Reconfiguration of Colorings and List Colorings: Proofs and Conjectures

Daniel W. Cranston
dcransto@gmail.com

16 May 2026
Cumberland Conference
Auburn

What is Reconfiguration?

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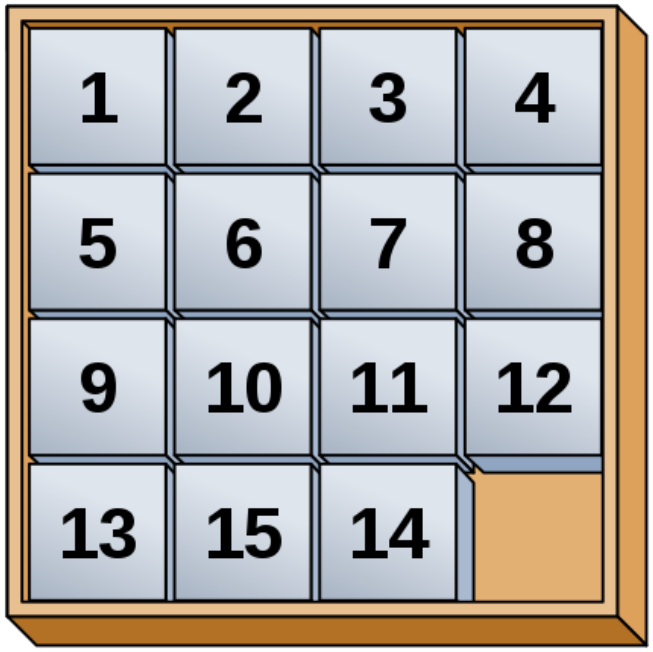

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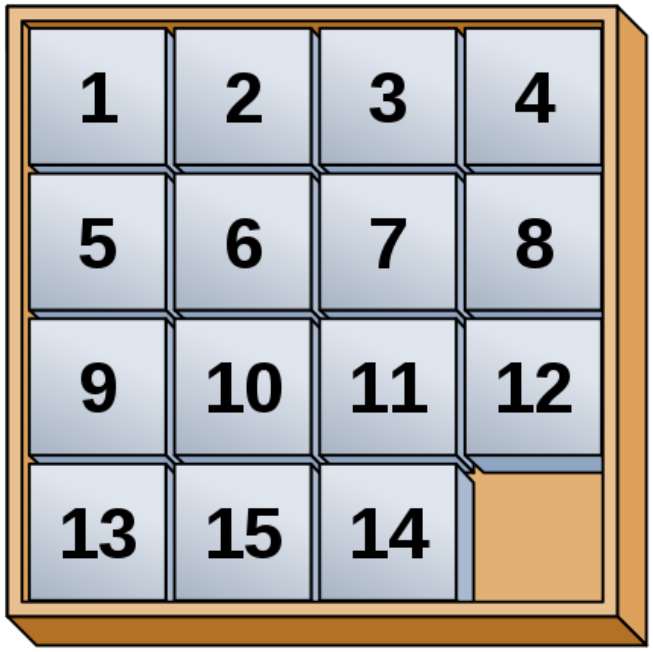

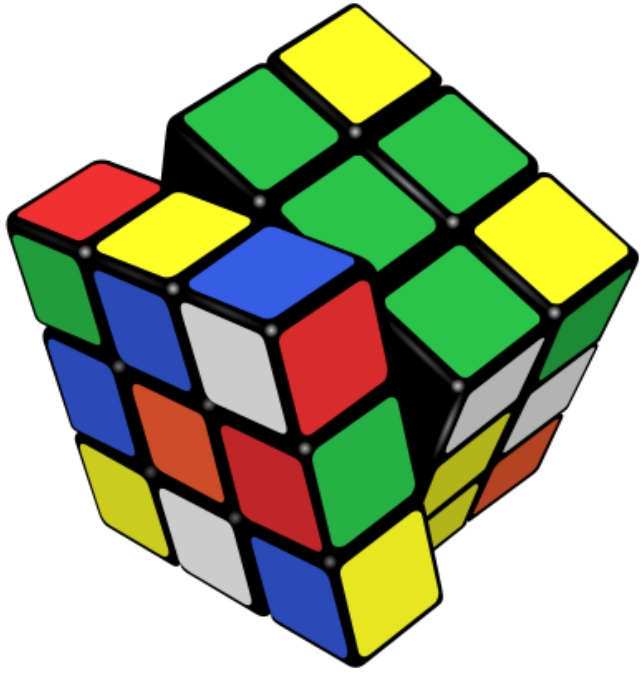

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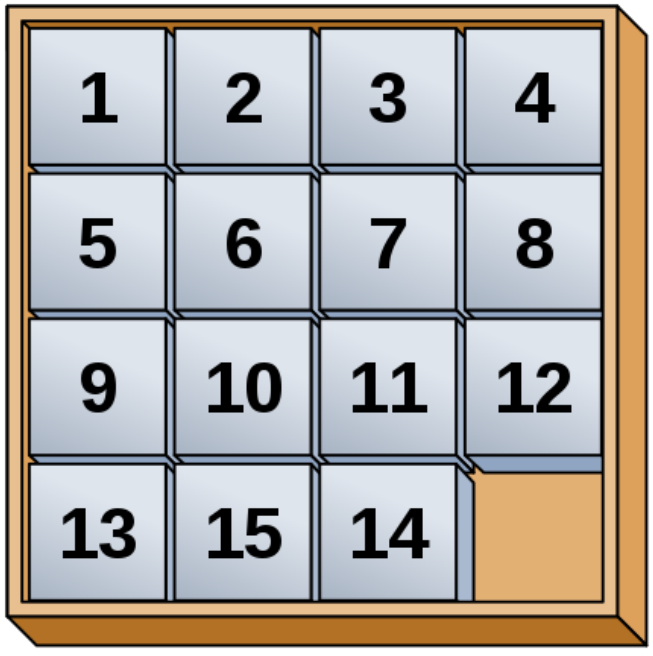

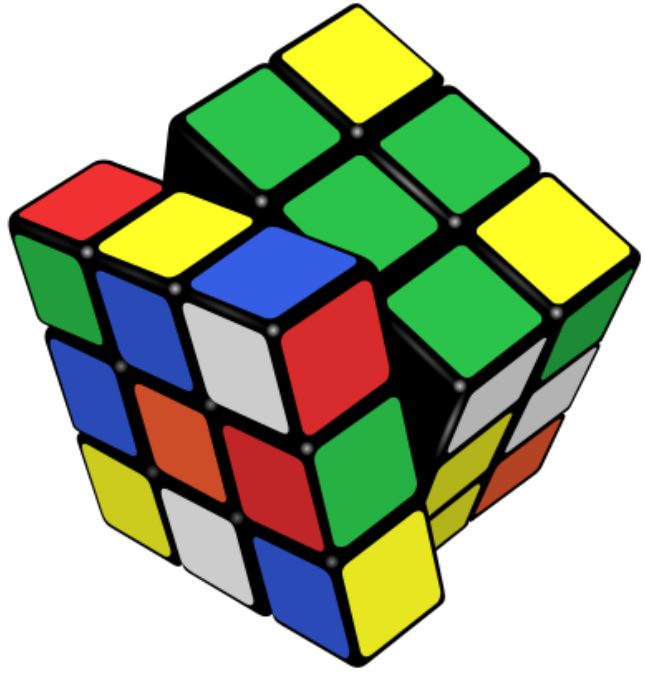

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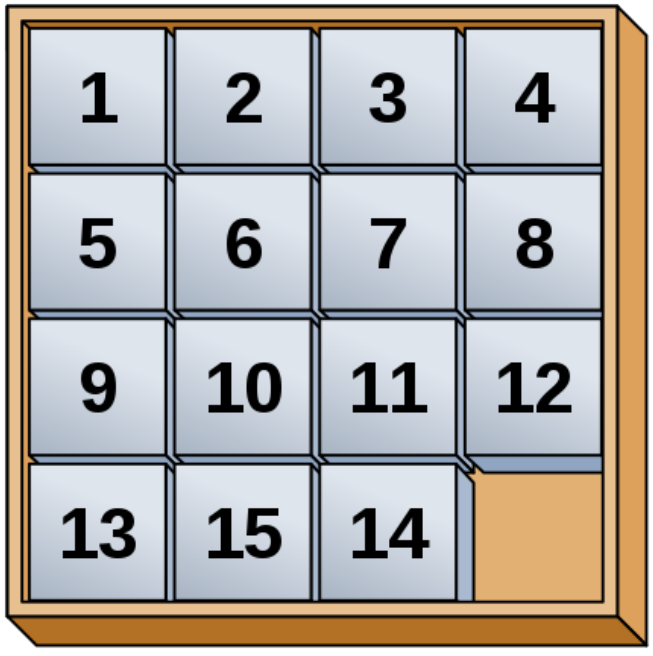

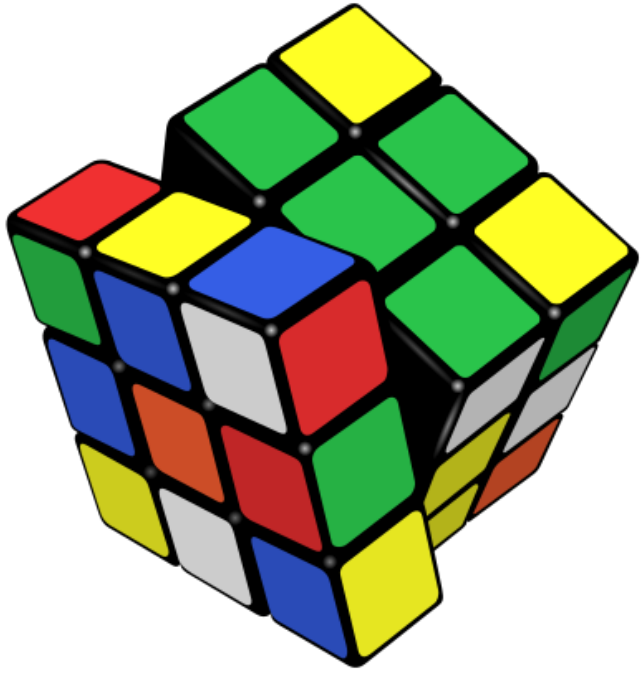

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Move from one instance to another

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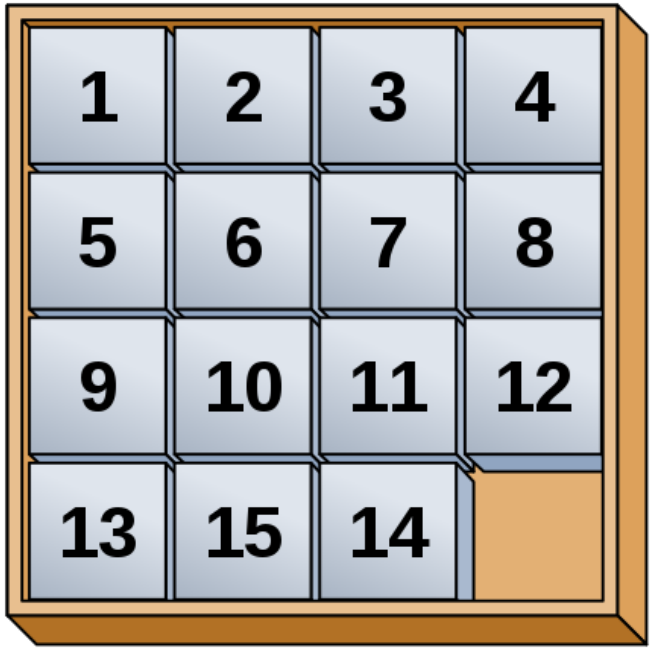

