In Most 6-regular Toroidal Graphs All 5-colorings are Kempe Equivalent

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Joint with Reem Mahmoud

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Kempe Swaps

Defn: Given k-coloring φ, $i, j \in \{1, \ldots, k\}$, and v with $\varphi(v) = i$, an (i, j)-swap at v recolors v's component of subgraph induced by color classes i and j, swapping those colors on that component.

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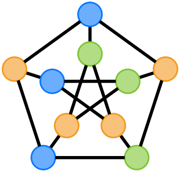

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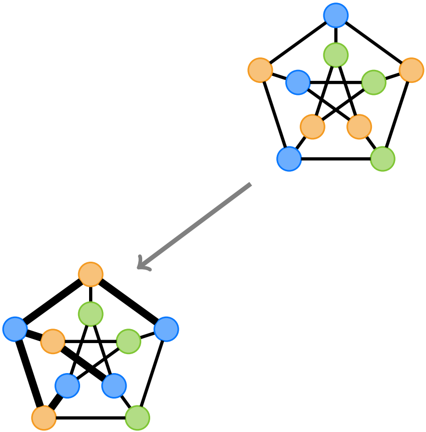

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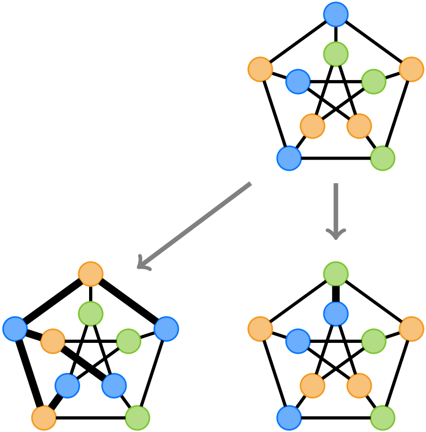

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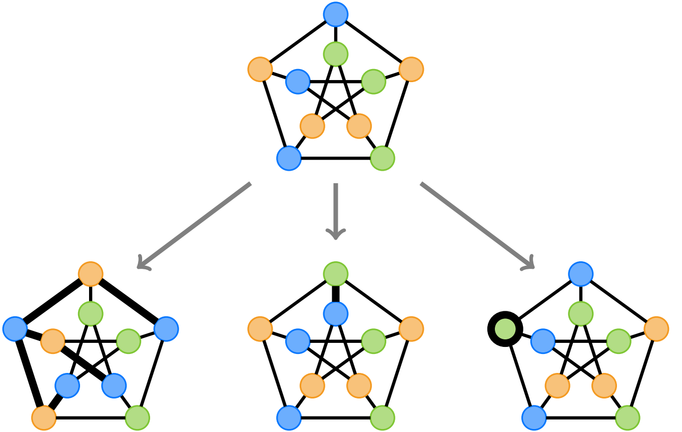

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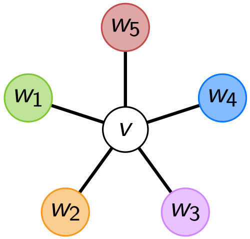

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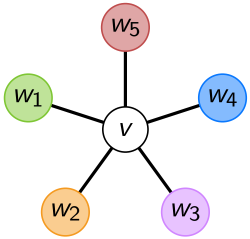

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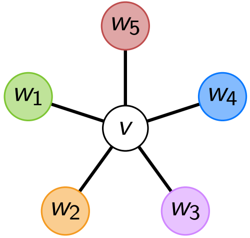

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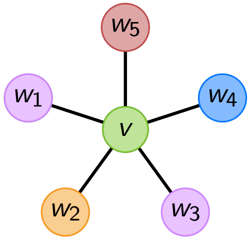

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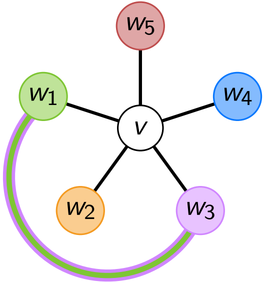

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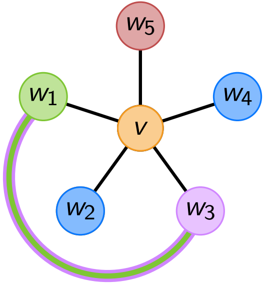

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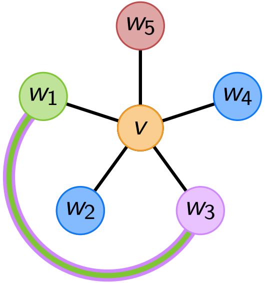

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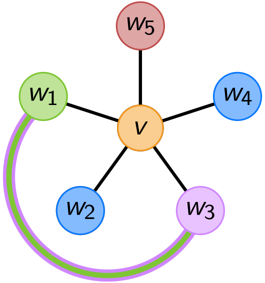

Note: Kempe swaps also give a short proof of Brooks' Theorem. For edge-coloring, Kempe swaps are extremely useful, since each Kempe component is a path or a cycle.

