Reconfiguration of Colorings and List Colorings: Proofs and Conjectures

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CanaDAM
21 May 2025

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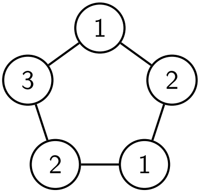

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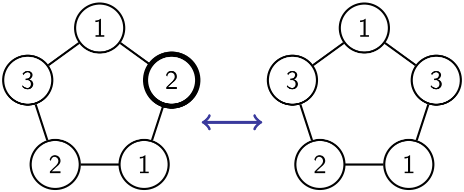

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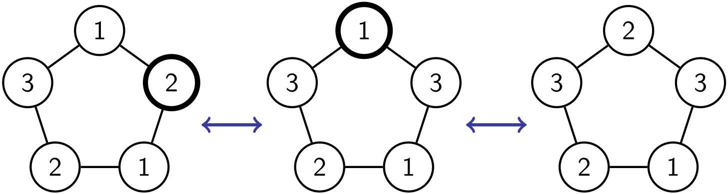

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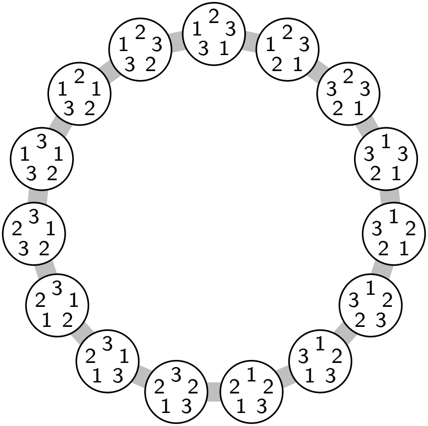

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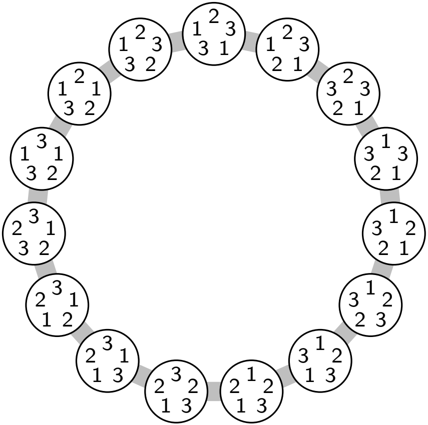

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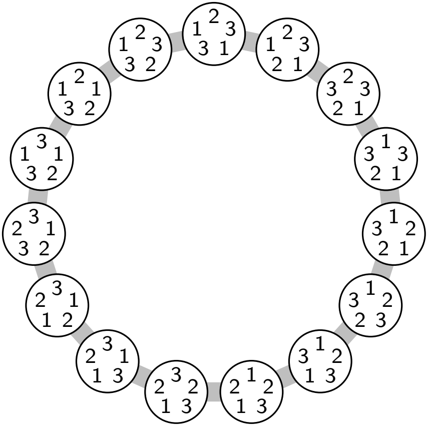

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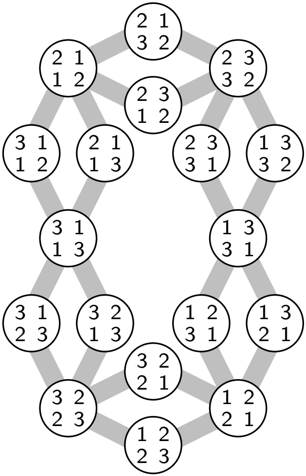

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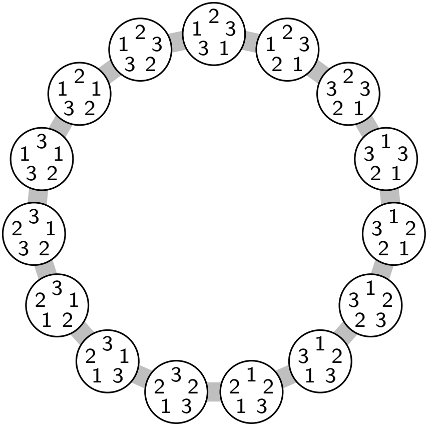