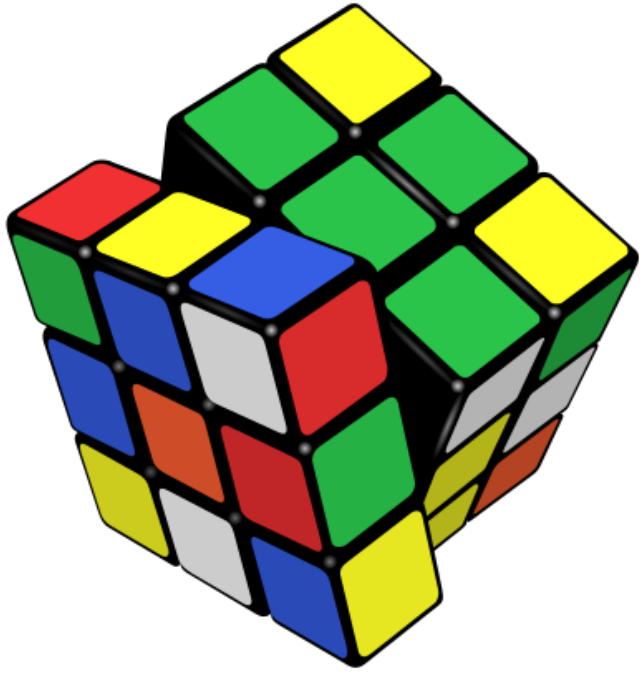

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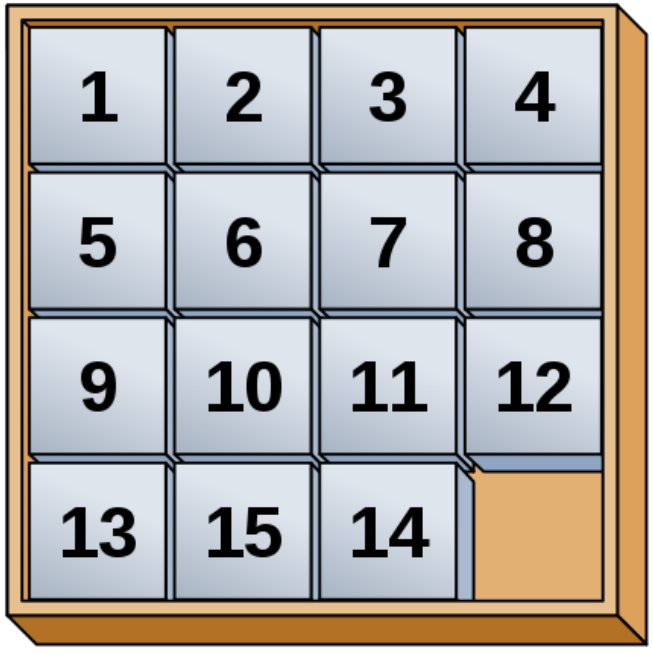

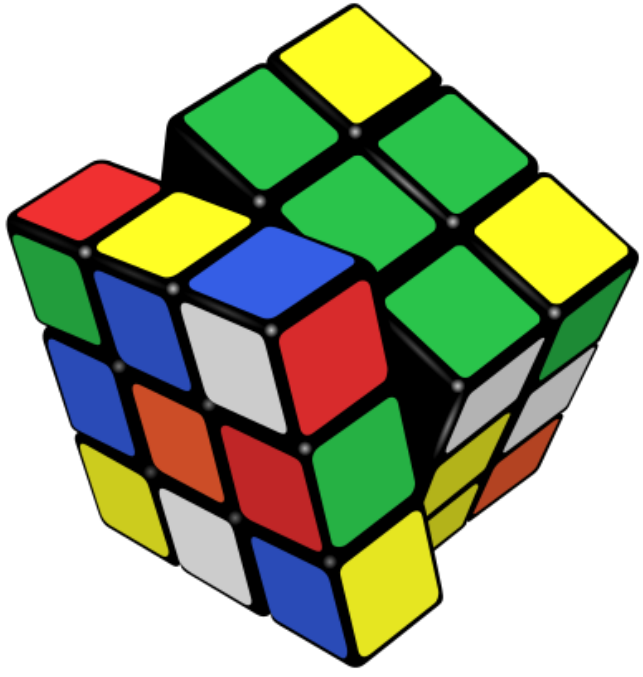

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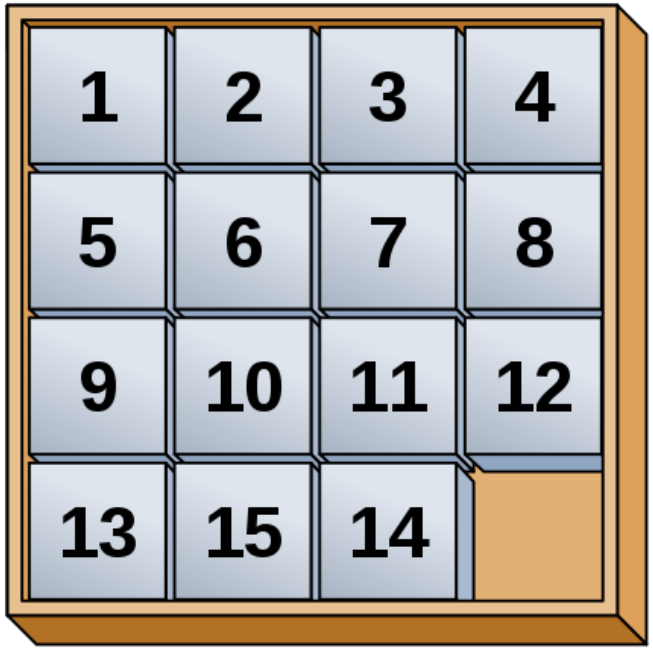

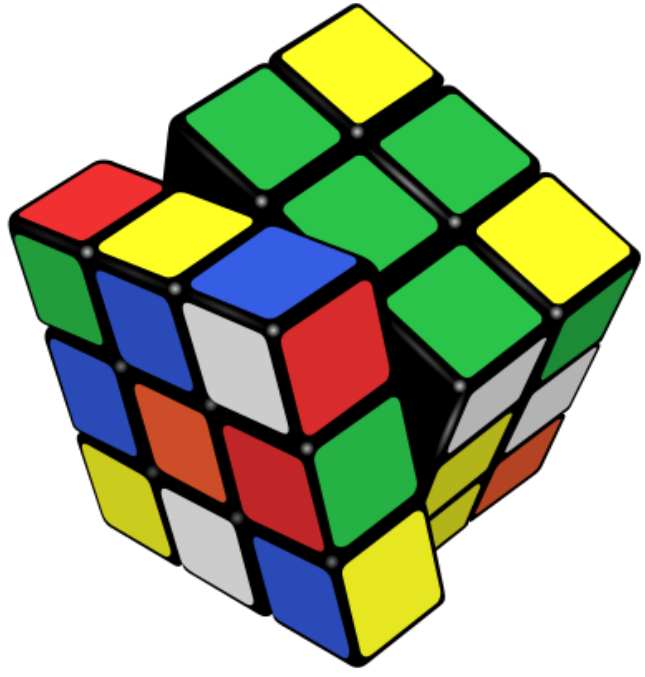

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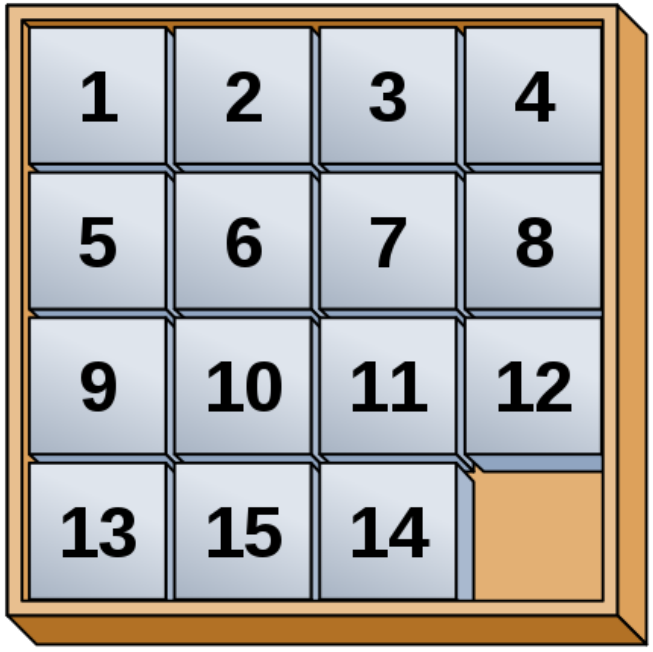

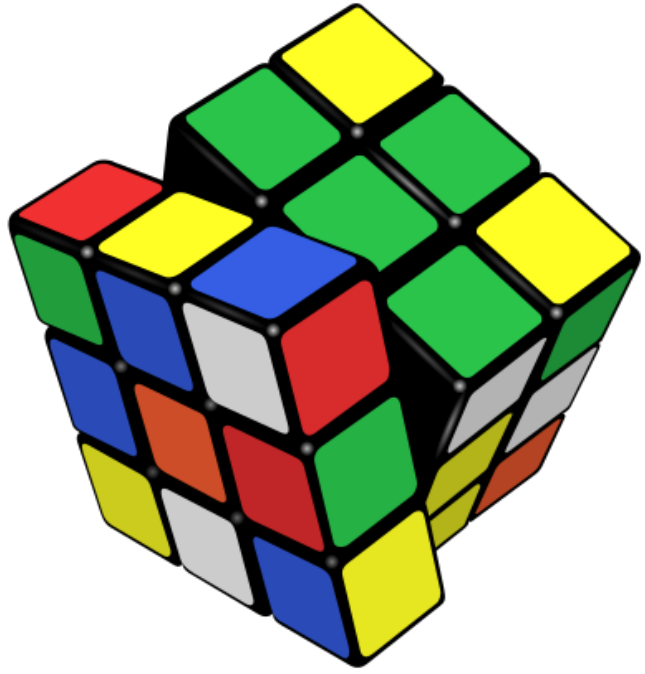

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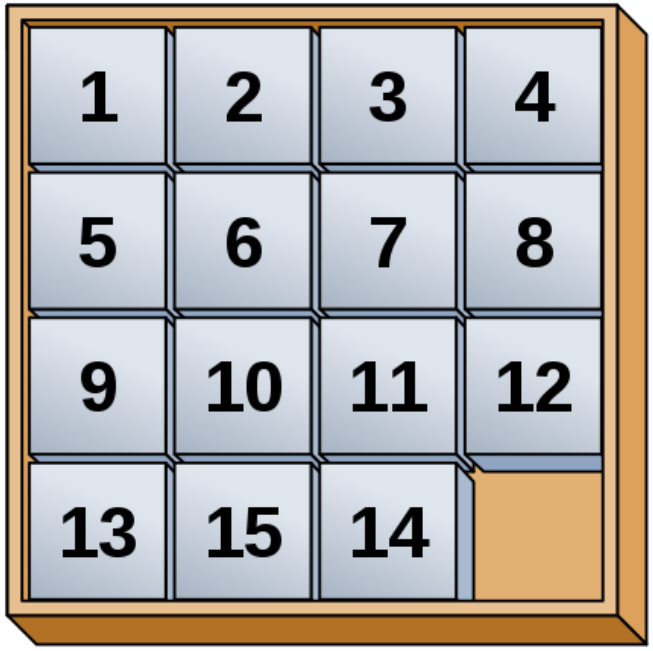

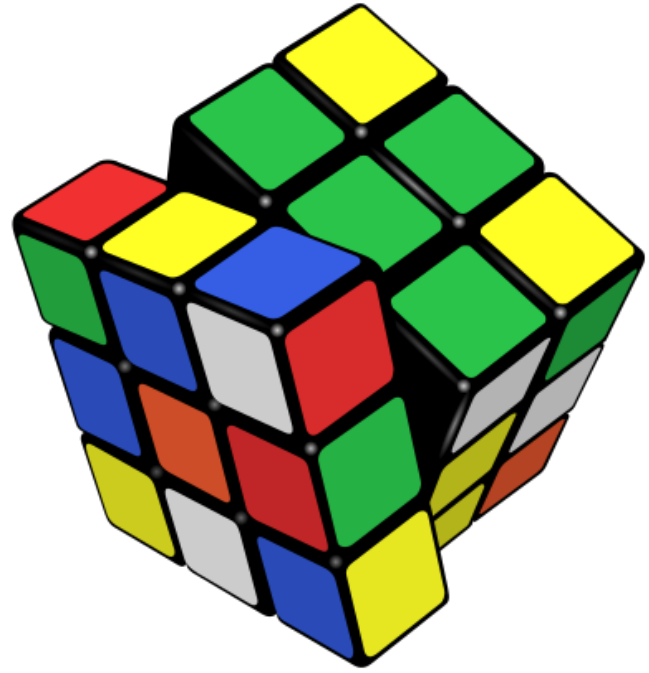

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Move from one instance to another by a sequence of small steps?

- Is it always possible?
- If so, how many moves do you need?
- Can you quickly find a short sequence from one to another?
- Can you quickly sample from all instances (nearly) uniformly?

