An Analogue of Mohar's Conjecture for List-Coloring

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Joint with Reem Mahmoud

William & Mary Math Colloquium
6 October 2023

Kempe Swaps

Defn: Given k-coloring φ, $i, j \in \{1, \ldots, k\}$, and v with $\varphi(v) = i$, an (i, j)-swap at v recolors v's component of subgraph induced by color classes i and j, swapping those colors on that component.

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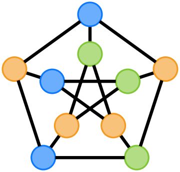

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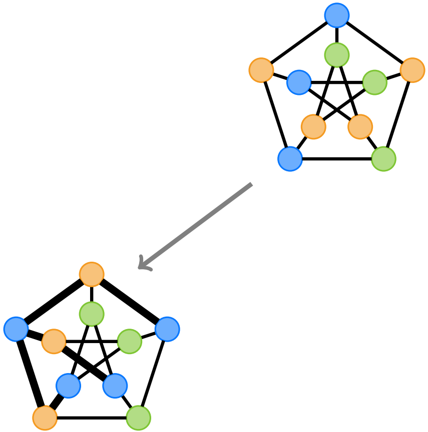

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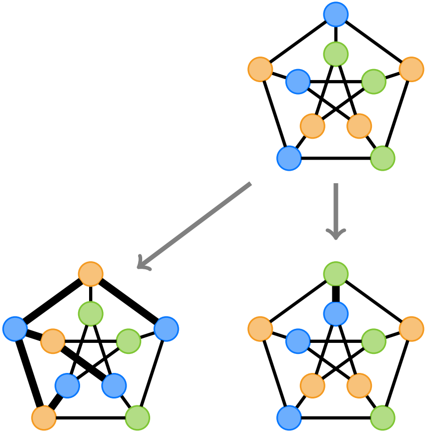

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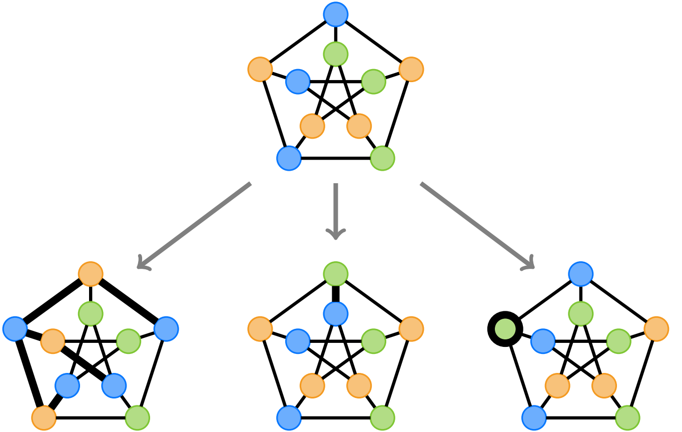

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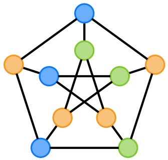

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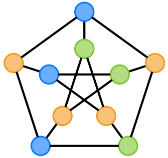

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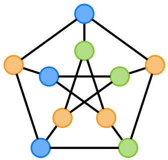

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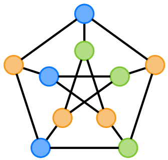

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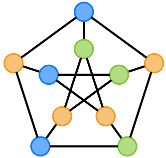

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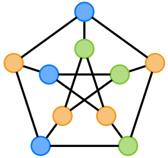

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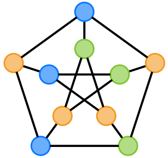

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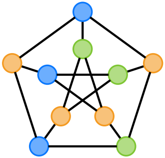

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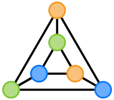

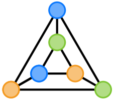

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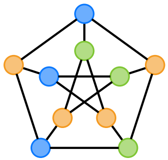

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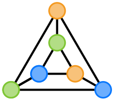

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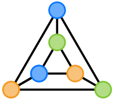

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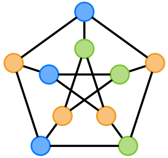

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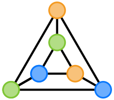

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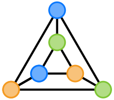
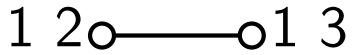


Plan: We will generalize this to list-coloring.

