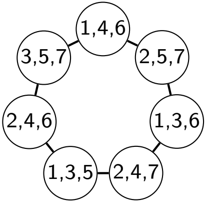

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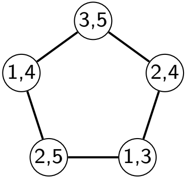

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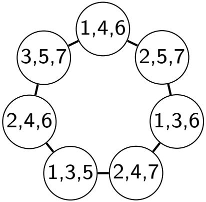

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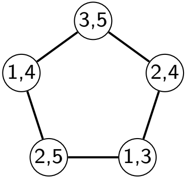

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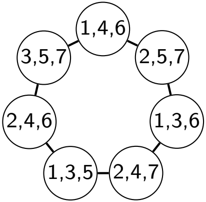

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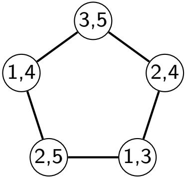

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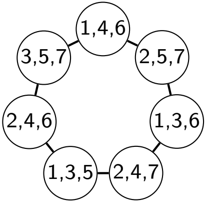

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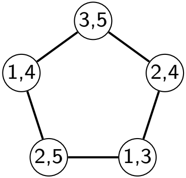

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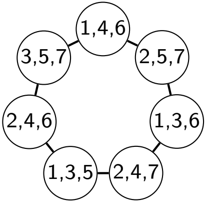

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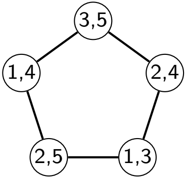

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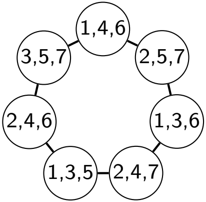

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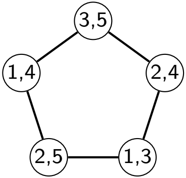

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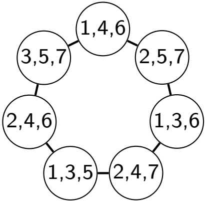

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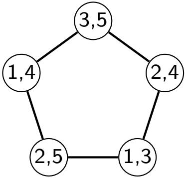

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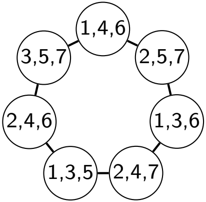

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$$\chi_f = \min_t \frac{\chi_t(G)}{t}.$$

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- Fractional total coloring: $\chi''_f(G) \leq \Delta(G) + 2$. [Kilakos–Reed '93]

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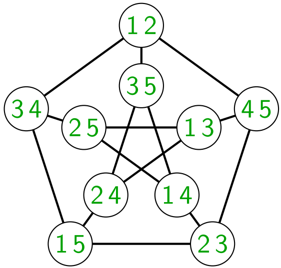

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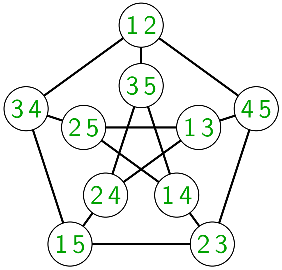

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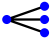
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3.

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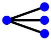
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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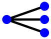
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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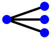
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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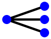
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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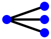
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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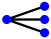
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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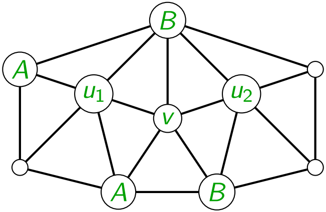

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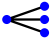
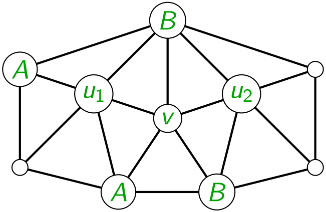
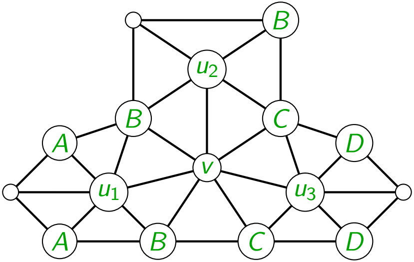
Key Fact: Denote the center vertex of  by v and the other vertices by u_1, u_2, u_3. If v has 5 allowable colors and each u_i has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

Pf: Give v a color available for at most one u_i, say u_1. $2(5) > 3(3)$ Now give v another color not available for u_1. Now color each u_i.





