Star-coloring planar graphs with high girth

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Joint with Craig Timmons and Andre Kundgen

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Thm. [Fetin-Raspaud-Reed 2001]
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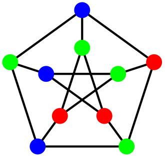

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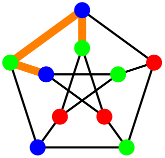

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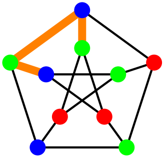

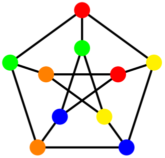

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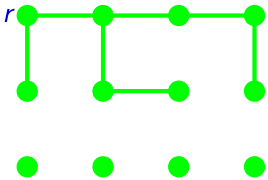

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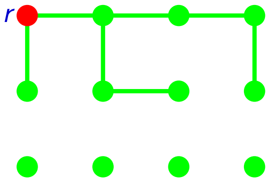

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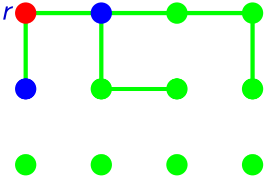

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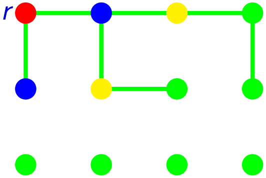

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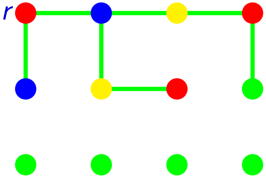

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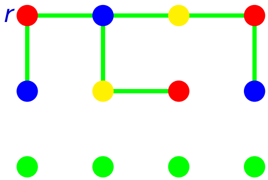

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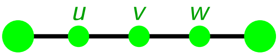

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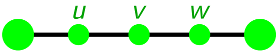

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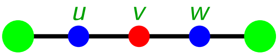

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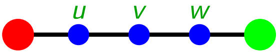

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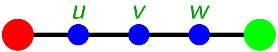

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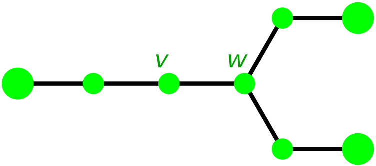

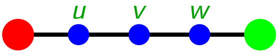

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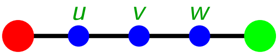

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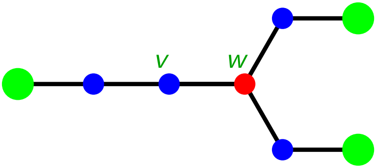

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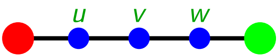

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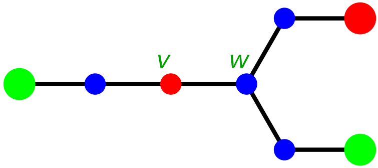

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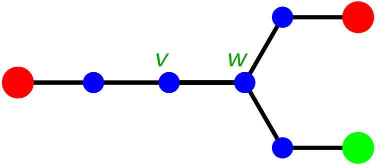

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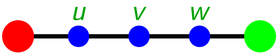

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Give charge $2l(f) - 28$ to each face f and
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Contradiction! So G contains a reducible configuration.

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Give charge $2l(f) - 26$ to each face f and
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$$\underbrace{\sum_{v \in V}(12d(v) - 28)}_{\textit{negative}} + \underbrace{\sum_{f \in F}(2l(f) - 28)}_{\textit{nonnegative}} = 28(|E| - |F| - |V|) = -56$$

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Contradiction! So G contains a reducible configuration.

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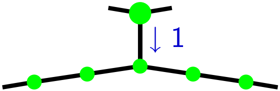

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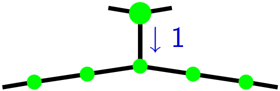

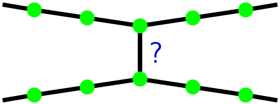

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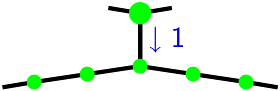

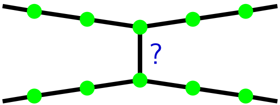

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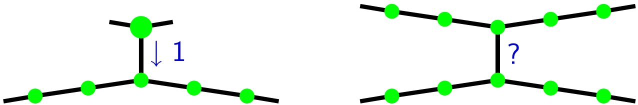

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Claim $\text{mad}(H) \leq 2$.

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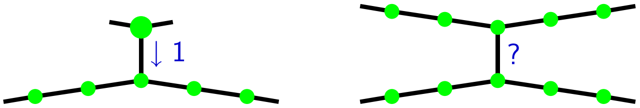

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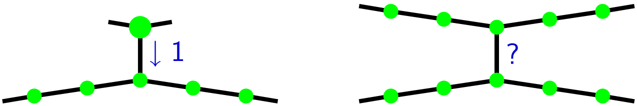

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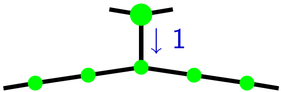

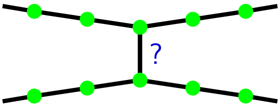

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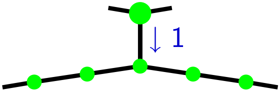

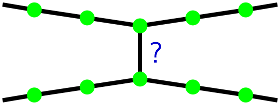

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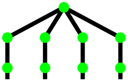

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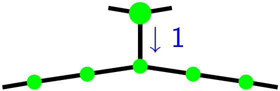

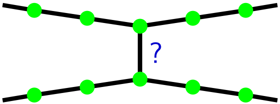

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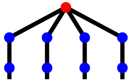

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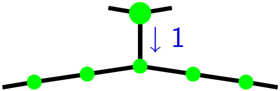

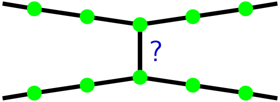

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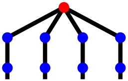

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