Definitions and Examples



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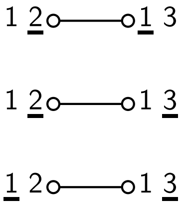

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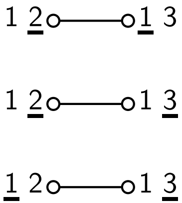

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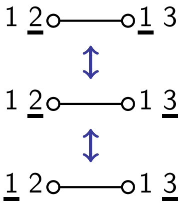

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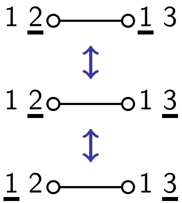

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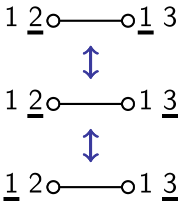

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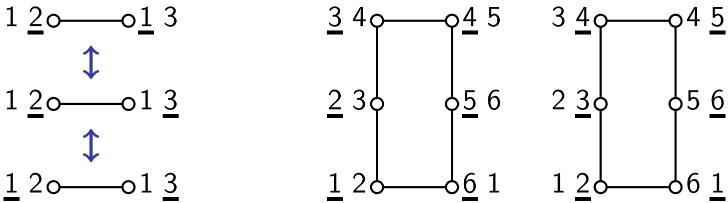

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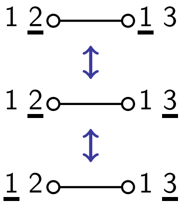

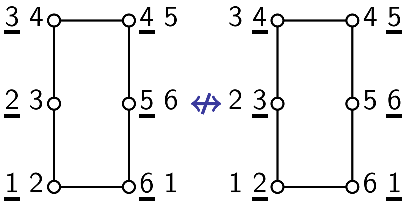

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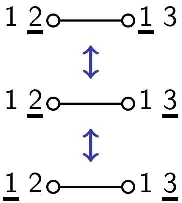

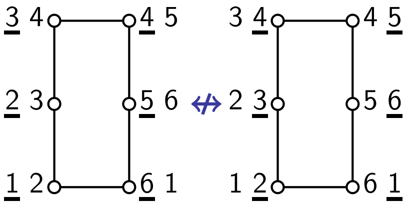

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Main Theorem [C-Mahmoud '23+]**:** If G is k-regular with $k \geq 3$ and G is connected, then G is k-swappable if $G \neq K_3 \square K_2$.

A Roadmap to a Proof, Inspired by Choosability

Degeneracy Lem: (extends [LV–M]) If G has a vertex v with $d(v) < k$, then G is k-swappable iff $G - v$ is k-swappable.

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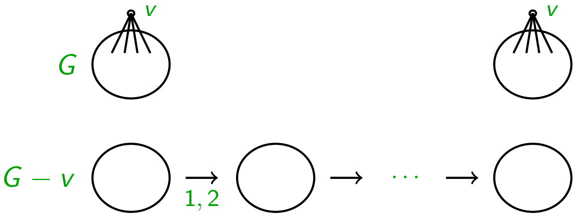

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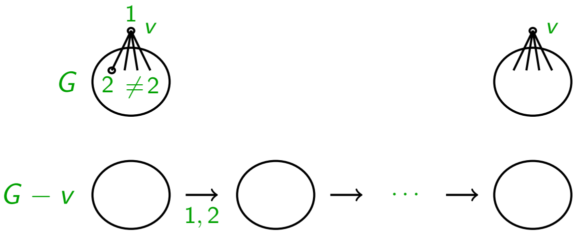

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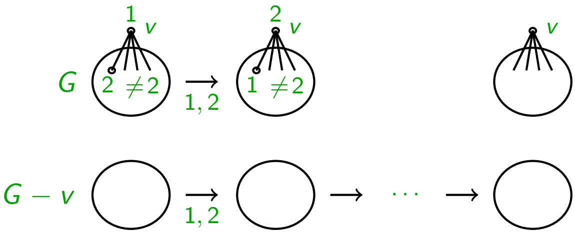