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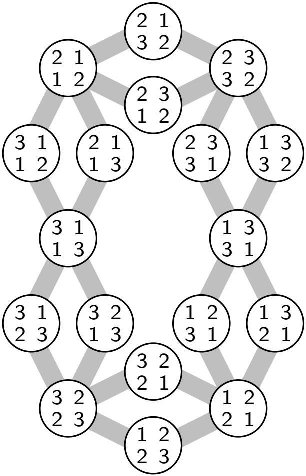

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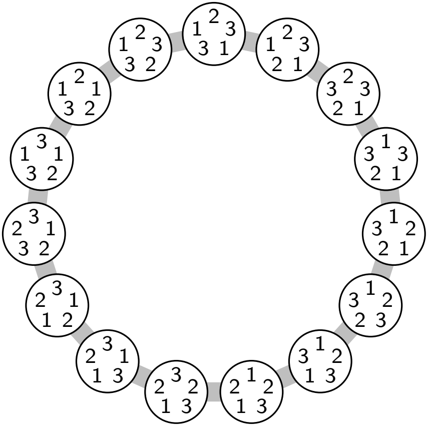

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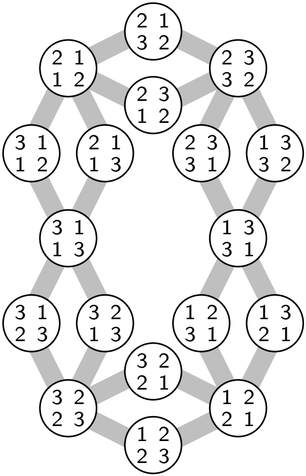

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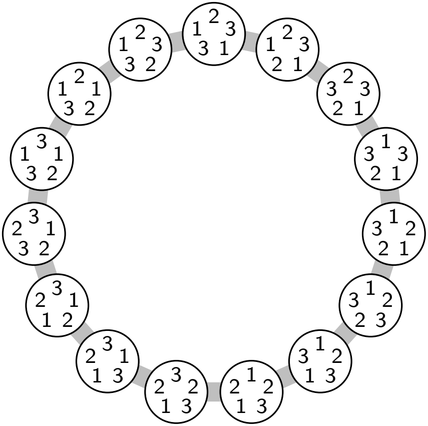

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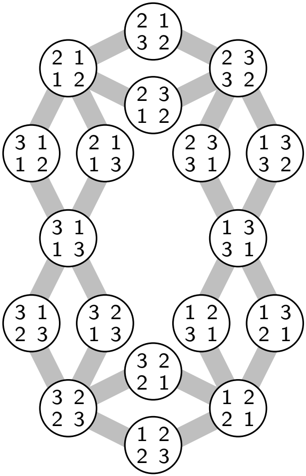

- “Reconfiguration graphs” of 3-colorings of 5-cycle and 4-cycle.
- Is the reconfiguration graph connected? What is its diameter?

What is List Coloring Reconfiguration?

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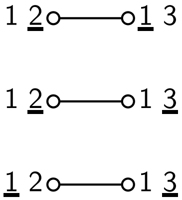

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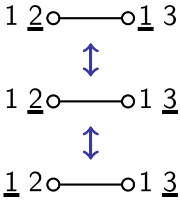

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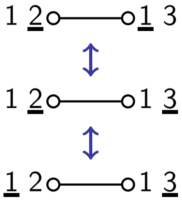

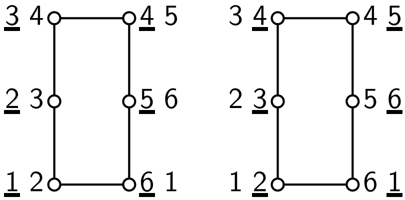

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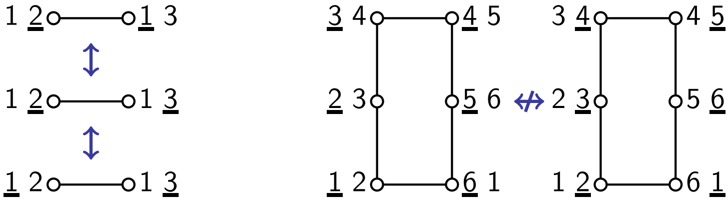

