Cliques in Squares of Graphs with Maximum Average Degree less than 4

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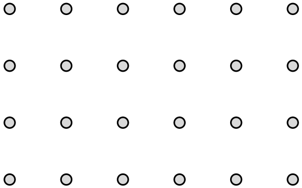

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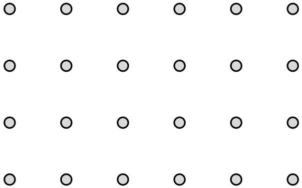

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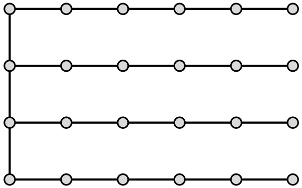

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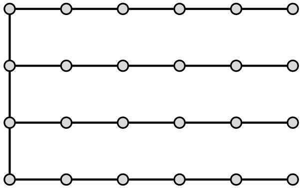

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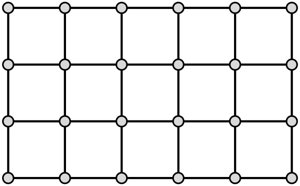

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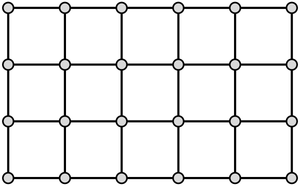

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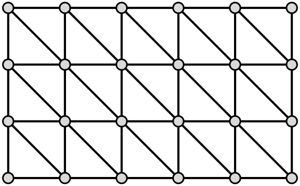

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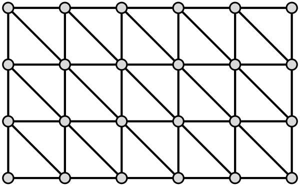

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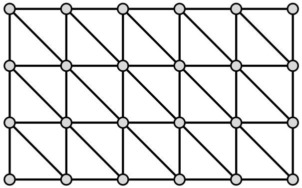

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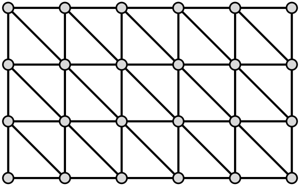

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Obs: If G is k-degenerate, then $\mathrm{mad}(G) < 2k$.

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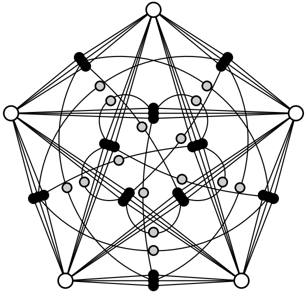

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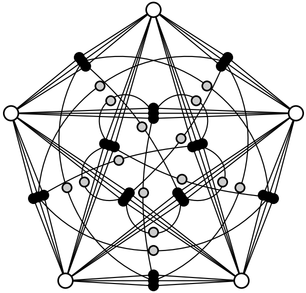

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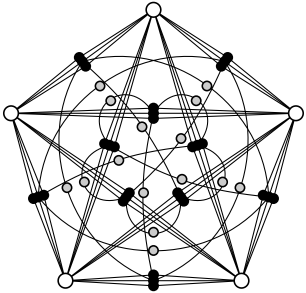

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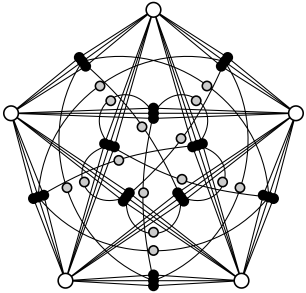

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Rem: Upper bound on degeneracy sharp up to constant.

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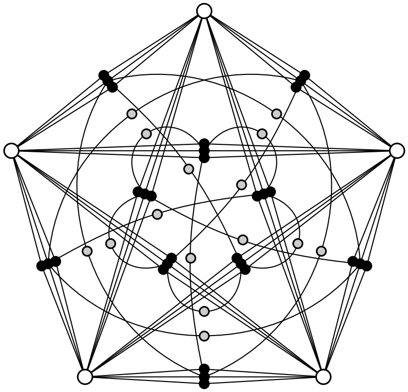

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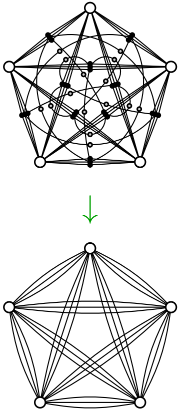
Thm 1': Let H be a multigraph with $\Delta(H) \leqslant D$. If each edge of H shares at least one end-point with all but at most $D-2$ other edges of H, then $|E(H)| \leqslant 5D/2$.