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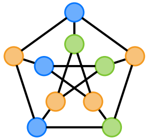

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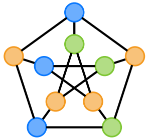

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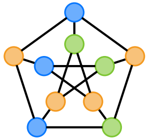

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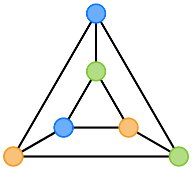

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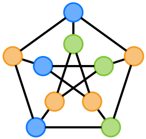

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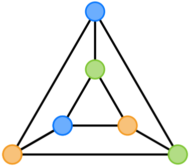

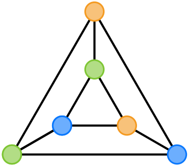

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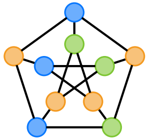

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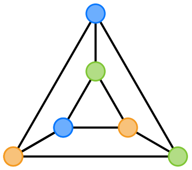

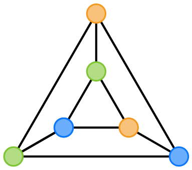

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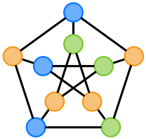

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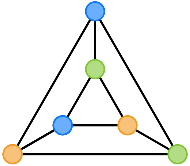

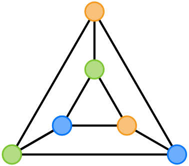

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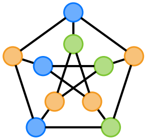

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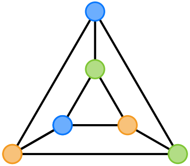

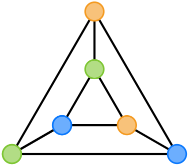

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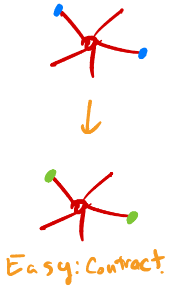

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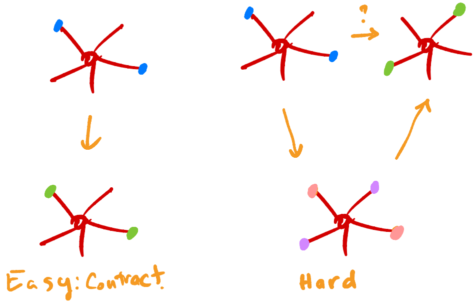

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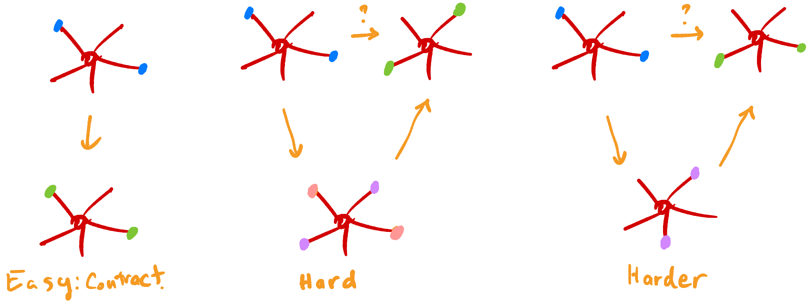

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  Ideas: From 5CT and finding "in-between" 5-coloring.

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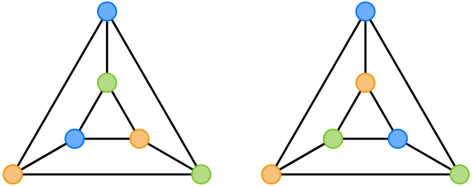

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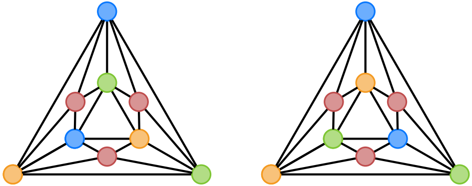

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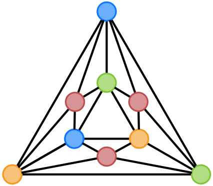

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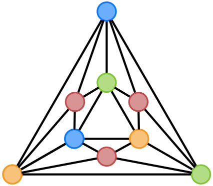

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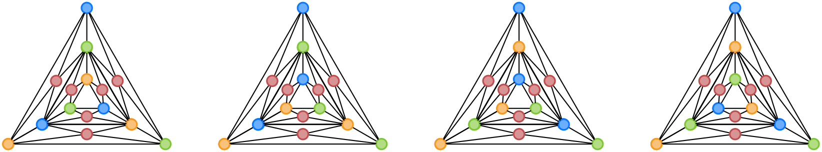