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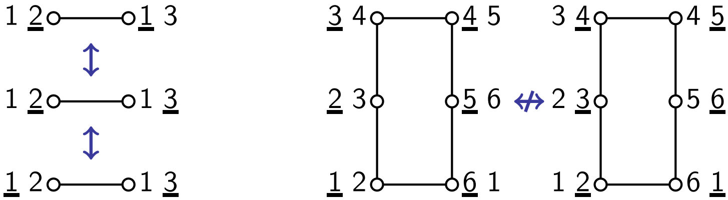

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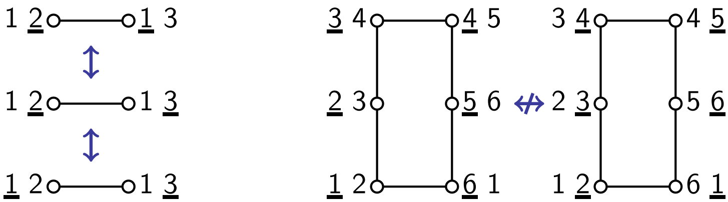

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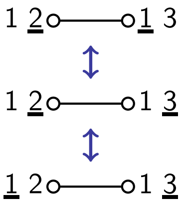

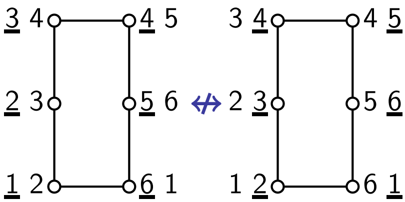

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- Given list-assignment L, can we transform every L-coloring α into every L-coloring β?
- If so, how many steps are needed in the worst case?

Lists of Size $d(v) + 2$

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Prop: For every G and every f, with $f(v) \geq 2$ for all v, there is list assignment L with $|L(v)| = f(v)$ for all v and L-colorings α and β where changing α to β needs $n(G) + \mu(G)$ moves.

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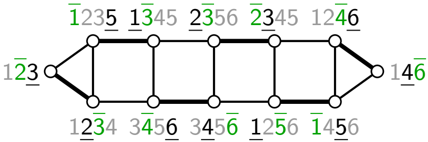
Pf: Every vert needs recolored; every edge of M needs extra step.



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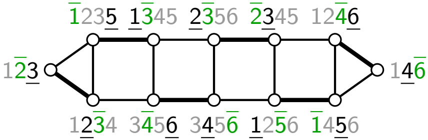

Thm:[Cambie–Cames van Batenburg–C.] `arXiv:2204.07928`

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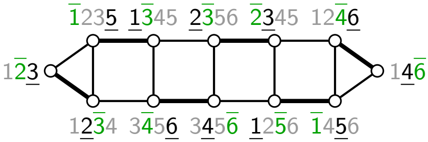

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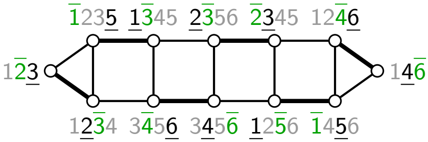

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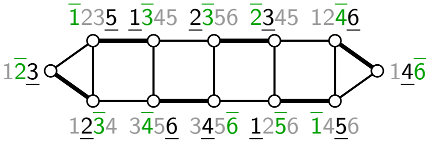

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Conj:[Cambie–Cames van Batenburg–C.] For list assignment L with $|L(v)| \geq d(v) + 2$ for all v and L-colorings α and β, can always change α to β in at most $n(G) + \mu(G)$ steps.

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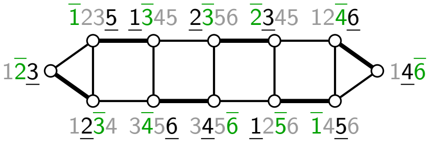

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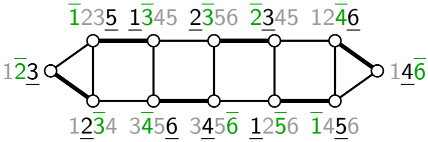

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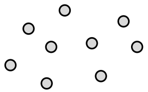

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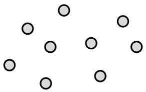

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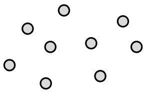

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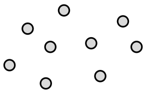

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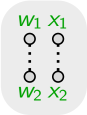