Coloring the Plane

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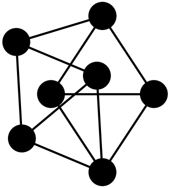

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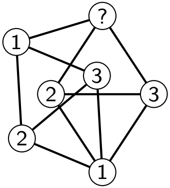

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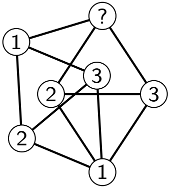

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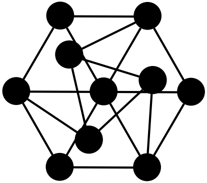

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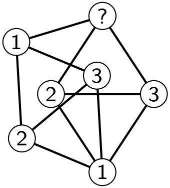

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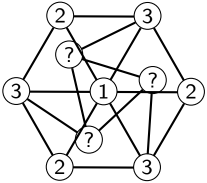

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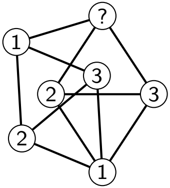

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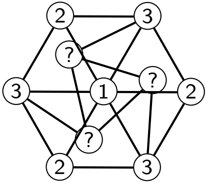

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So $\chi(\mathbb{R}^2) \geq 4$

Coloring the Plane: an Upper Bound

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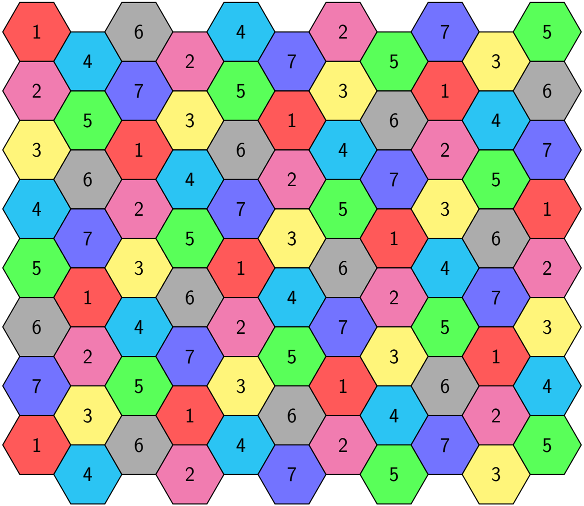
Also, $\chi(\mathbb{R}^2) \leq 7$ [Isbell early '50s]



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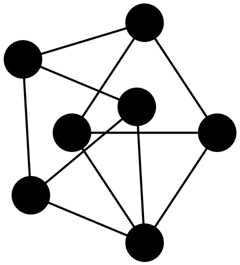

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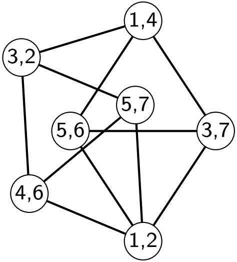

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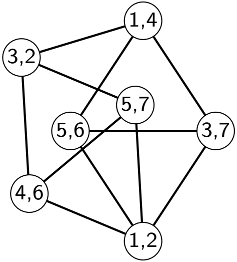

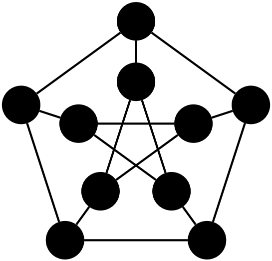

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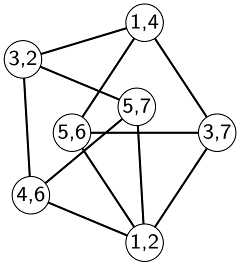

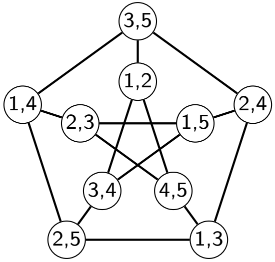

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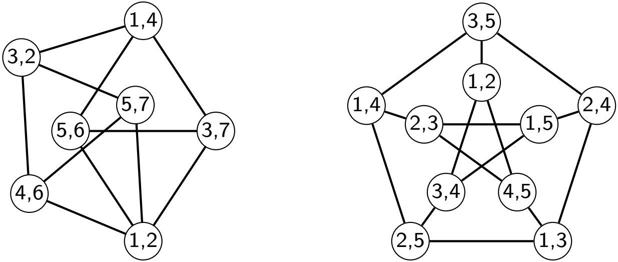

More generally, for every weight function μ,

$$\chi_f(G) \geq |V_\mu(G)|/\alpha_\mu(G).$$

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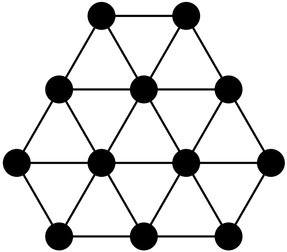