Rem: Thm 1' implies Thm 1.

Pf: Given G, S, σ, delete all vertices before S in order σ, and contract one edge incident to each vertex of S. Now $|E(H)| = |S|$.

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If G is 2-degenerate with $\Delta(G) \leqslant D$, then $\omega(G^2) \leqslant 5D/2 + 72$.

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Intuition: If $v \in S$ doesn't get adjacency (in G^2) to lots of S via vertices earlier in σ, then both neighbors of S later in σ give v lots of adjacencies in G^2 to verts of S.

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Q: Why can't most verts in S get much help from verts earlier in σ?
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Q: How to formalize all of this?
A: Tokens!

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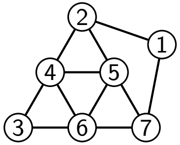

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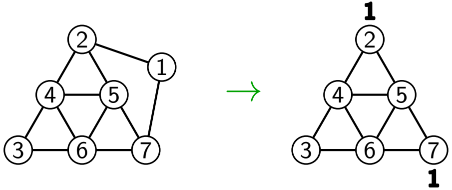

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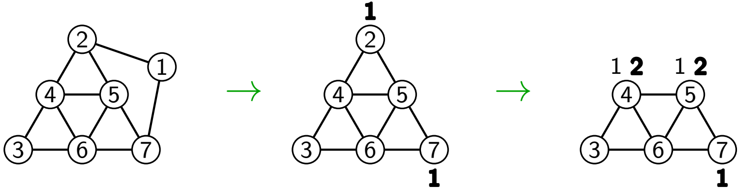

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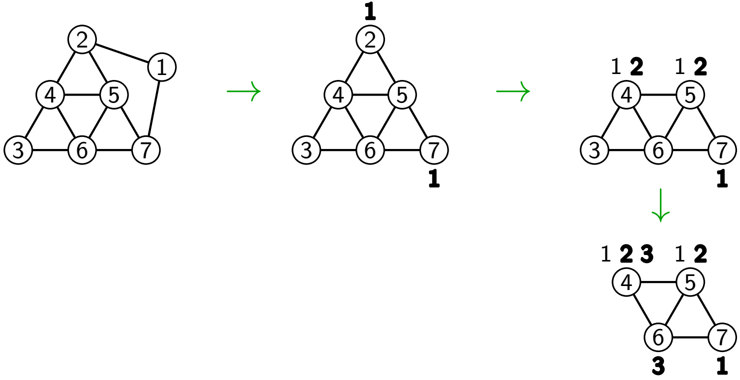

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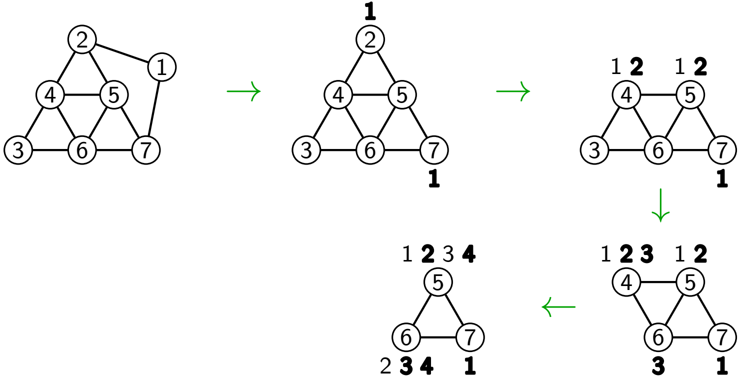

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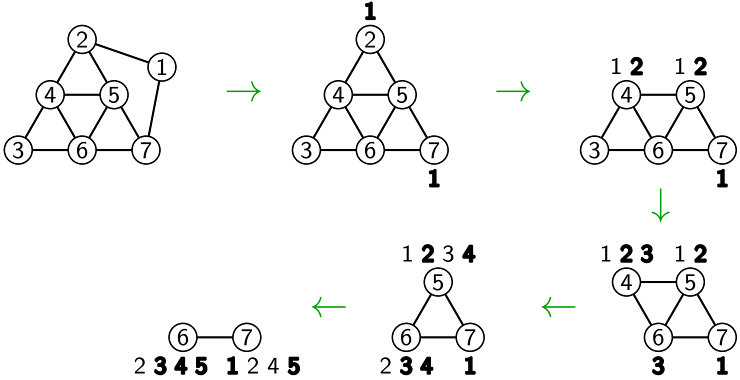

