Revolutionaries and Spies

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Slides available on my preprint page
Joint with Jane Butterfield, Greg Puleo,
Cliff Smyth, Doug West, and Reza Zamani

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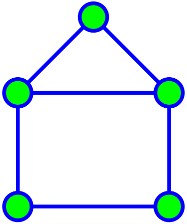

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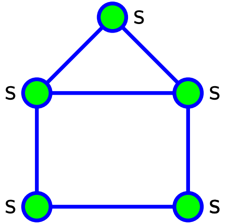

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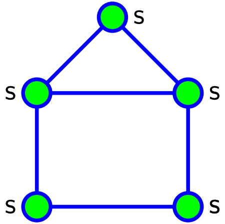

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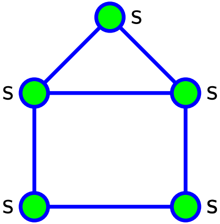

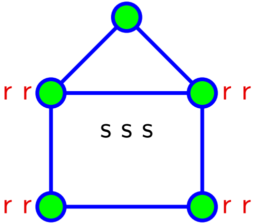

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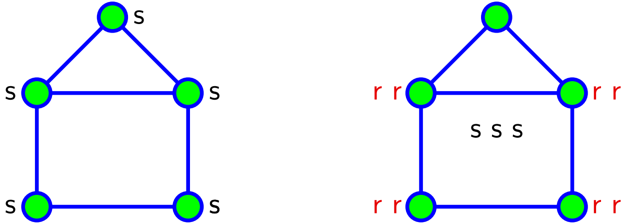

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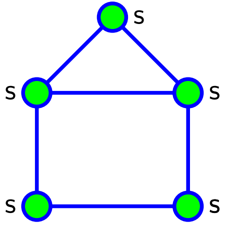

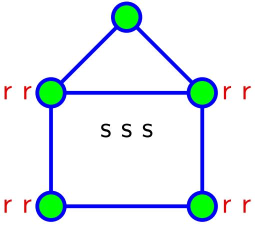

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Def: $\sigma(G, m, r)$ is minimum number of spies needed to win on G

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$$\sigma(G,2,r) = \frac{7}{10} r = \frac{7}{5}\frac{r}{2}$$

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Conj: As m grows: $\sigma(G,m,r) \sim \frac{3}{2}\frac{r}{m}$

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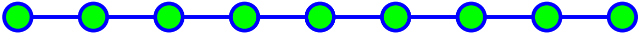

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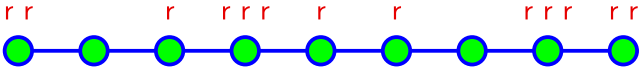

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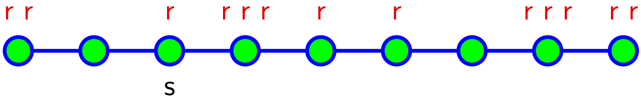

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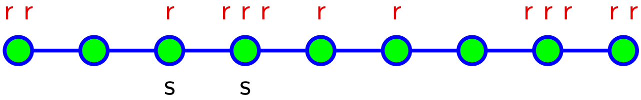

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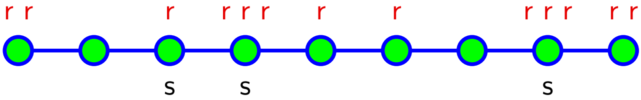

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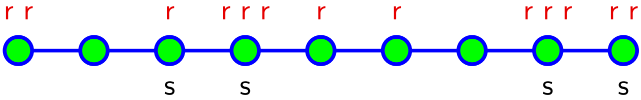

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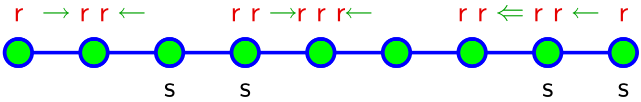

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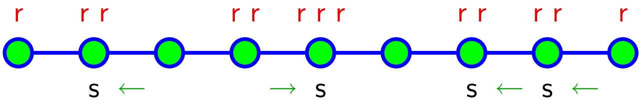