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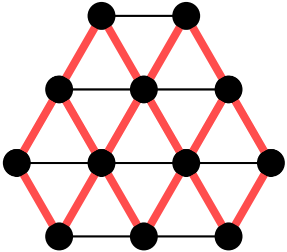

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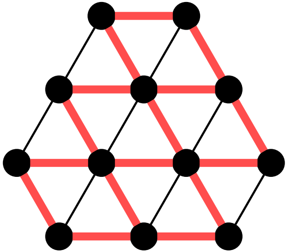

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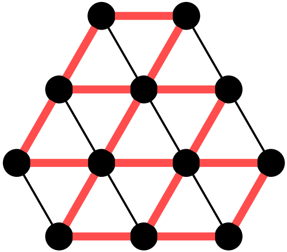

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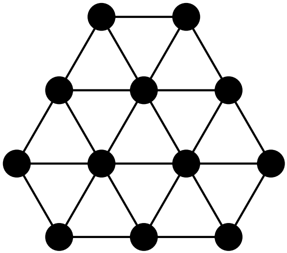

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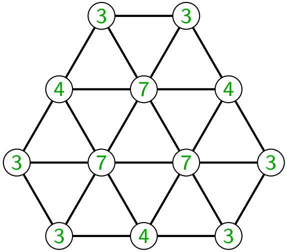

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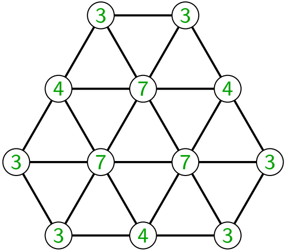

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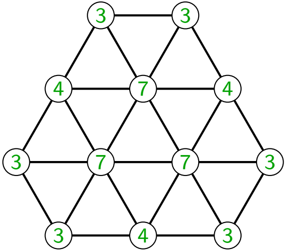

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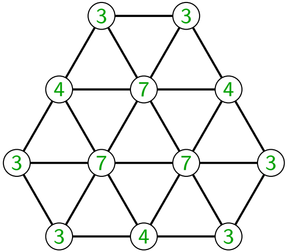

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$$\chi_f(H) \geq 96/27 = 32/9 = 3.5555\ldots$$

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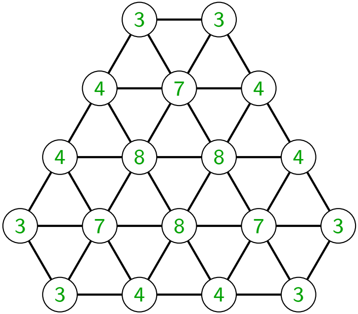

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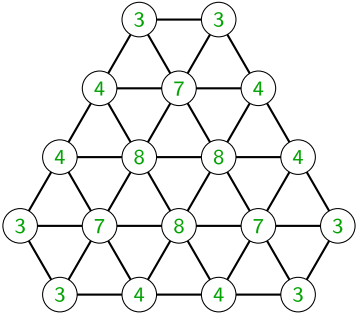

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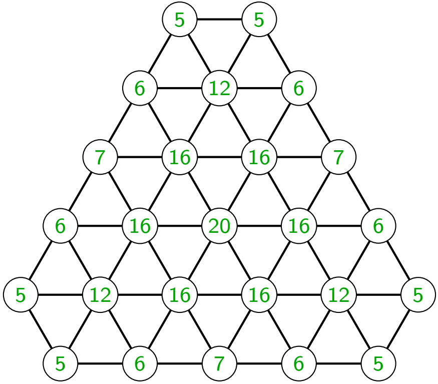

Spindle weight 2 gives
$\chi_f \geq \frac{491}{137} \approx 3.5839$

Our Biggest Core

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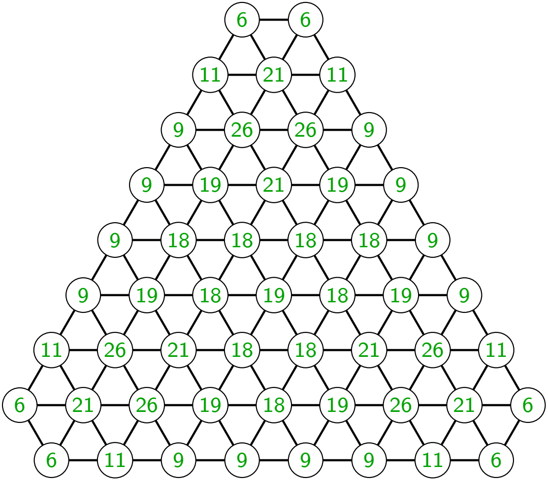

Spindle weight 3 gives $\chi_f \geq \frac{1732}{481} \approx 3.6008$

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Big Idea: Extend same approach to entire plane.