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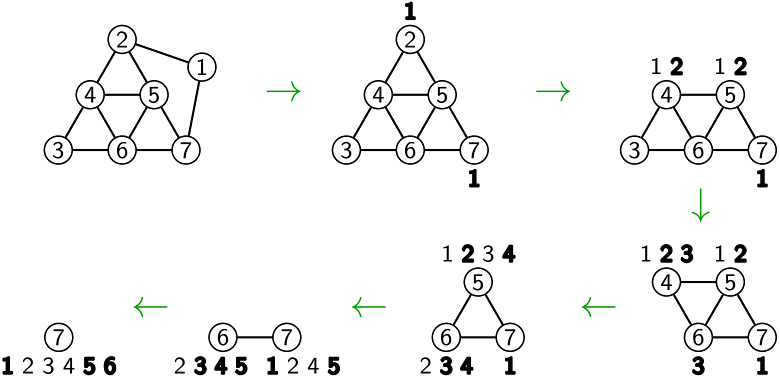

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Defn: Let $\mathrm{tok}(v)$ and $\mathrm{prim}(v)$ denote number of tokens and number of primary tokens held by v when v is deleted.

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Cor: $\textsc{Basic}$ is independent set of size at least $|S| - 72$.

Open Questions

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Is this achieved by k-degenerate graphs?

Open Questions

Defn: Let $\omega^2(2k, D)$ and $\chi^2(2k, D)$ be max $\omega(G^2)$ and max $\chi(G^2)$ over all G such that $\text{mad}(G) < 2k$ and $\Delta(G) \leqslant D$.

Conj: For D big enough $\chi^2(4, D) = 5D/2$.

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(Possibly $c_k = 0$ works.)

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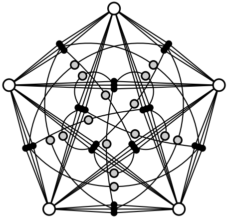

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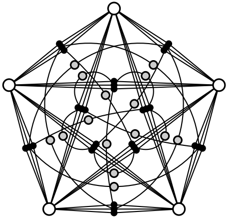

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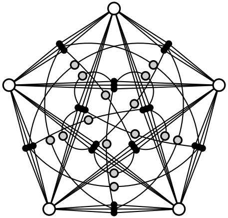

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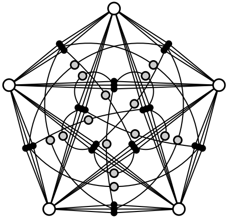

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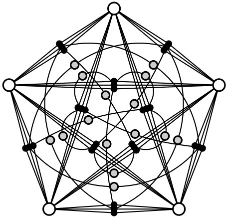

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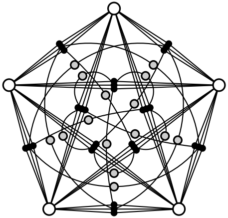

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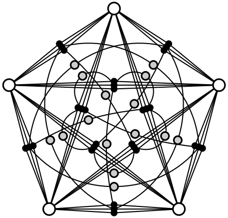

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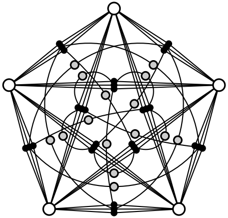

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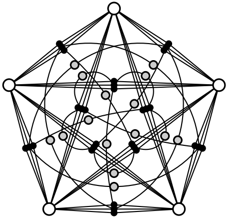

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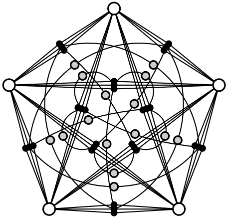

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Rem: Many interesting open questions!

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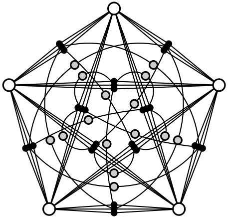

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Read more: `arXiv:2305.11763`