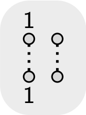

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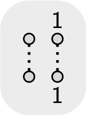

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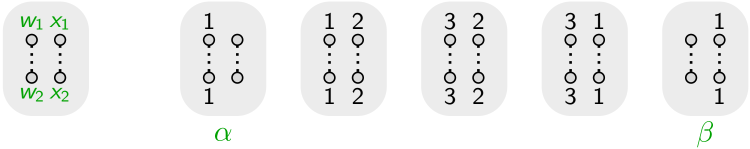

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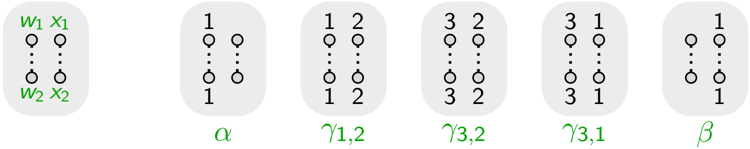

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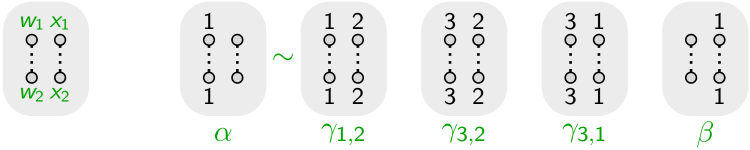

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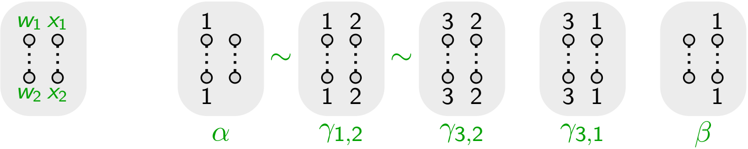

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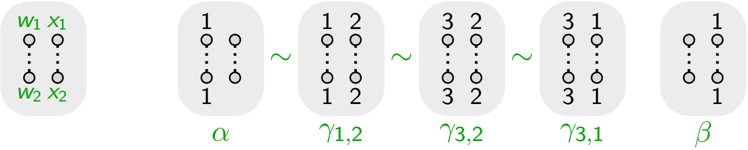

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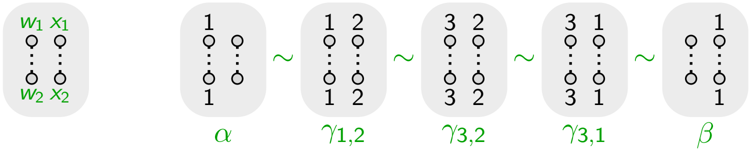

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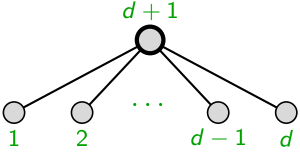

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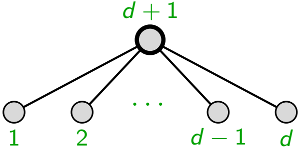

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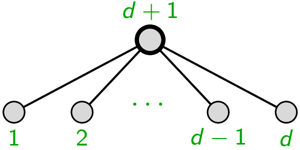

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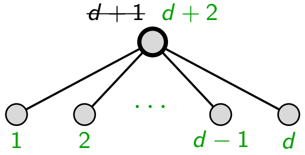

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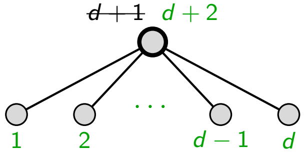

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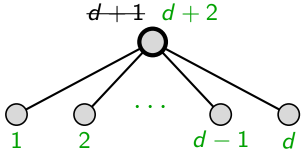

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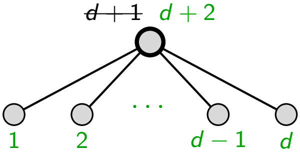

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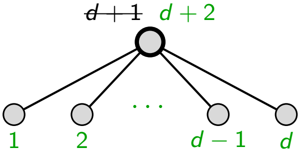

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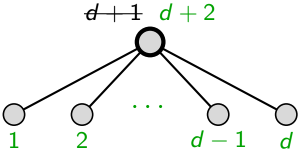

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Correspondence: Analogues of theorems are true.