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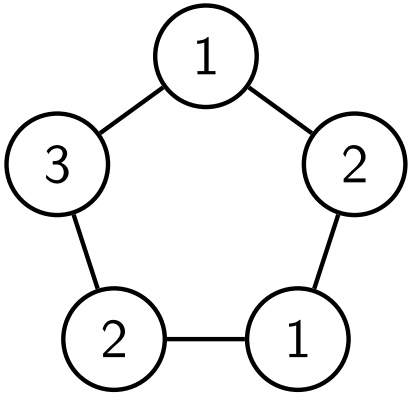

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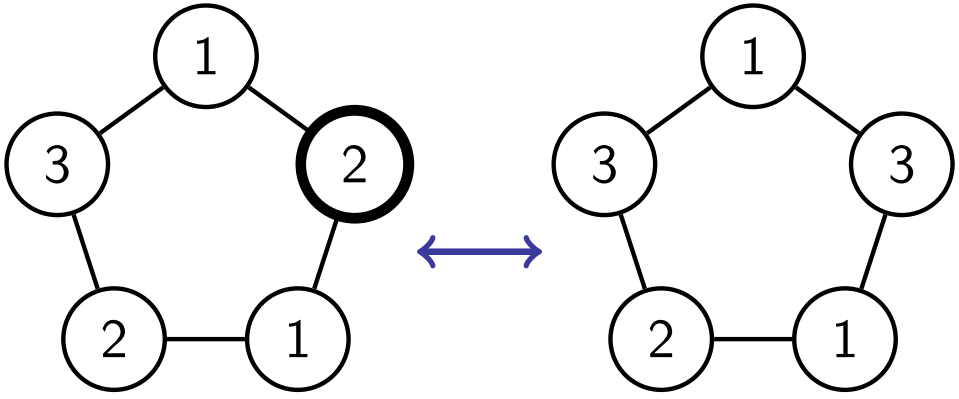

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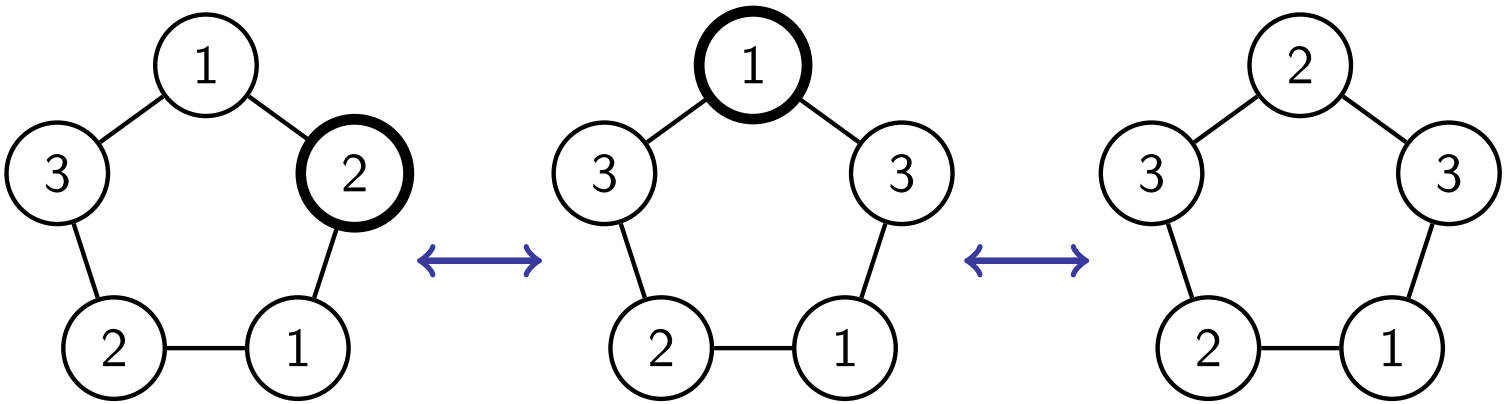

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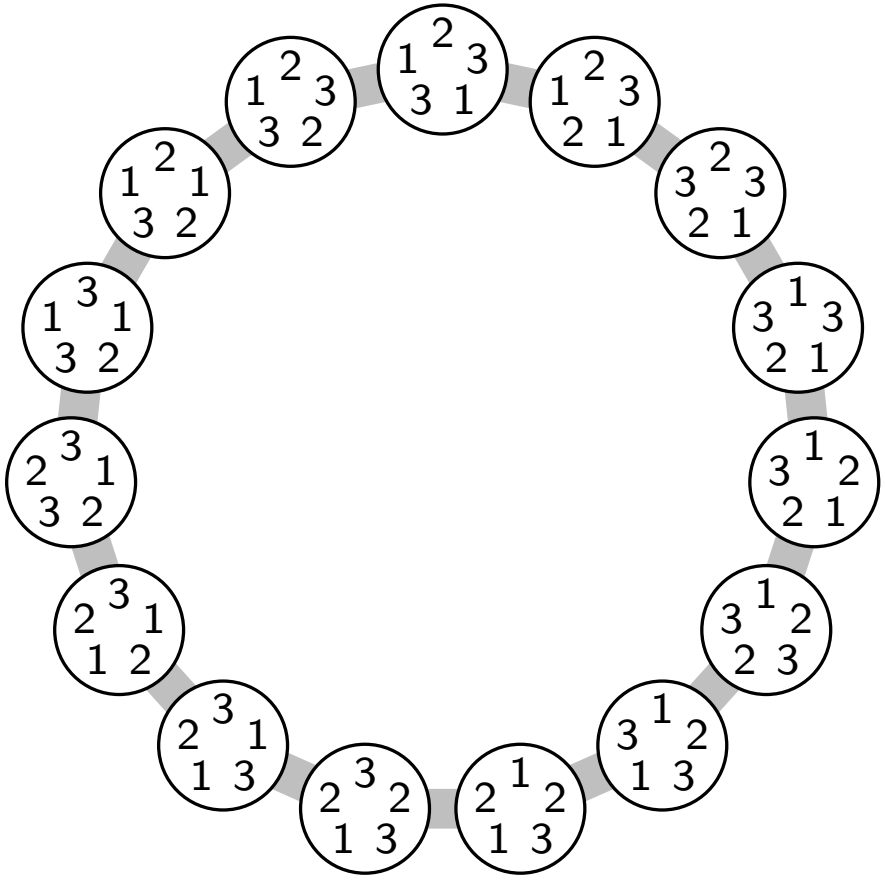

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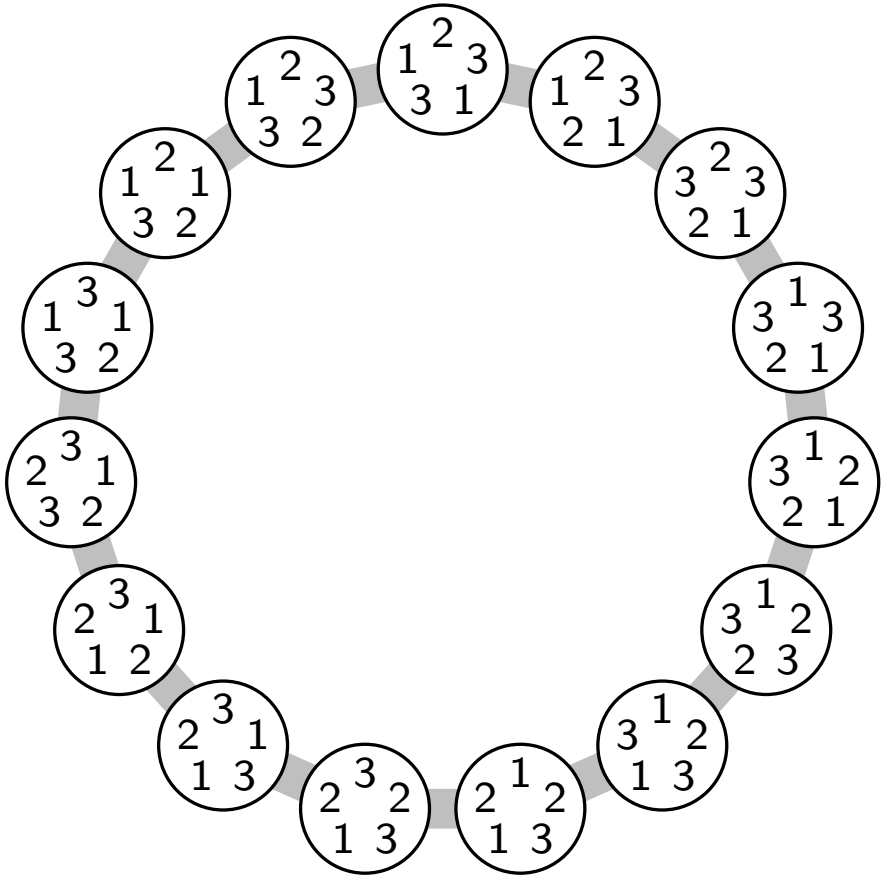

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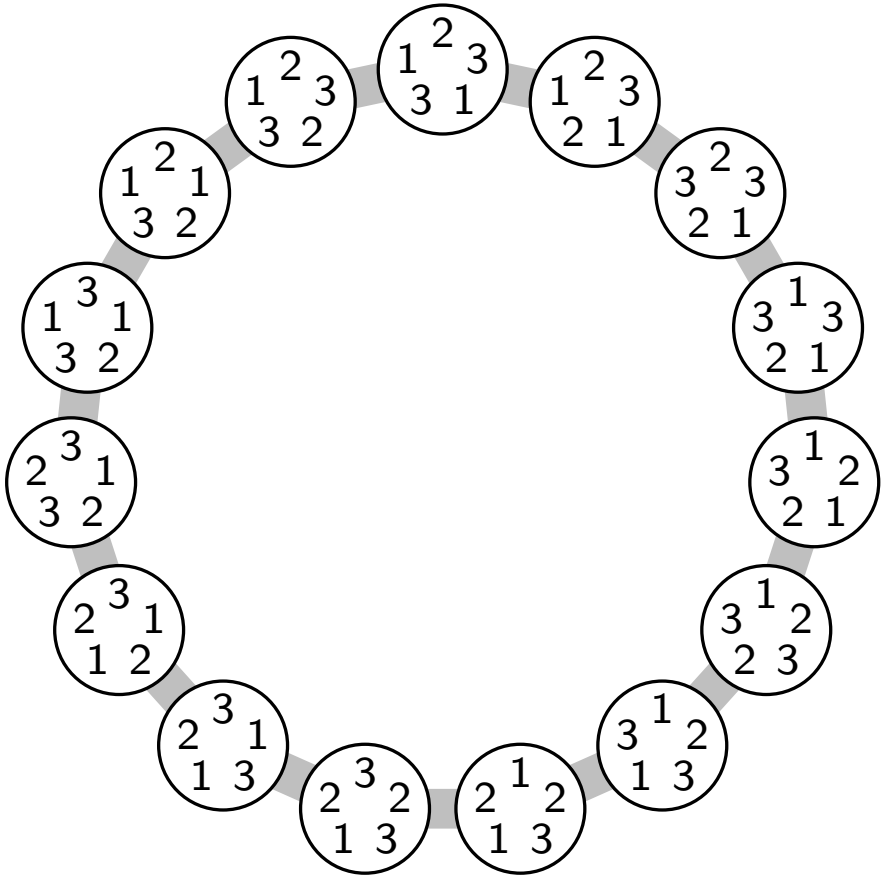

