List-coloring the Square of a Subcubic Graph

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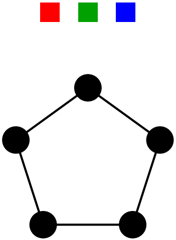

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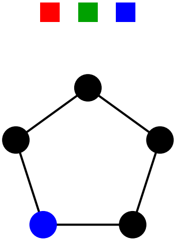

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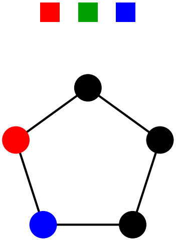

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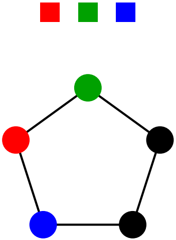

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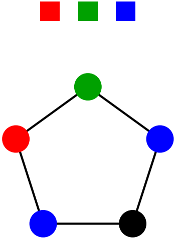

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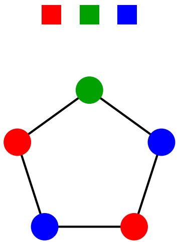

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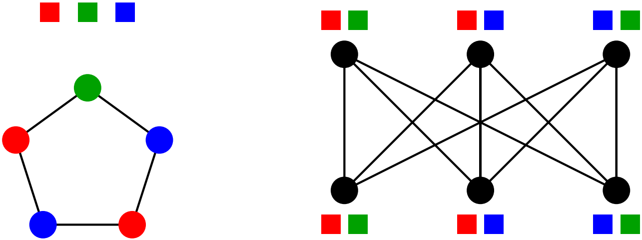

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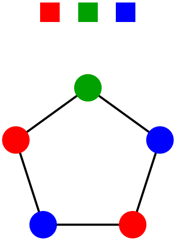

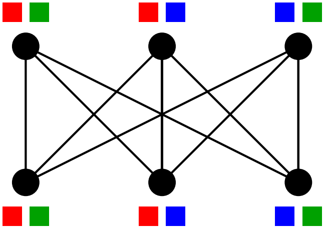

Def. G^2 (square of G): formed from G by adding edges between vertices at distance 2.

Results: Old and New

Thm. [Thomassen '08?] $\chi(G^2) \leq 7$ if G is planar and $\Delta(G) = 3$.

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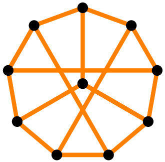

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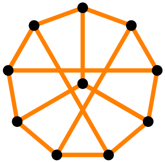

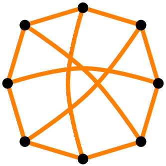

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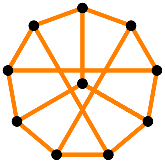

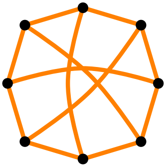

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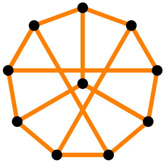

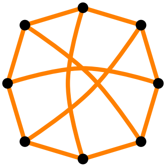

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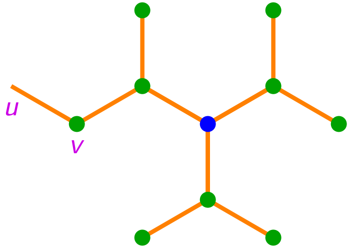

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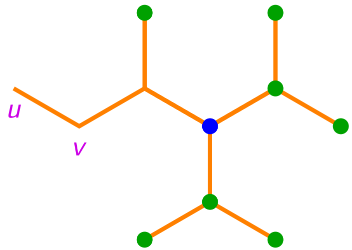

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Lem. Suppose that G has a partial coloring from its lists. Let H be the subgraph induced by uncolored vertices. Suppose that H is connected. If H contains adjacent vertices u and v such that $\text{ex}(u) \geq 1$ and $\text{ex}(v) \geq 2$, then we can complete the coloring.

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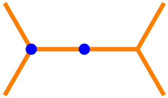

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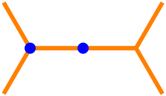

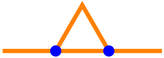

Cor. If G is Petersen-free and girth(G)=3, then $\chi_l(G^2) \leq 8$.