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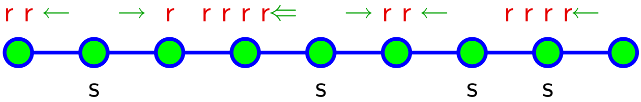

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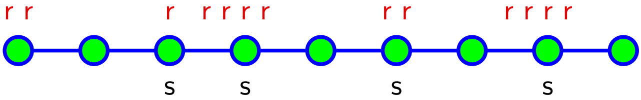

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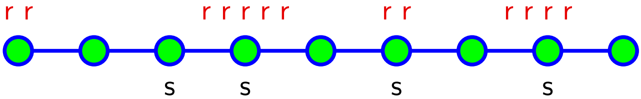

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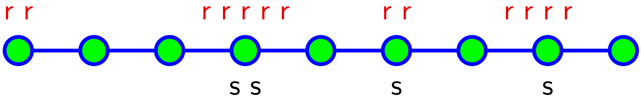

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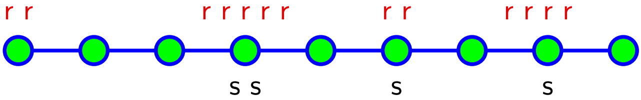

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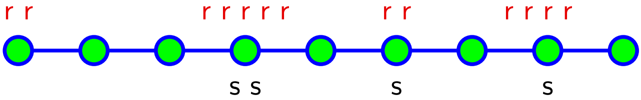

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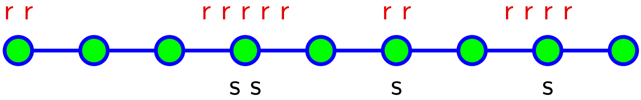

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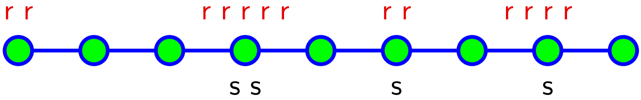

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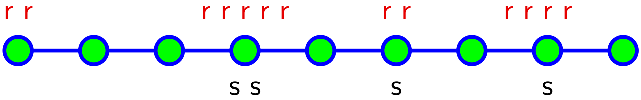

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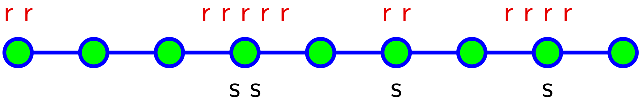

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3. $\sum_{v \in T} s(v) = \left\lfloor \frac{w(u)}{m} \right\rfloor = \left\lfloor \frac{r}{m} \right\rfloor$

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X: spies off u → $x_1 \cdots x_k$ | $x_{k+1} \cdots x_s$ ← X': spies on u

Y: new meetings → $y_1 \quad \cdots \quad y_{k'}$

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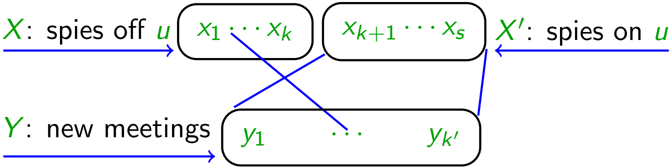

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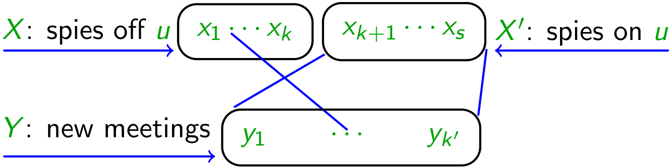

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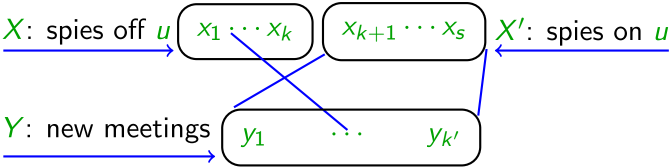

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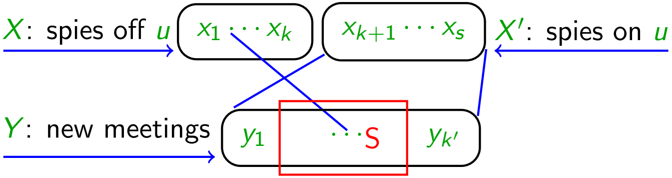