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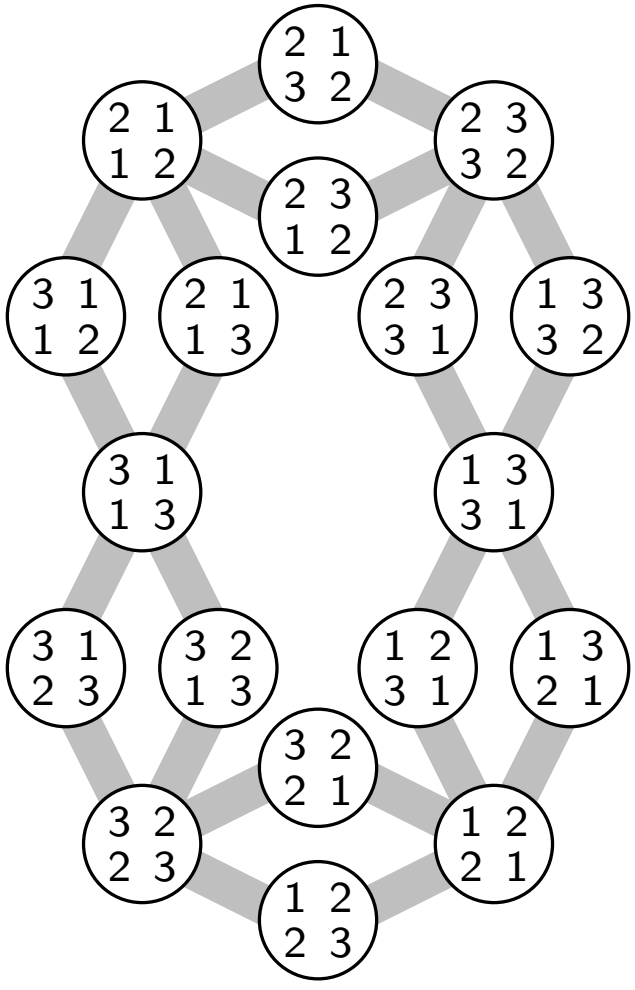

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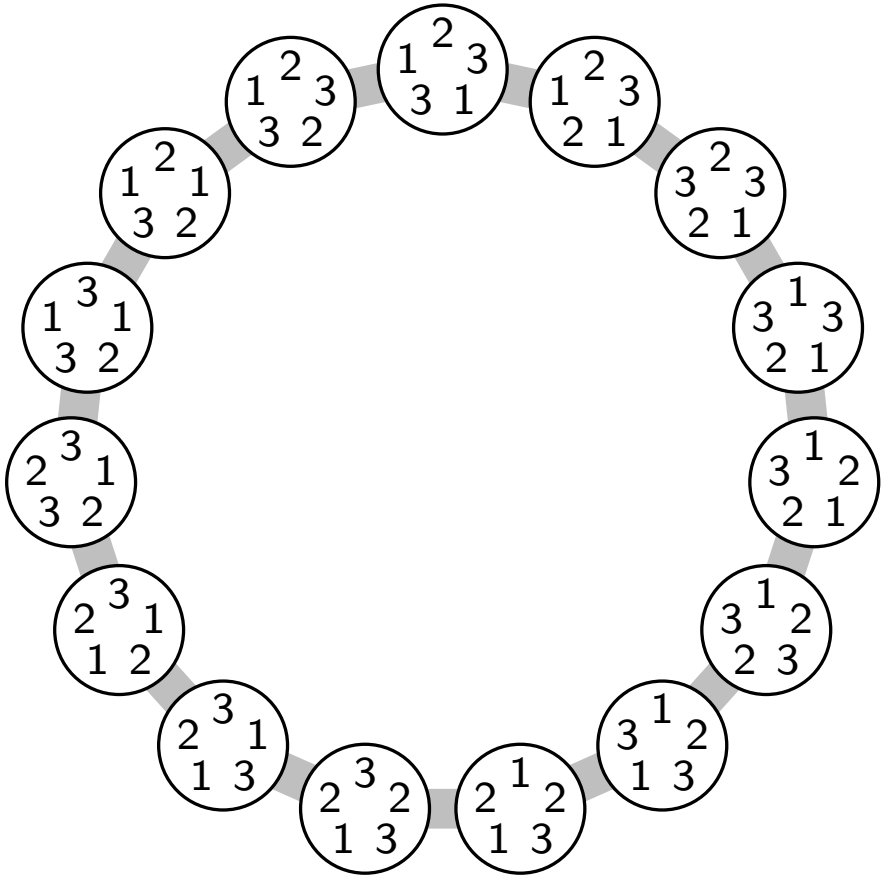

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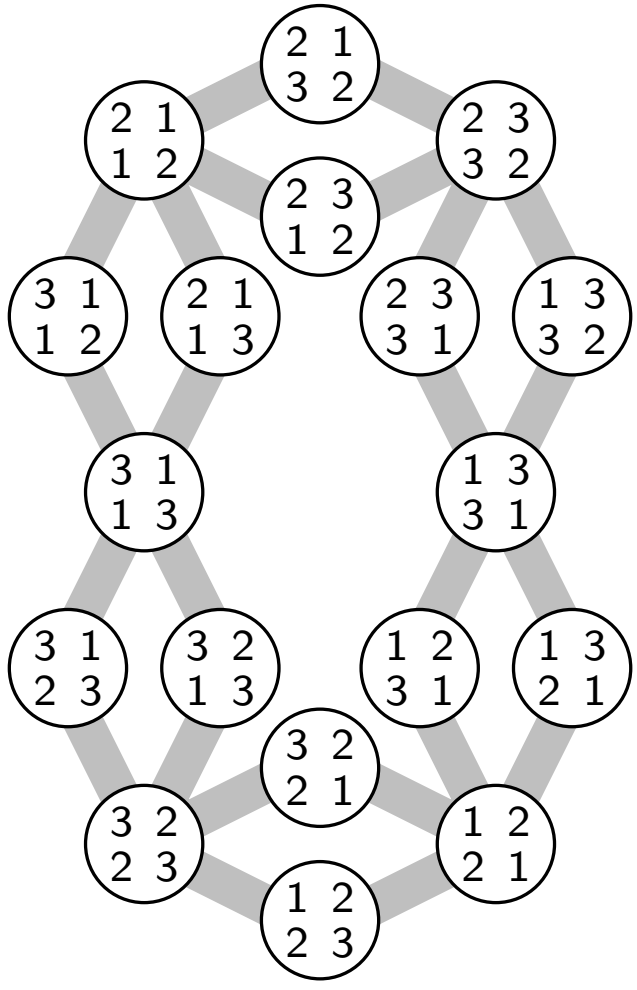

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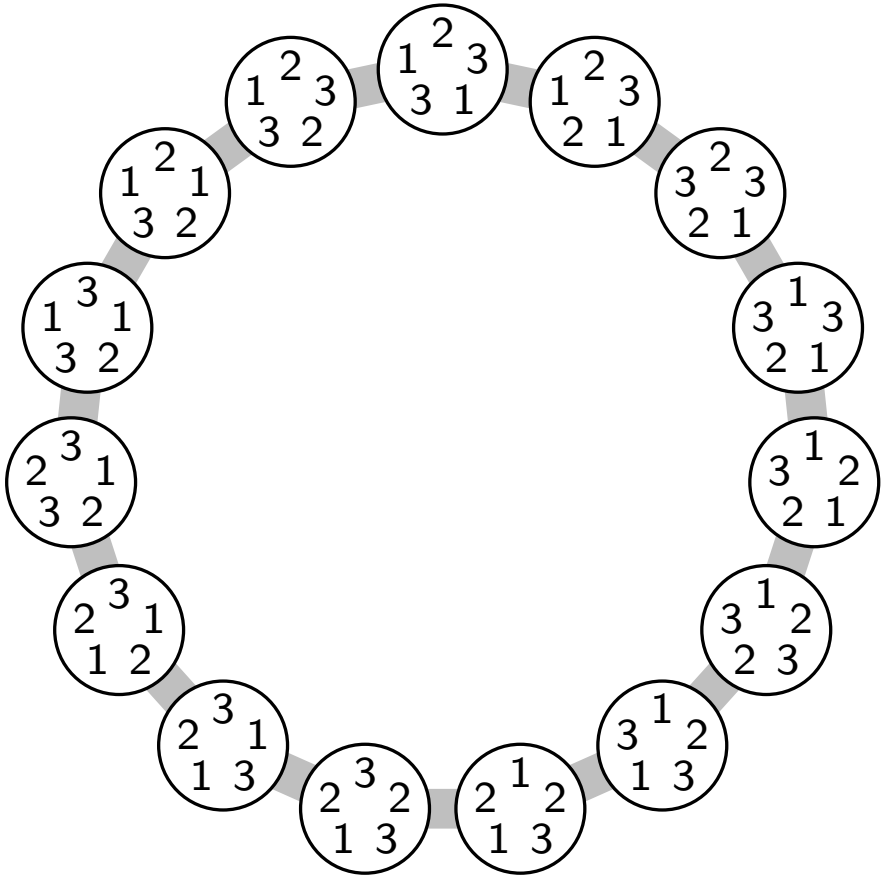

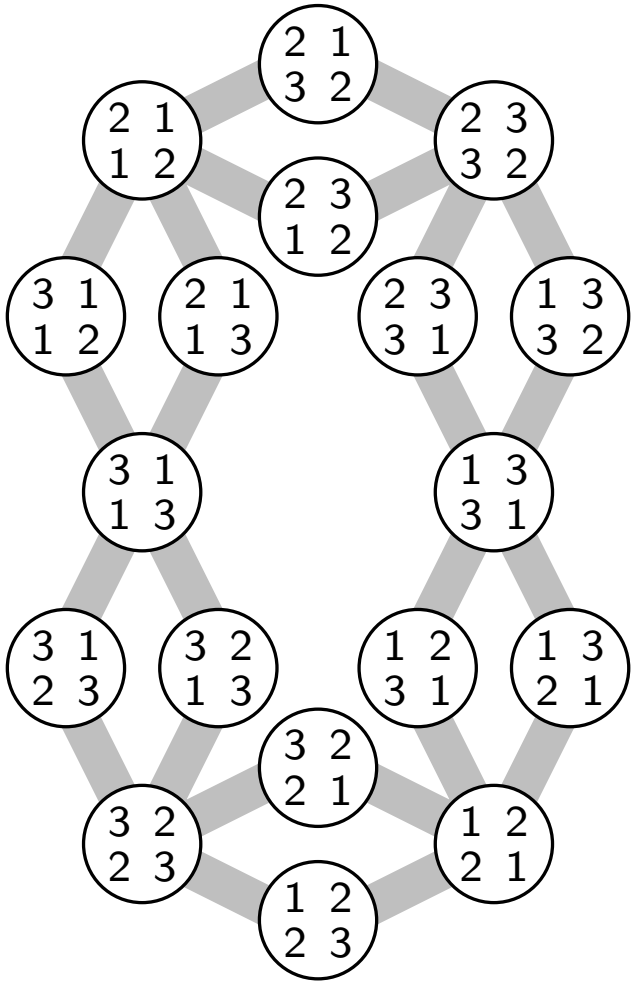

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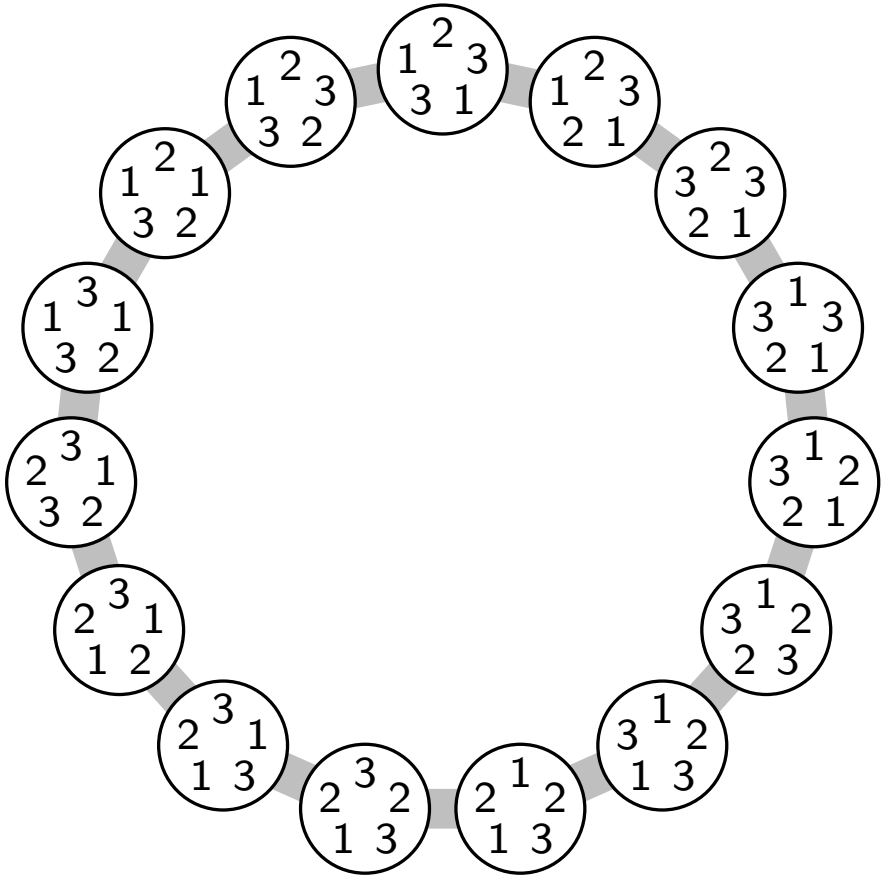