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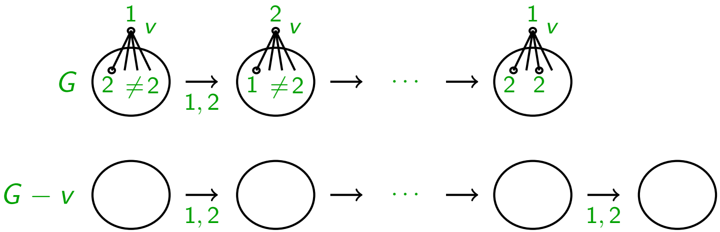

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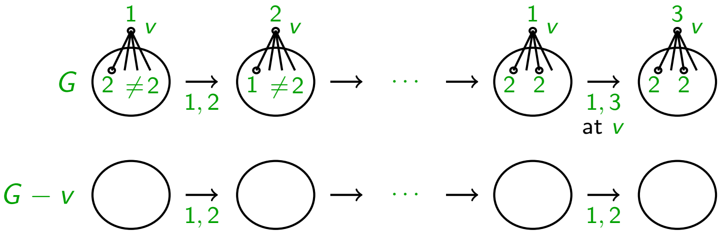

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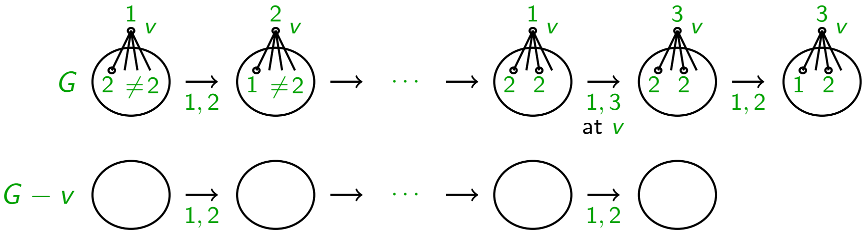

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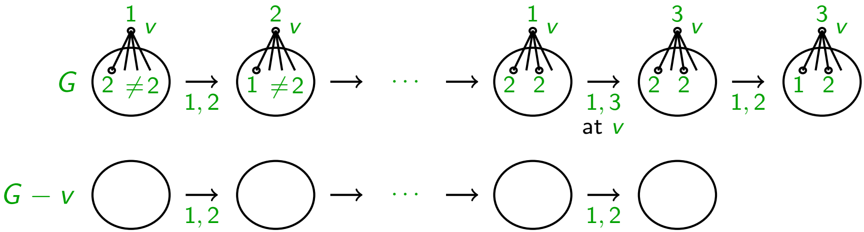

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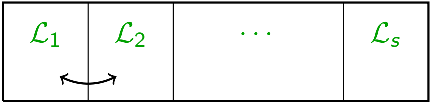

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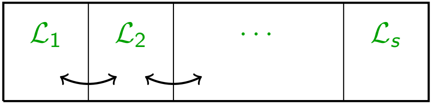

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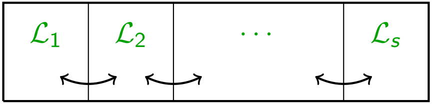

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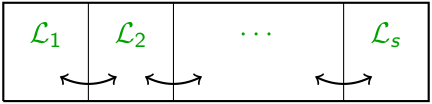

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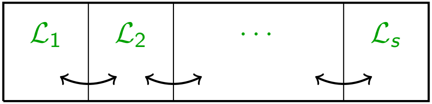

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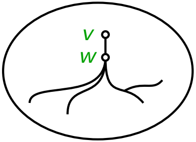

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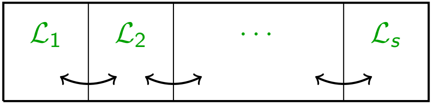

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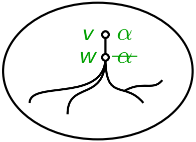

