Planar Graphs of Girth at least 5 are Square $(\Delta + 2)$-Choosable

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Joint with Marthe Bonamy and Luke Postle
Slides available on my webpage

SIAM Discrete Math
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Background

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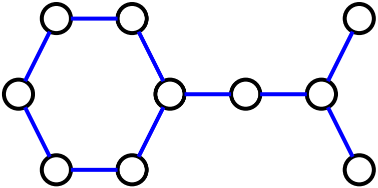

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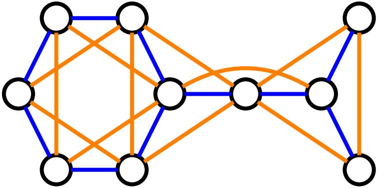

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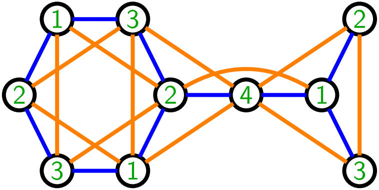

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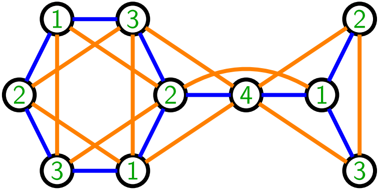

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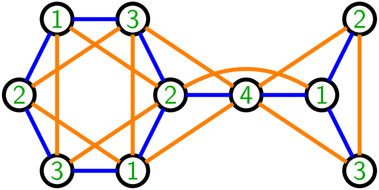

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Wegner's Conjecture:

If G is planar with maximum degree Δ,

$$\chi(G^2) \leq \begin{cases} 7 & \text{if } \Delta = 3 \\ \Delta + 5 & \text{if } 4 \leq \Delta \leq 7 \\ \left\lfloor \frac{3}{2}\Delta + 1 \right\rfloor & \text{if } \Delta \geq 8 \end{cases}$$

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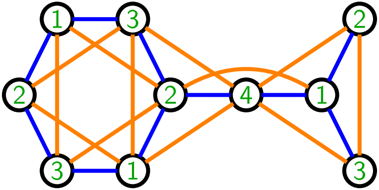

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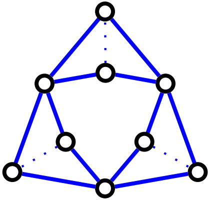

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In particular, true for $\Delta \geq 2,689,601$.

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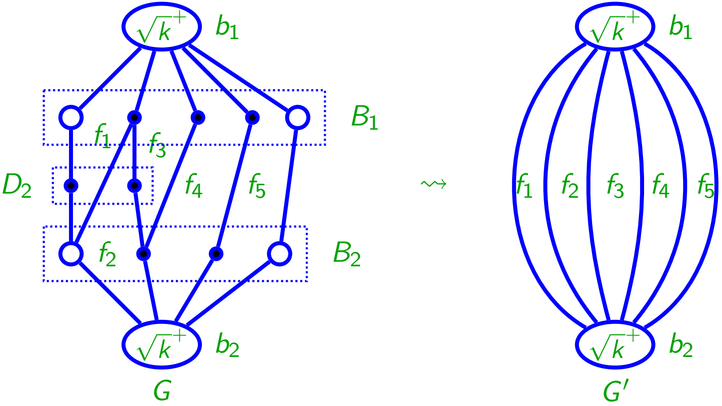
r-regions



$\sqrt{k}^+$
b_1
B_1
f_1
f_3
D_2
f_4
f_5
f_2
B_2
$\sqrt{k}^+$
b_2
G
$\rightsquigarrow$
$\sqrt{k}^+$
b_1
f_1
f_2
f_3
f_4
f_5
$\sqrt{k}^+$
b_2
G'