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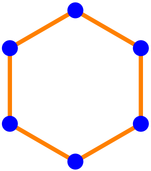
$H =$ 

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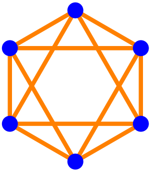
$H^2 =$ 

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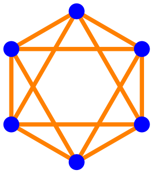
$H^2 =$ 

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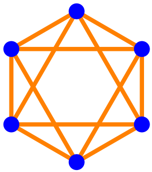
$H^2 =$ 

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Cycle + Triangle Thm
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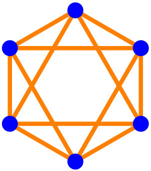
$H^2 =$ 

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Cycle + Triangle Thm
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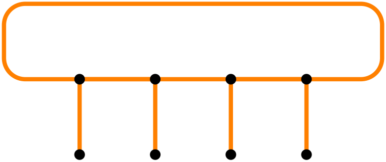
$\chi_l(C_{6k}^2) = 3$
[Juvan, Mohar, Skrekovski '98]

Large girth



Obs. If girth(G)≥ 7 and C is a shortest cycle in G, then any two vertices that are each adjacent to the cycle are nonadjacent.

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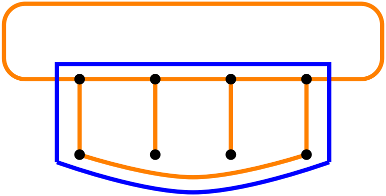

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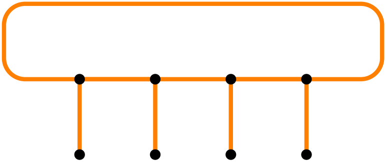

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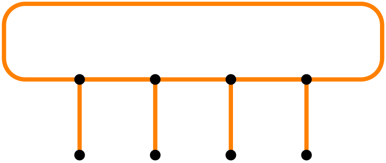

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Pf. Let H be a shortest cycle and neighbors. Color $G^2 \setminus V(H)$.

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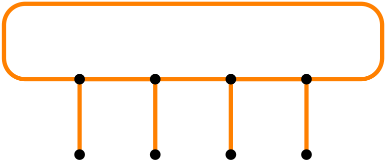

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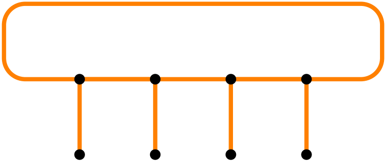
Pf. Let H be a shortest cycle and neighbors. Color $G^2 \setminus V(H)$.
Two cases depending on whether
there exist $i \neq j$ s.t. $|i-j| \leq 2$ and $L(u_i) \cap L(u_j) \neq \emptyset$
or there exists i s.t. $L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i)$

Large girth



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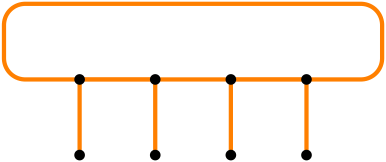
1) Suppose so:

Large girth



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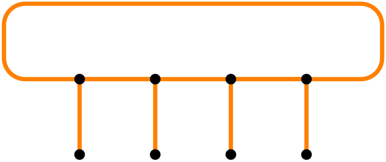
1) Suppose so: We can color more vertices so that for some i, ex(v_i)≥ 1 and ex(v_{i+1})≥ 2. Then use our main lemma.

Large girth



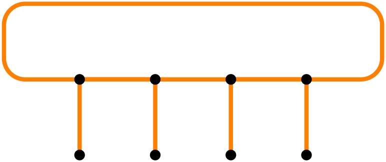
Lem. If girth(G)≥ 7, then $\chi_l(G^2) \leq 8$.

Pf. Let H be a shortest cycle and neighbors. Color $G^2 \setminus V(H)$.
Two cases depending on whether
there exist $i \neq j$ s.t. $|i-j| \leq 2$ and $L(u_i) \cap L(u_j) \neq \emptyset$
or there exists i s.t. $L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i)$

1) Suppose so: We can color more vertices so that for some i,
ex(v_i)≥ 1 and ex(v_{i+1})≥ 2. Then use our main lemma.

2) Suppose not:

Large girth



Lem. If girth(G)≥ 7, then $\chi_l(G^2) \leq 8$.

Pf. Let H be a shortest cycle and neighbors. Color $G^2 \setminus V(H)$.
Two cases depending on whether
there exist $i \neq j$ s.t. $|i-j| \leq 2$ and $L(u_i) \cap L(u_j) \neq \emptyset$
or there exists i s.t. $L(u_{i-1}) \cup L(u_i) \cup L(u_{i+1}) \not\subseteq L(v_i)$

1) Suppose so: We can color more vertices so that for some i, ex(v_i)≥ 1 and ex(v_{i+1})≥ 2. Then use our main lemma.

2) Suppose not: Choose $c(u_i)$ arbitarily from $L(u_i)$. Choose $c(v_i)$ from $L(u_i) - c(u_i)$.

Thank you!

Any Questions?