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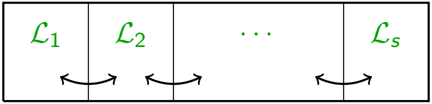

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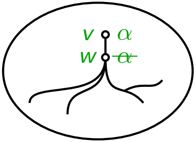

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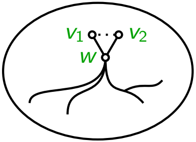

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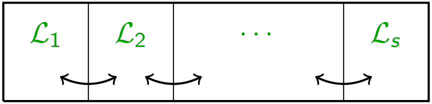

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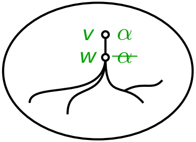

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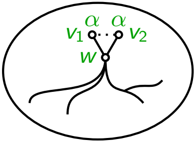

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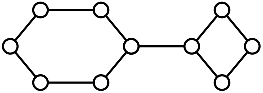

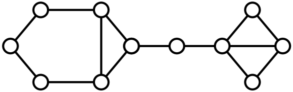

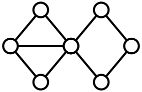

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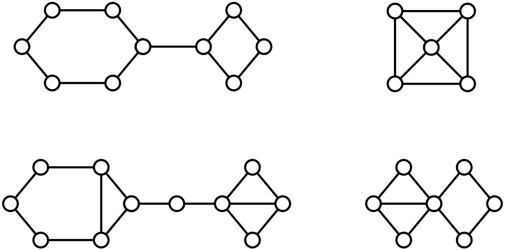

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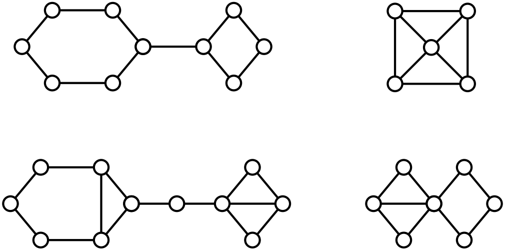

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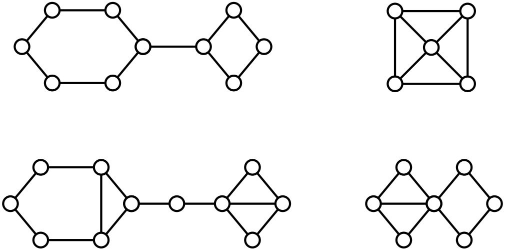

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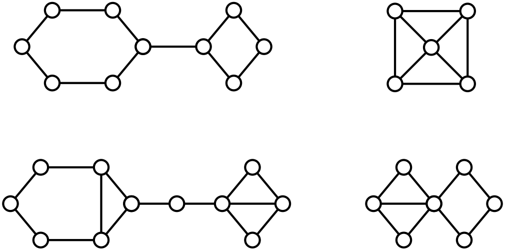

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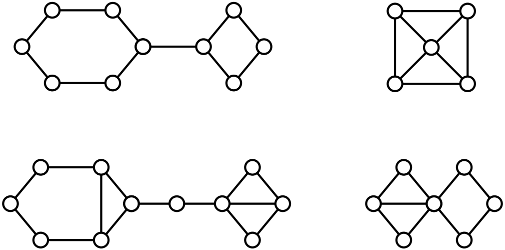

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- True except $K_2 \square K_3$. Extend to list-coloring.

- G is L-swappable if we can turn any L-coloring into any other by sequence of Kempe swaps, keeping L-coloring at each step.
- **Key Lemma:** If H is degree-swappable and a connected G has H as induced subgraph, then G is also degree-swappable.
  - Also useful for list-edge-swappability of planar graphs (Δ big).

- **Thm:** [C-Mahmoud] If G is k-regular with $k \geq 3$ and G is connected, then G is k-swappable if $G \neq K_3 \square K_2$.
  - If G has small vertex cut S, then $G - S$ looks "almost" like Gallai forest. So we find induced degree-swappable subgraph.
  - If G contains 4-wheel, then done.
  - If G contains no 4-wheel, but is 4-connected, then explicitly find paths between any two L-colorings.

- Read more at `arXiv:2112.07439`