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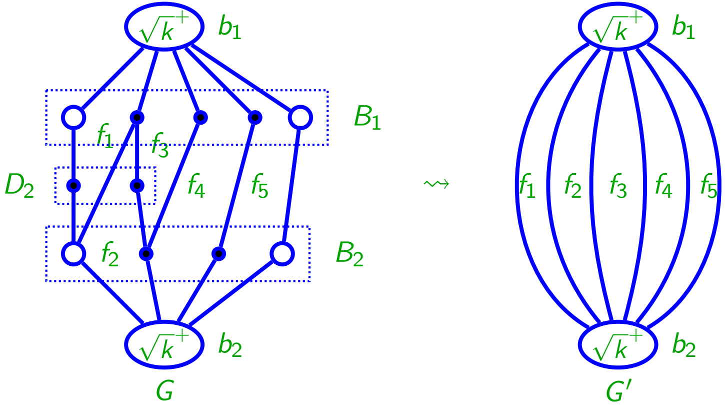

Rem: In the square, B_1 and B_2 are both cliques; each vertex has at most 3 neighbors in the other clique.

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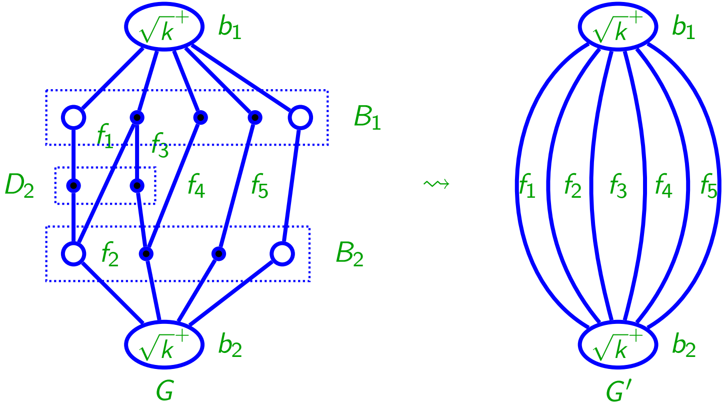

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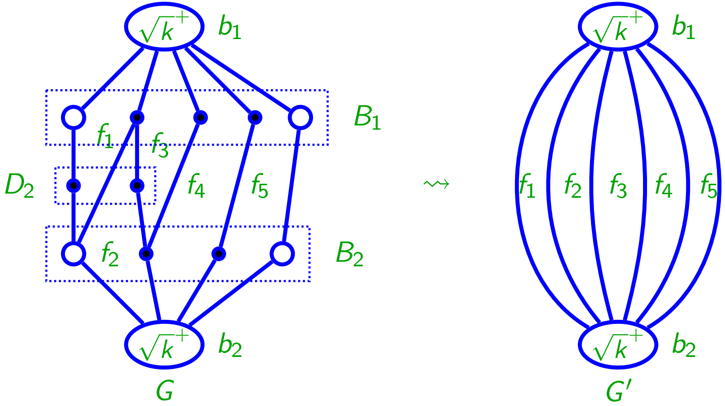

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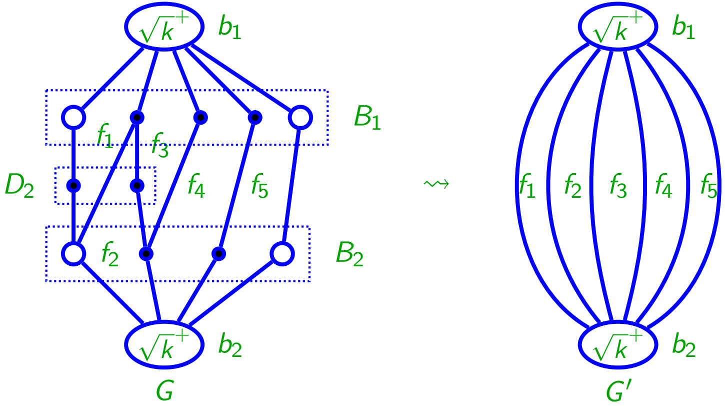

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Structural Thm Idea

Structural Thm: Let G be plane graph with girth ≥ 5 and $\Delta \geq 2.7 \times 10^6$. Let $k = \Delta$ and $B = \{v \in V(G) \;:\; d(v) \geq \sqrt{k}\}$. Now G contains one of the following configurations.

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Pf Idea: Let $S = \{v \in V \ : \ d(v) < \sqrt{k}\}$ and $S_i = \{v \in S \ : \ d_B(v) = i\}$.

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