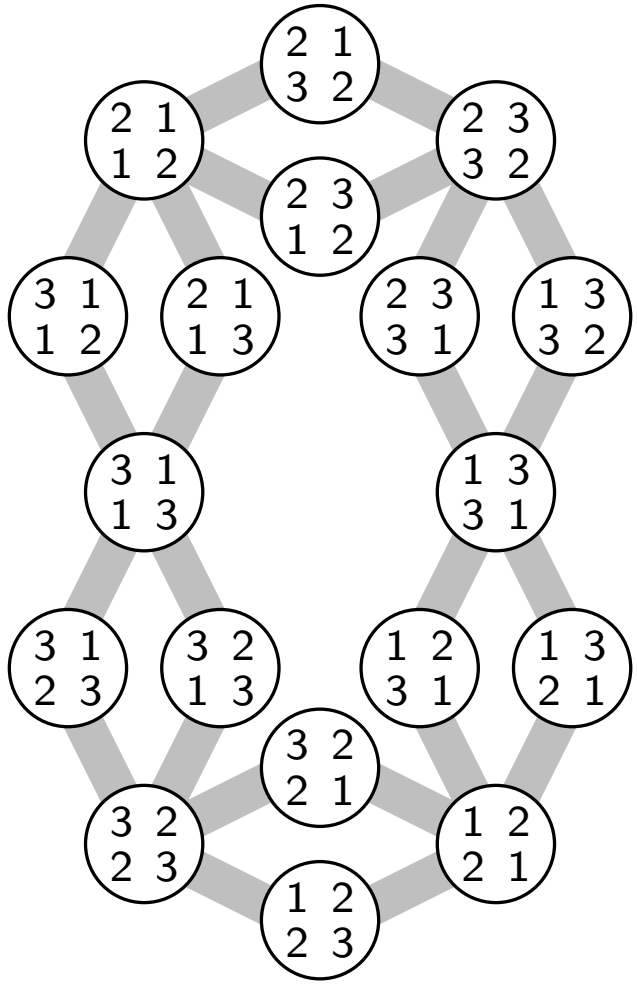

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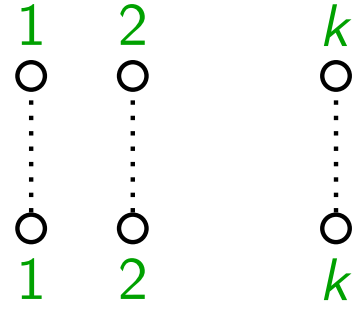
- "Reconfiguration graphs" of 3-colorings of 5-cycle and 4-cycle.
- Is the reconfiguration graph connected? What is its diameter?

Enlightening Examples



$G_k := K_{k,k} - kK_2$

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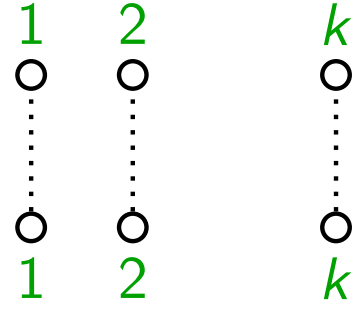

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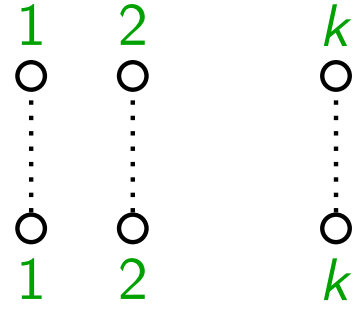

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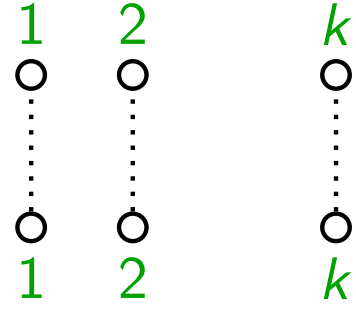

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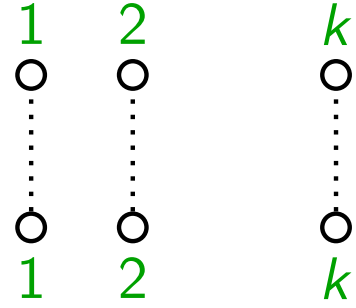

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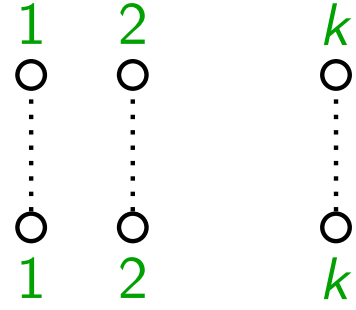

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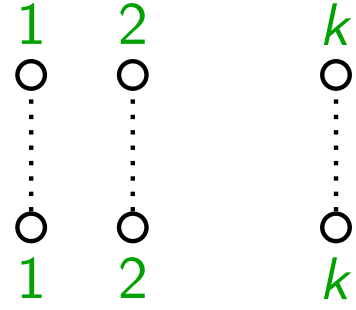

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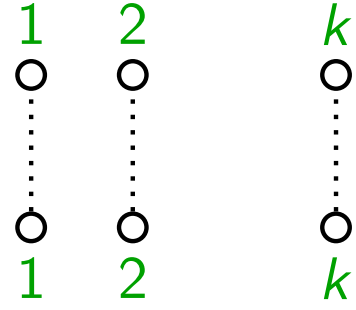

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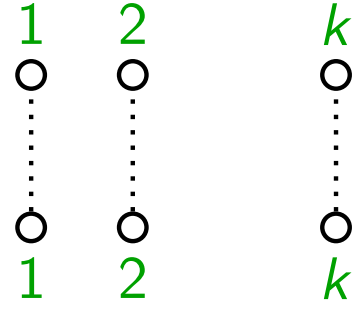

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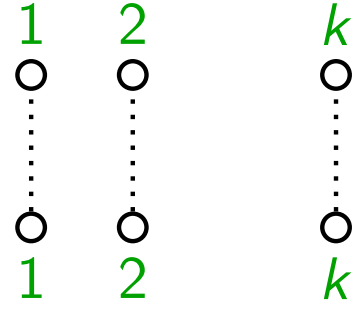

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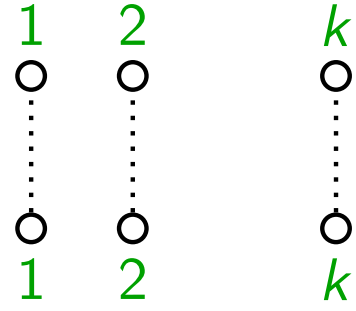

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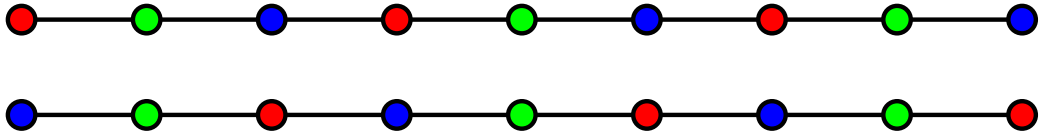

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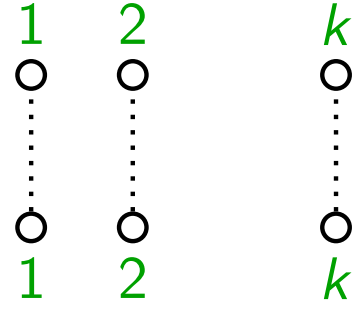

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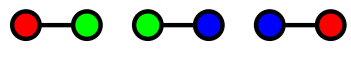
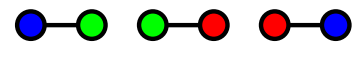
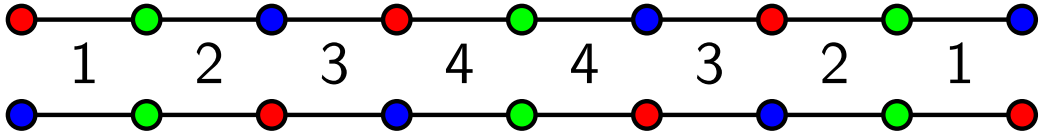
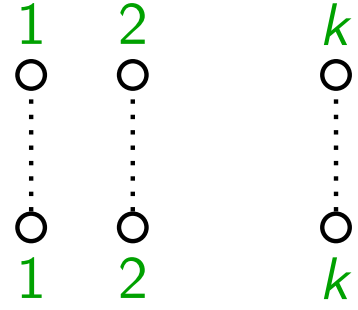
How about "nice" graphs? '+':  '−': 



Enlightening Examples



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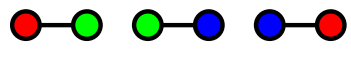
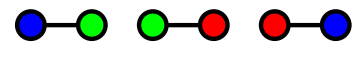
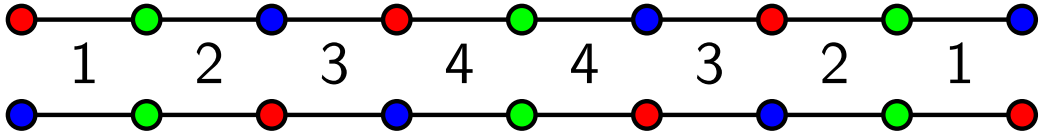
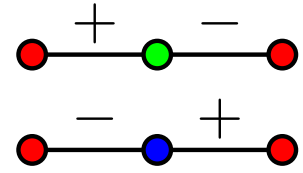
How about "nice" graphs? '+':  '−': 





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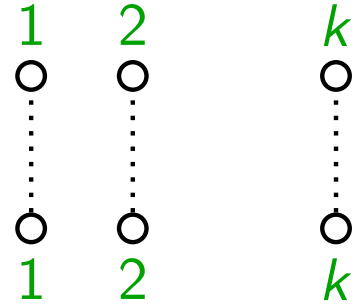

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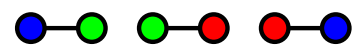
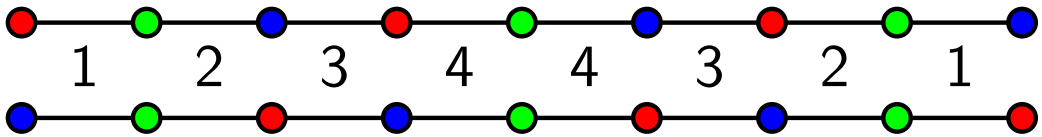
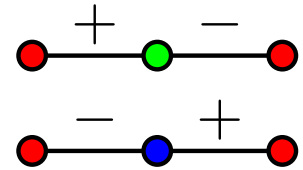
How about "nice" graphs? '+':  '−': 





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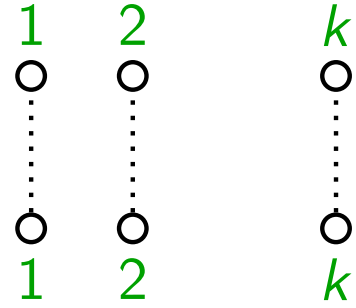
Enlightening Examples



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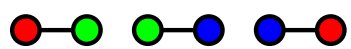
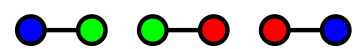
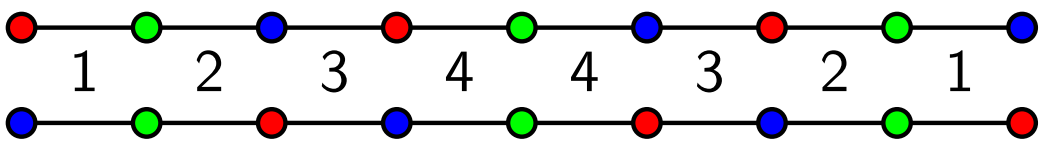
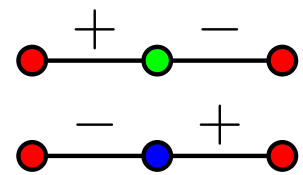
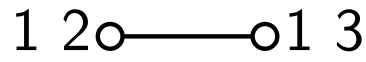
How about "nice" graphs? '+':  '−': 





- So $\mathrm{diam}(\mathcal{C}_3(P_n)) = \Theta(n^2)$ and $\mathrm{diam}(\mathcal{C}_k(P_n \vee K_{k-3})) = \Theta(n^2)$

What is List Coloring Reconfiguration?