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$$\chi_f \geq 21M/(6M) = 7/2 = 3.5$$

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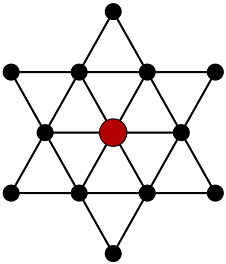

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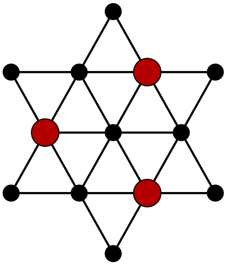

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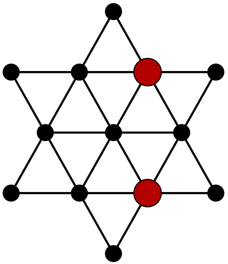

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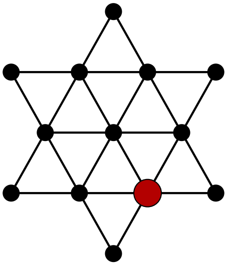

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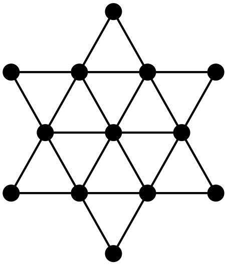

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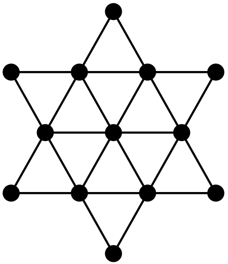

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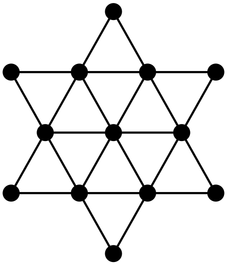

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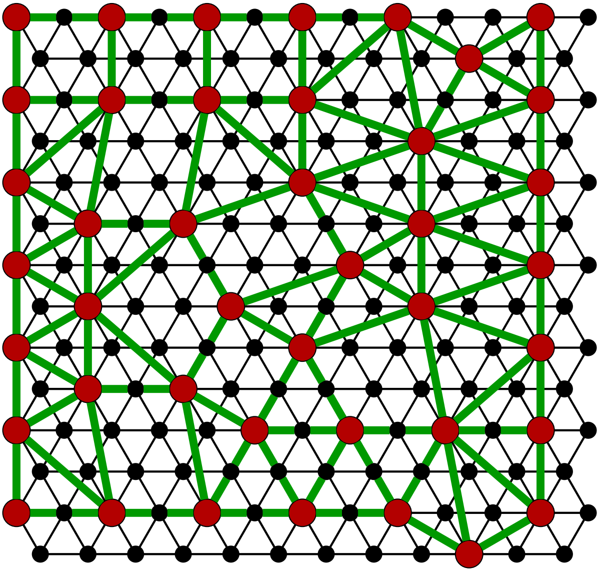

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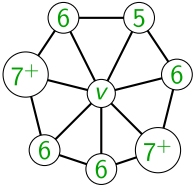

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If $d_{H_v}(w) = 0$, then w is isolated nbr of v;
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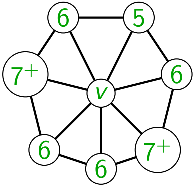

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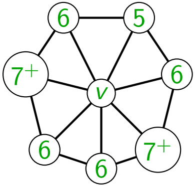
Def: H_v is subgraph induced by 6^--nbrs of v.
If $d_{H_v}(w) = 0$, then w is isolated nbr of v;
otherwise w is non-isolated nbr of v.
A non-isolated 5-nbr of vertex v is crowded (w.r.t. v) if it has two 6-nbrs in H_v.



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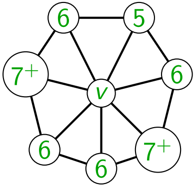

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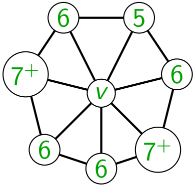

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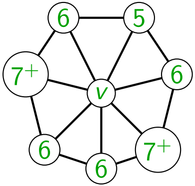

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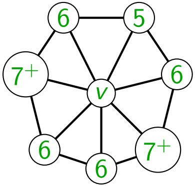

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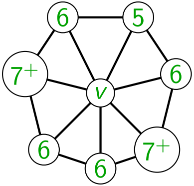

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Now show that $ch^*(v) \geq 0$ for all v.