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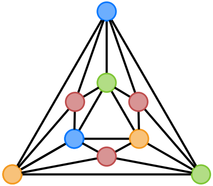

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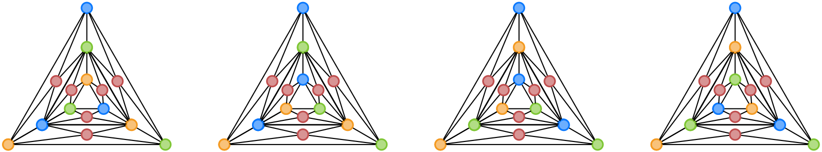

Obs: Gluing along triangles creates 4-chromatic planar graphs with arbitrarily many 4-equivalence classes.

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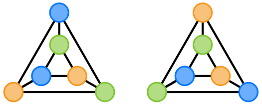

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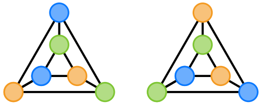

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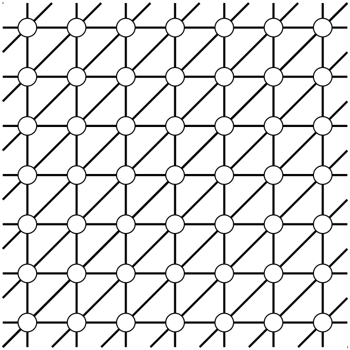

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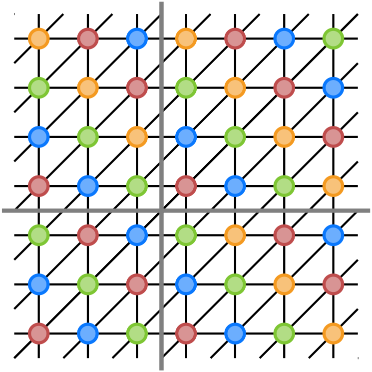

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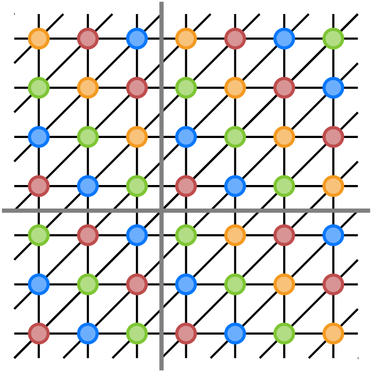

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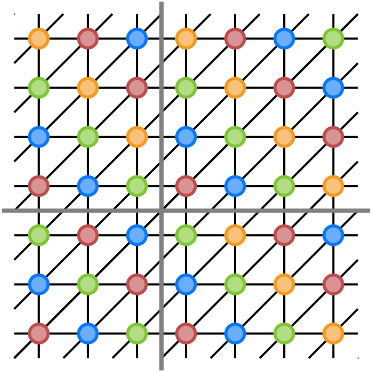

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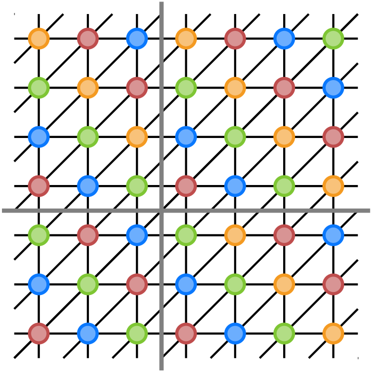

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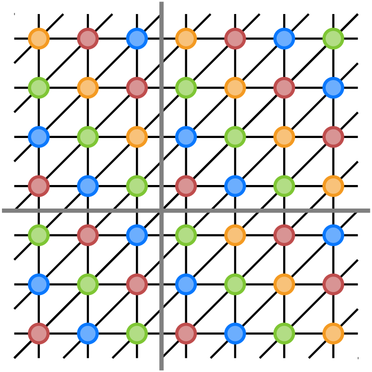

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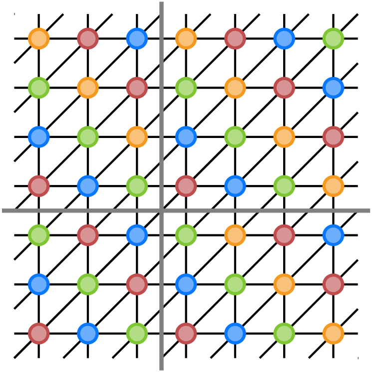

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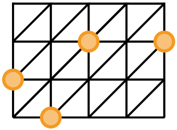

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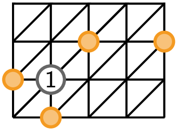

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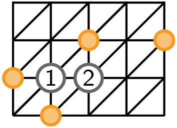