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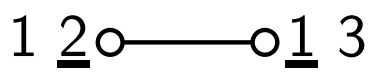

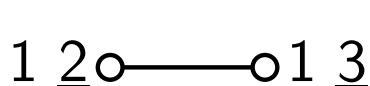

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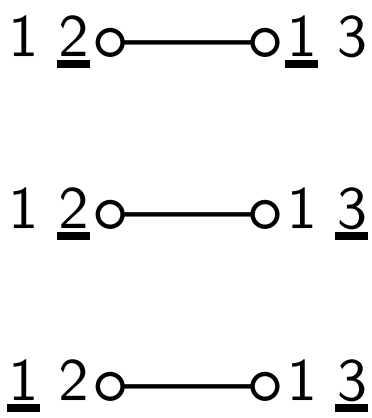

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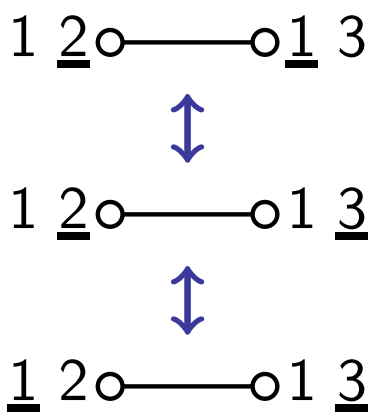

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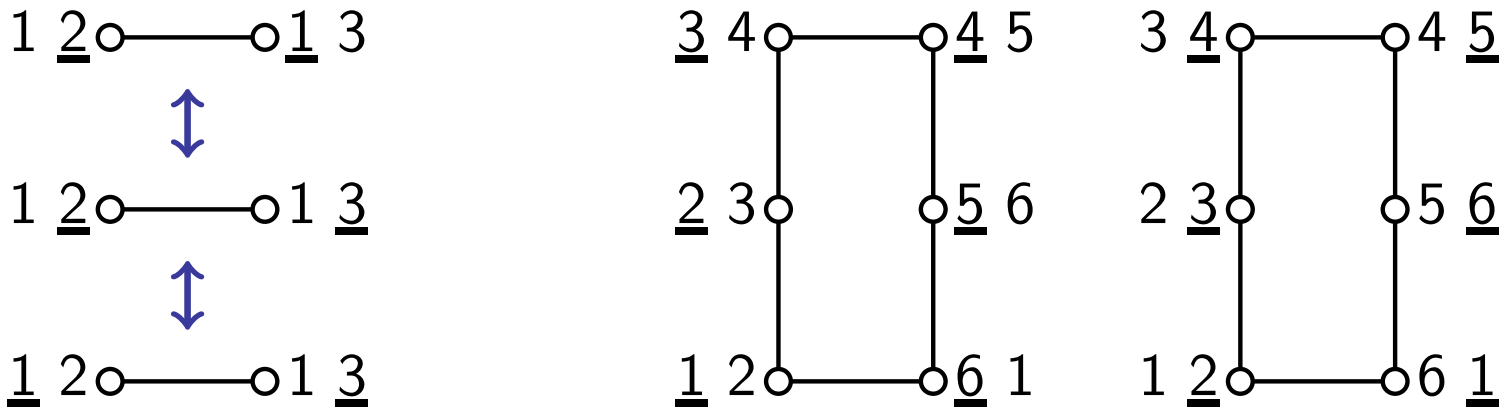

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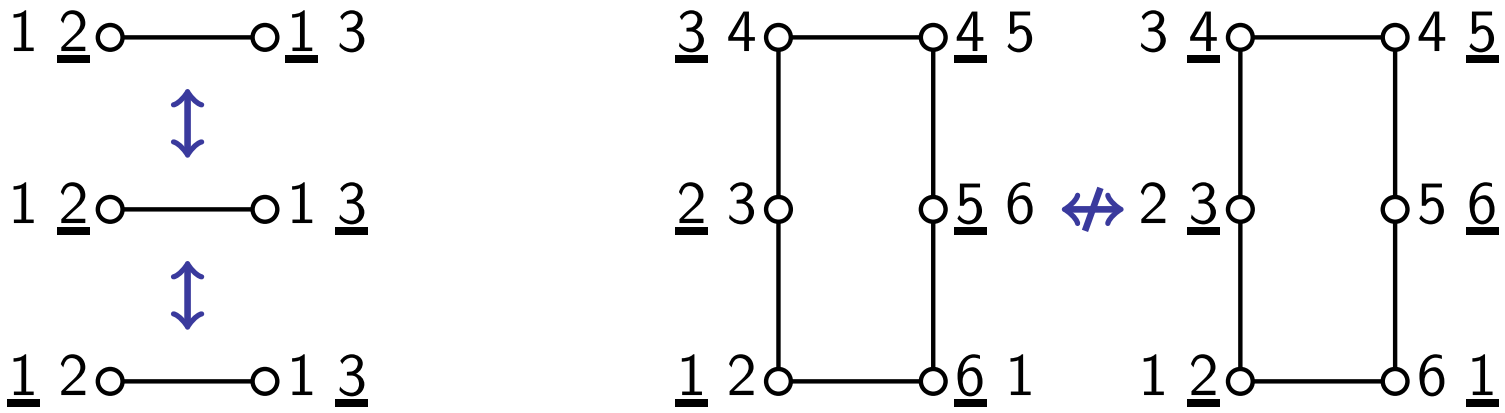

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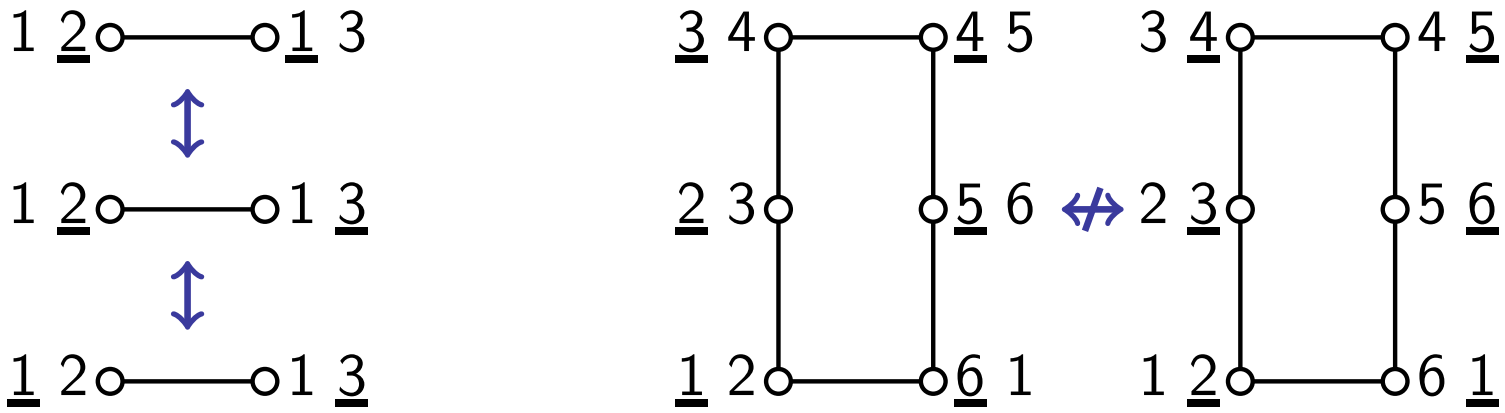

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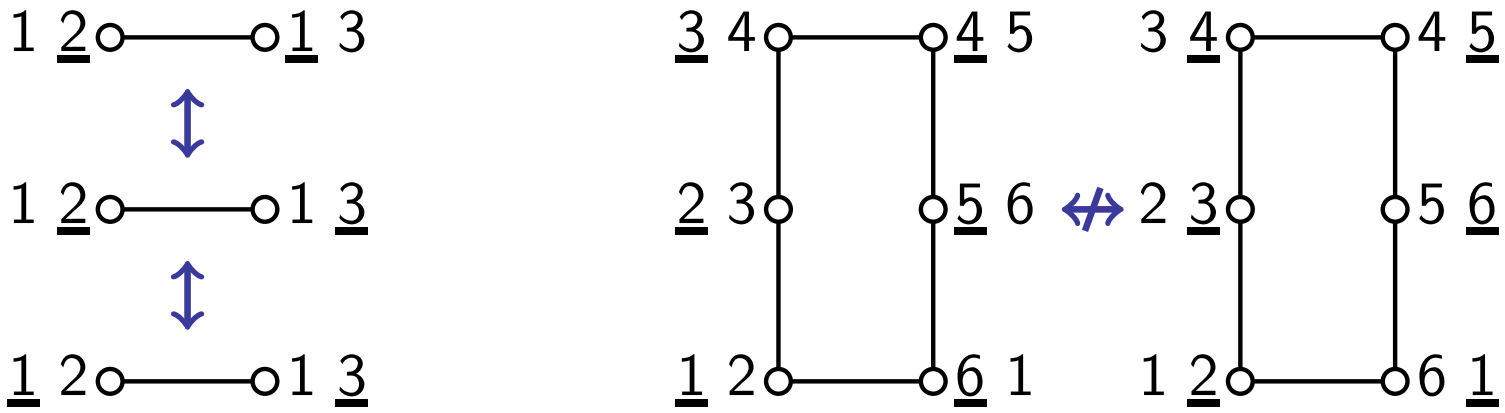

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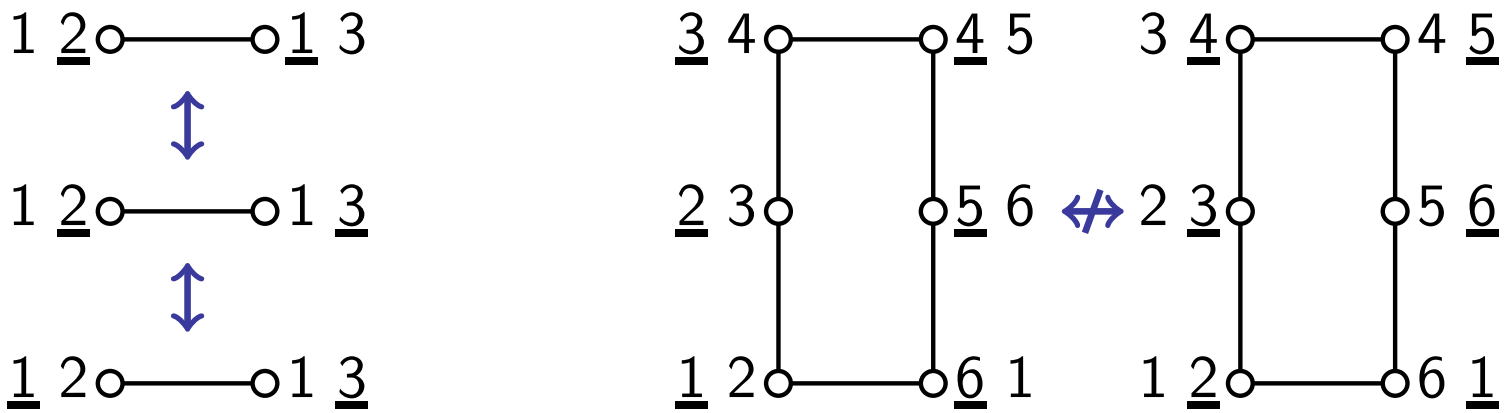

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Lists of Size $d(v) + 2$

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Prop: For every G and every f, with $f(v) \geq 2$ for all v, there is list assignment L with $|L(v)| = f(v)$ for all v and L-colorings α and β where changing α to β needs $n(G) + \mu(G)$ moves.