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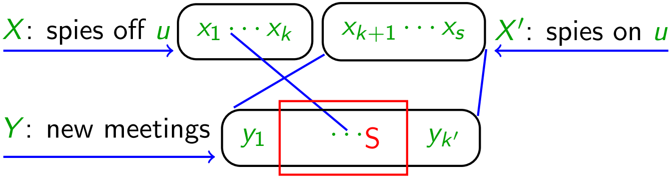

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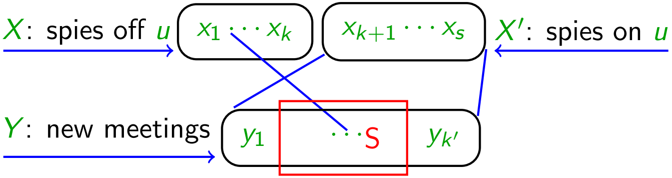

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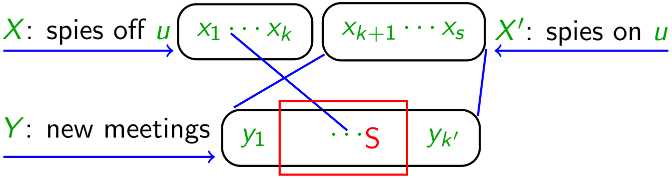

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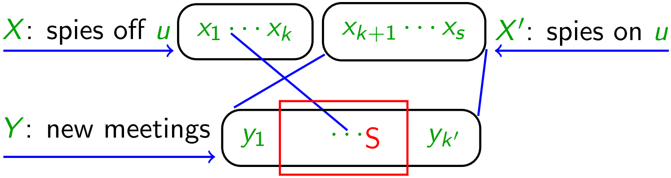

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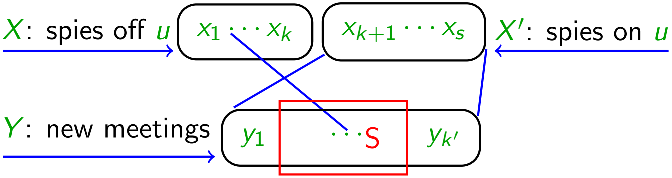

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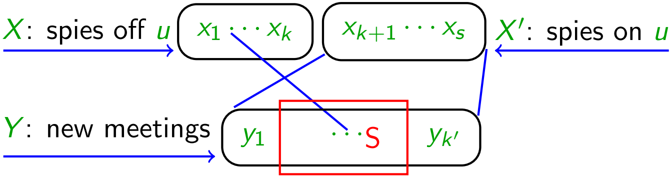

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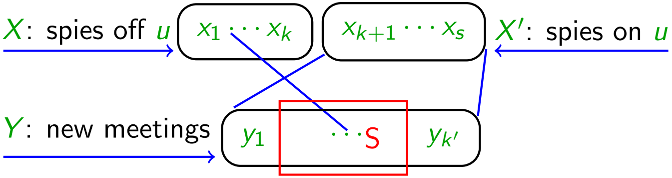

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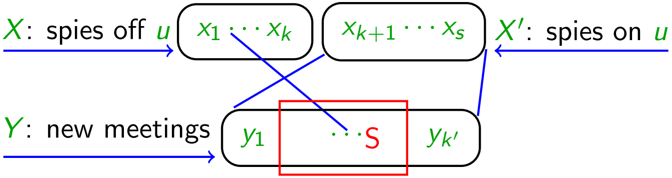

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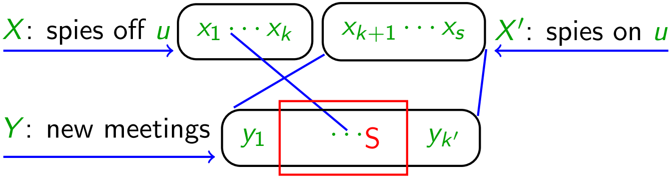

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So spies have a stable position at time $t + 1$, and G is spy-good.

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