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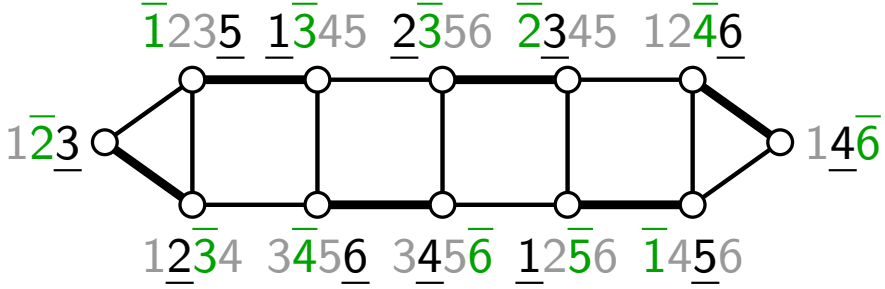

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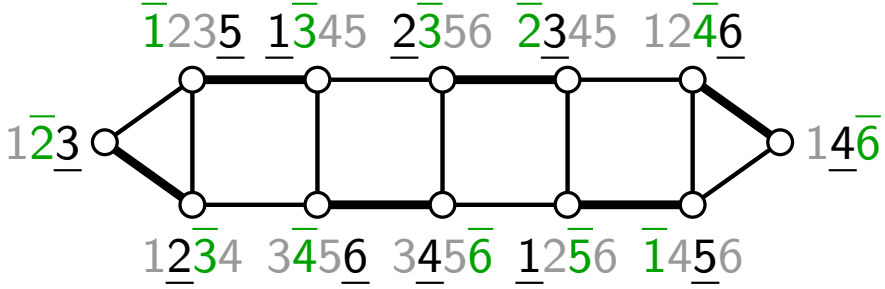

Thm:[Cambie–Cames van Batenburg–C.] `arXiv:2204.07928`

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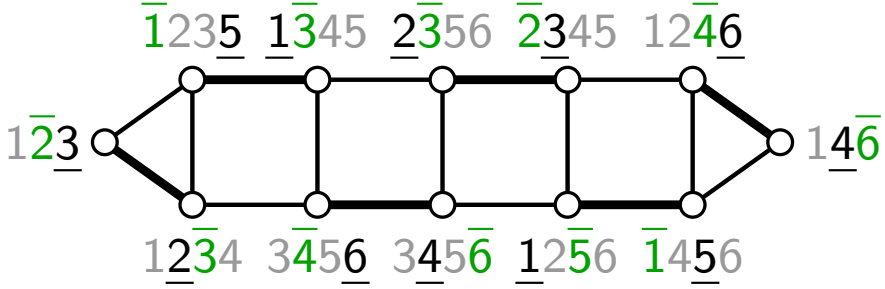

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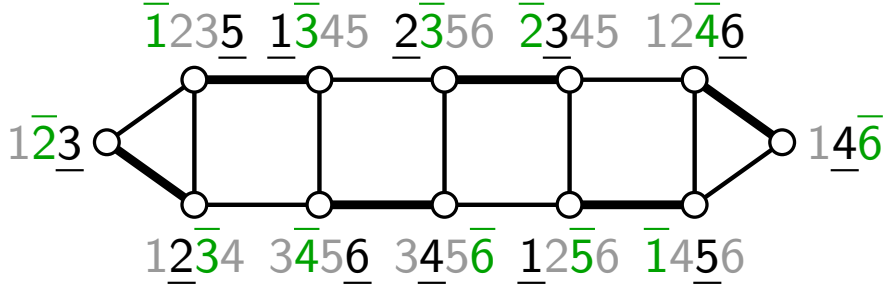

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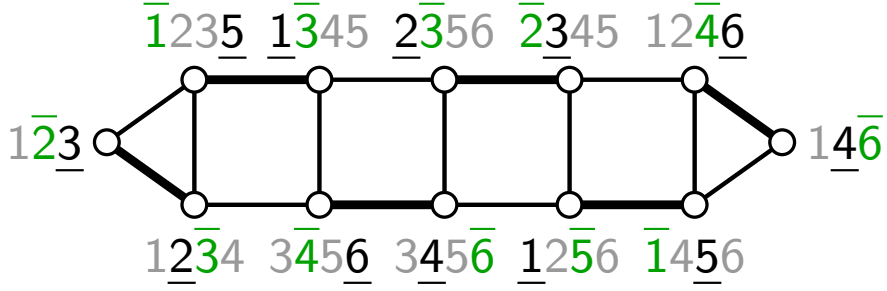

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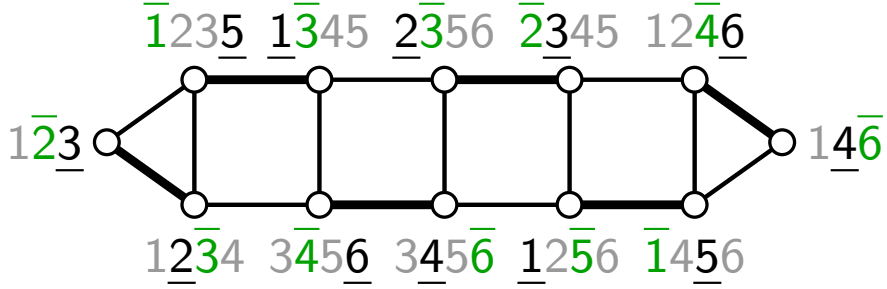

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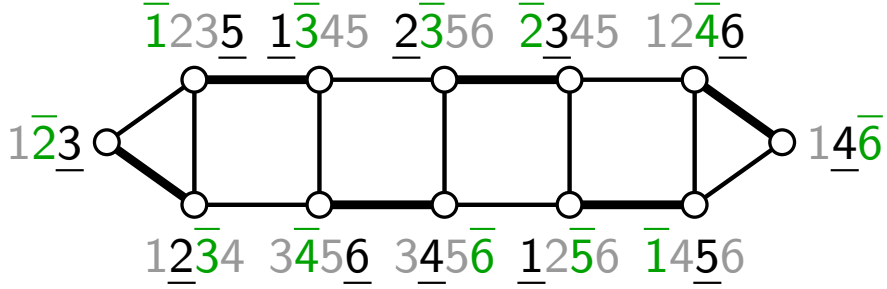

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Let G be connected with n verts and lists L. Let $\mathcal{C}(G, L)$ be the reconfig. graph, and $\widehat{\mathcal{C}}(G, L)$ be with all frozen colorings deleted.

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For coloring: by [FJP]; and $O(n)$ by [BFHR] `arXiv:2203.08885`

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Main Thm: If $L|(v)| \geq d(v)+1$ for all v and G has $\Delta \geq 3$, then $\widehat{\mathcal{C}}(G,L)$ is connected with diameter $O(n^2)$. `arXiv:2505.08020`
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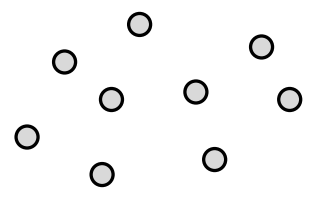

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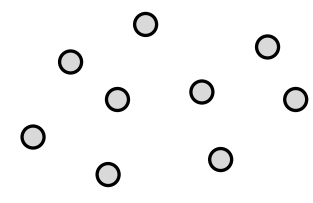

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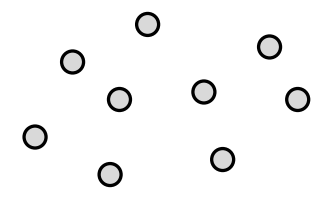

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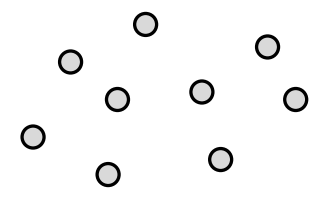

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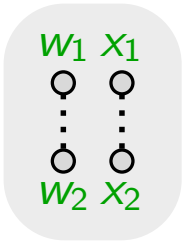
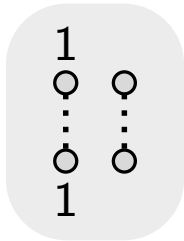
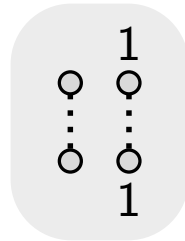




α



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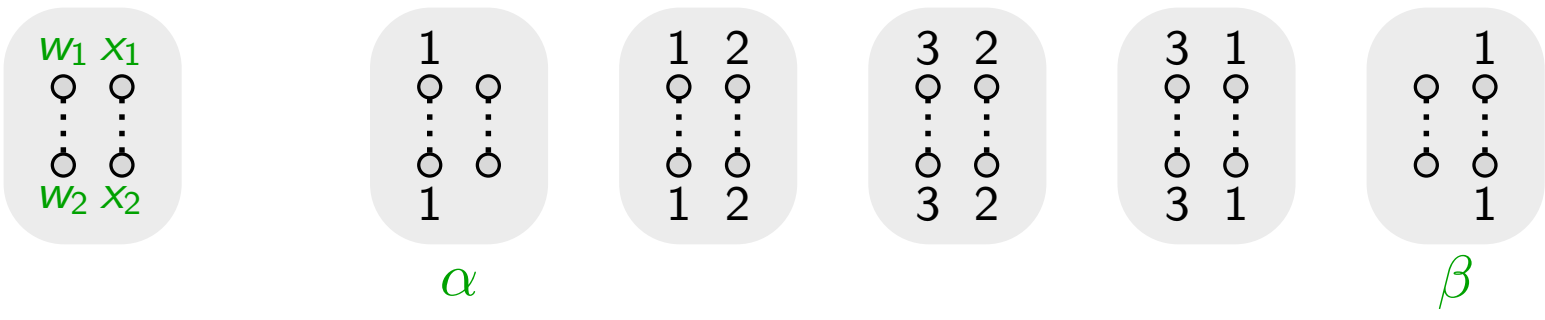

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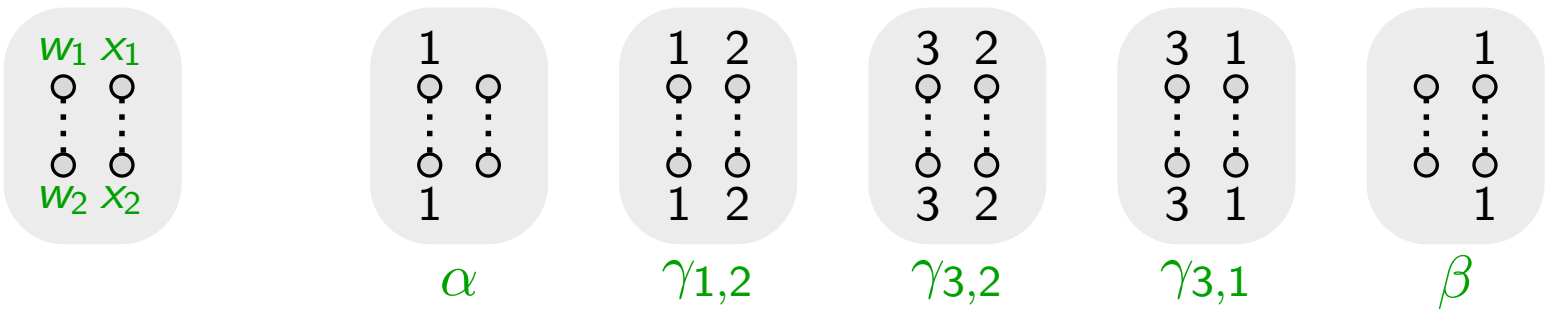

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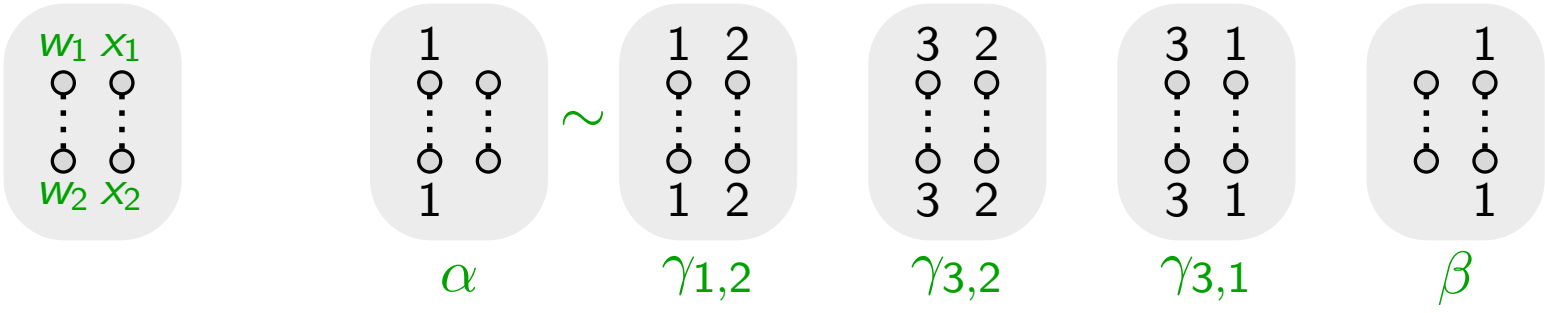

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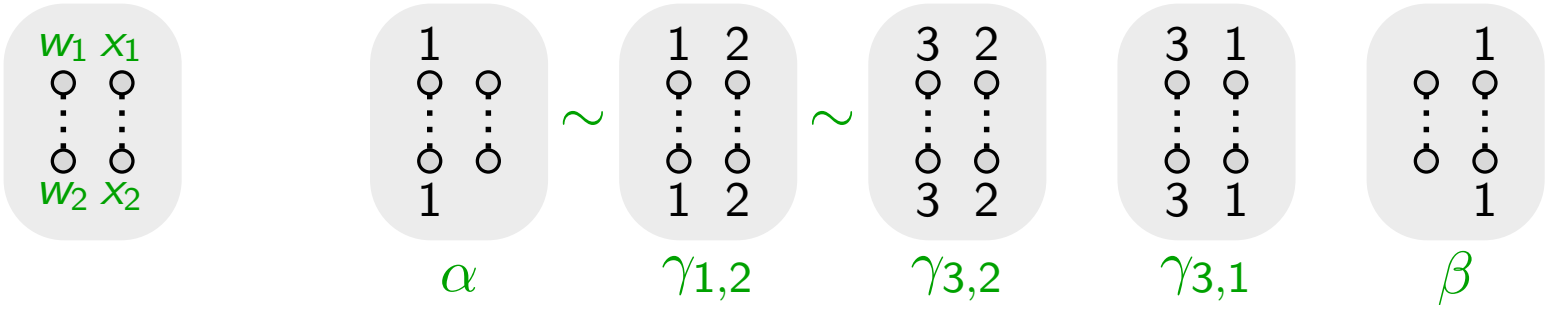

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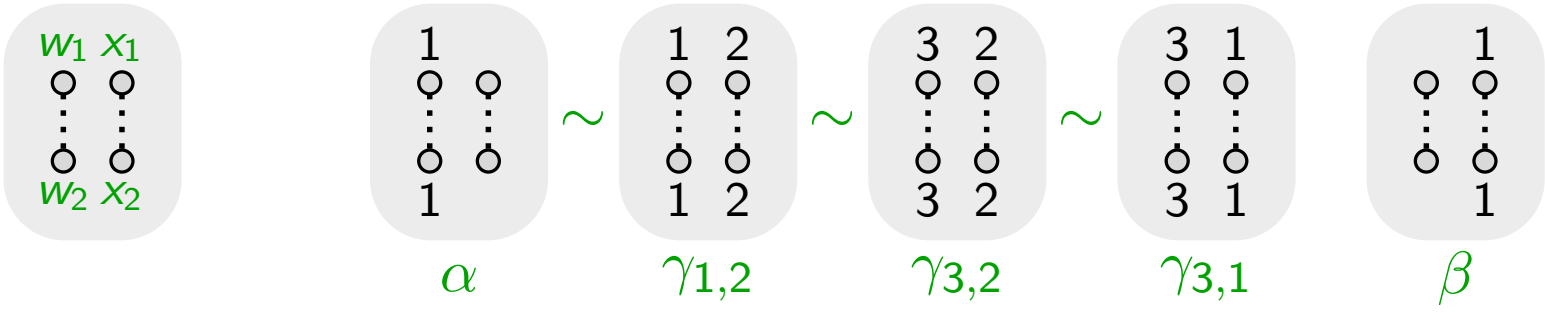

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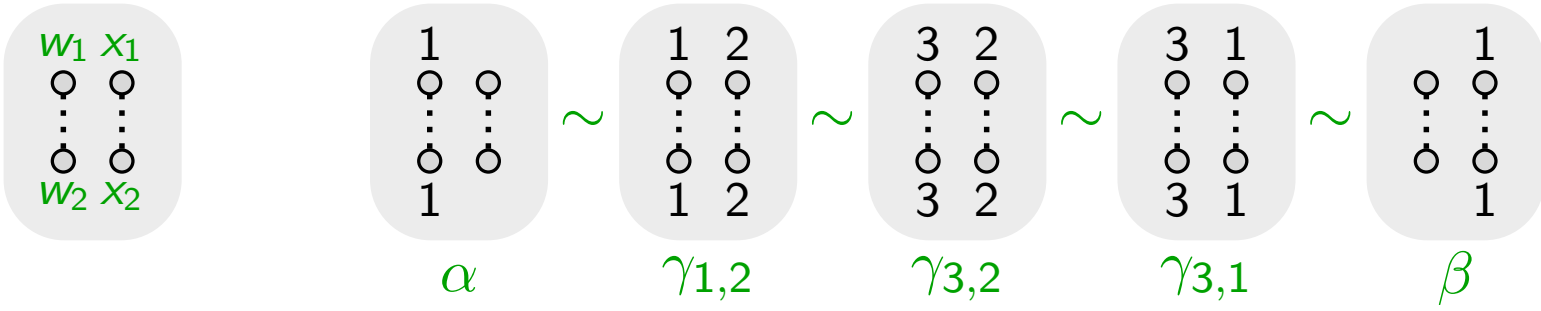

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- $\mathcal{G}$ is graphs with $\text{mad}(G) < a$, for some a; or
- $\mathcal{G}$ is planar graphs with girth at least g, for some g.

Main Tool for Sparse Graphs

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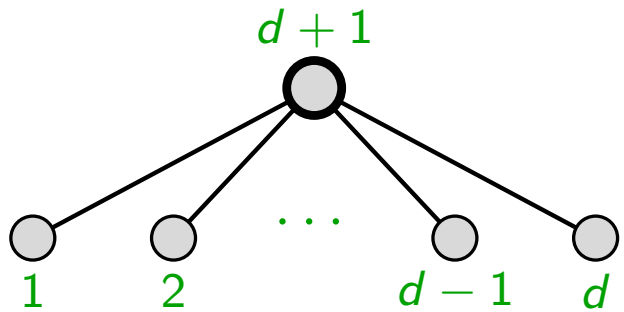

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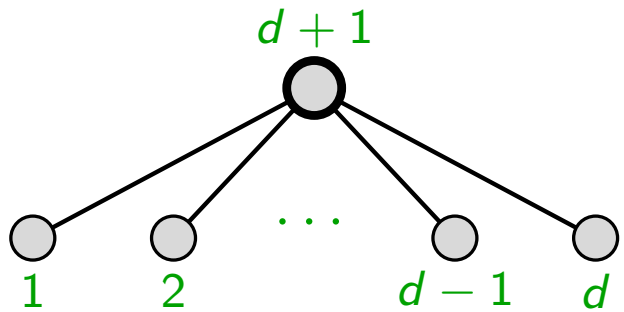

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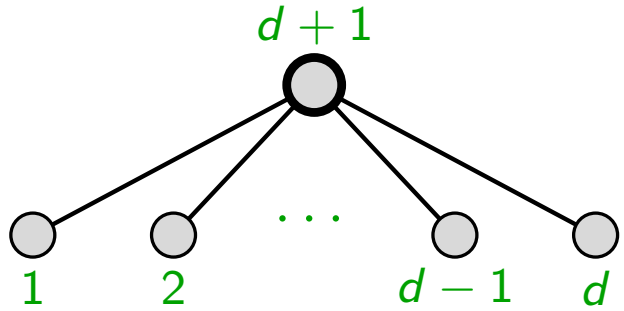

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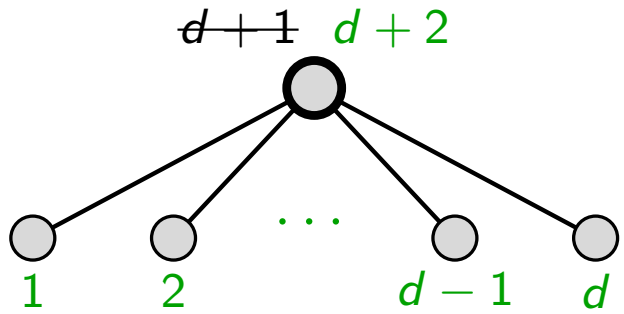

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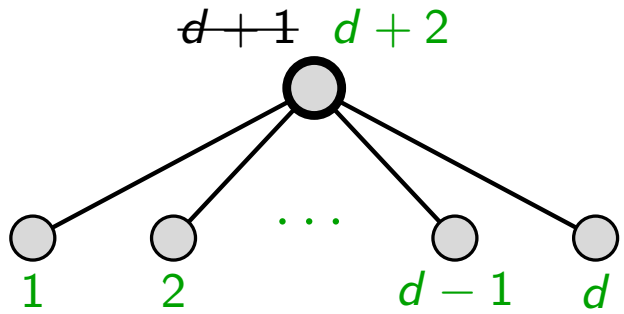

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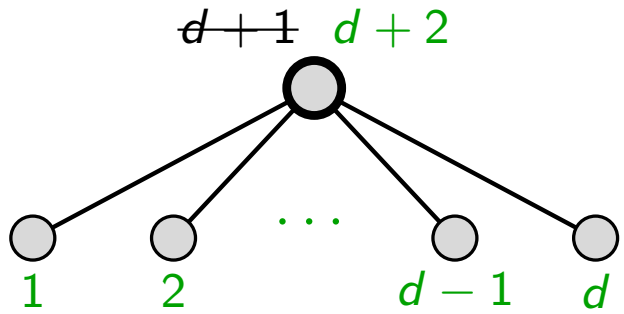

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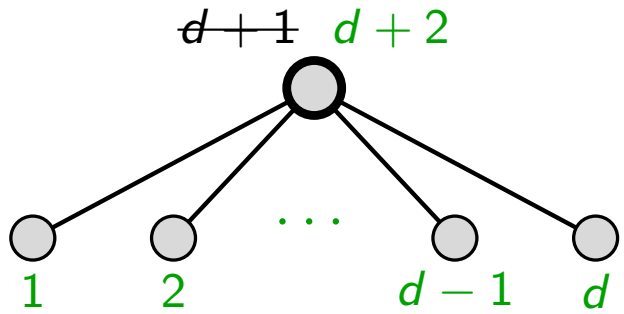

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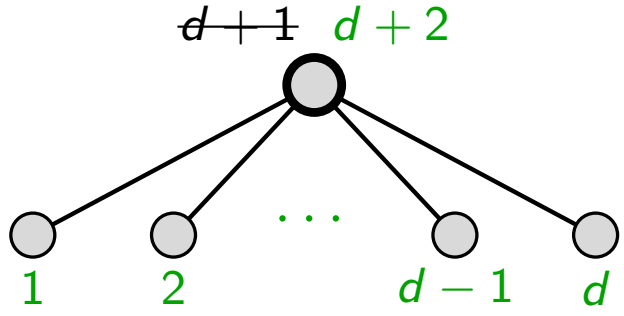

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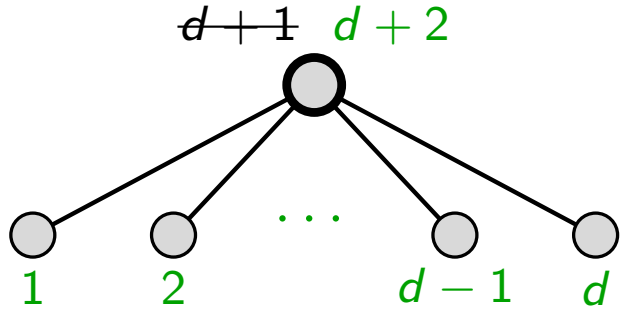

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Thm: If $|L(v)| \geq d(v) + 1$ and $\Delta \geq 3$, then diam $\widehat{\mathcal{C}}(G, L) \leq O(n^2)$.
Conj: If also $\delta(G) \geq 3$, then diam $\widehat{\mathcal{C}}(G, L) \leq O(n)$.
Correspondence: Analogue is false (as shown by cliques).

Thm: If planar and $|L(v)| \geq 10$, then diam $\mathcal{C}(G, L) = O(n)$.
Thm: If planar, ∇-free and $|L(v)| \geq 7$, then diam $\mathcal{C}(G, L) = O(n)$.
Conj: If d-degen and $|L(v)| \geq d + 3$, then diam $\mathcal{C}(G, L) = O(n)$.
Conj: Let $d := \text{mad}(G)$. If $|L(v)| \geq \lceil d + 2 \rceil$, then diam $\mathcal{C}(G, L) = O_d(n)$.
Correspondence: Analogues of theorems are true.