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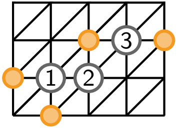

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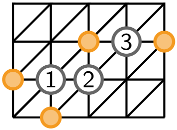

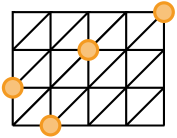

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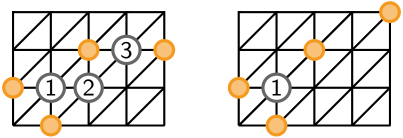

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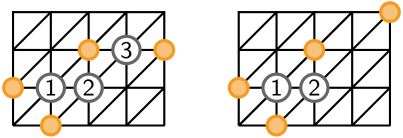

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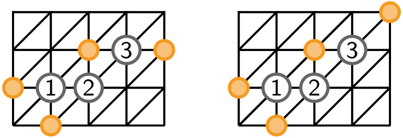

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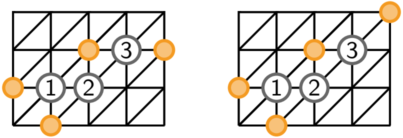

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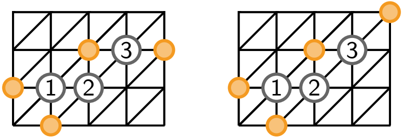

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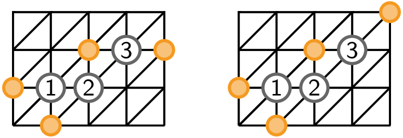

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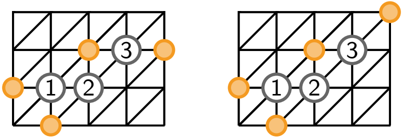

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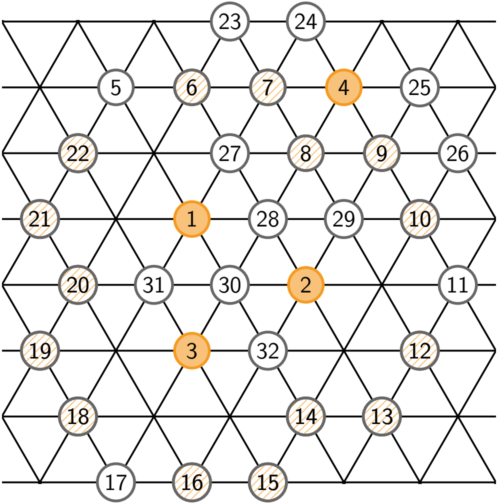
Finding a Good 4-Template: A Pretty Picture

Lem: If φ has a triple (such as vertices 1, 2, 3 below), then φ is 5-equivalent to a coloring with a good 4-template.



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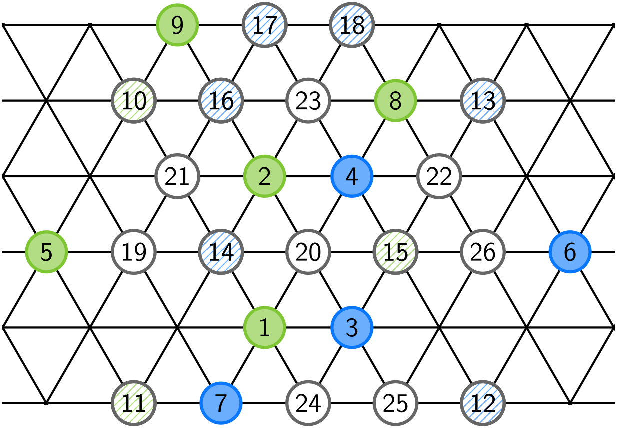

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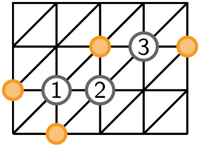

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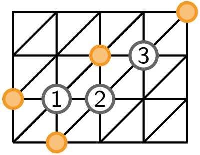

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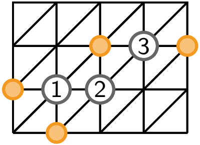

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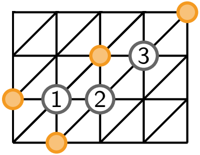

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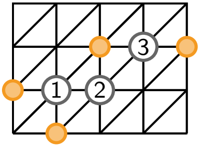

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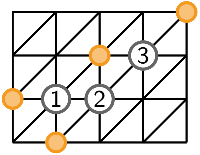

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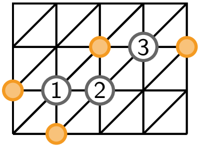

Ques: Is triangulated toroidal grid 5-ergodic?

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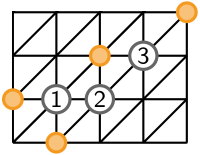

Thm: [C-Mahmoud '21] If G is 6-regular toroidal graph and each non-contractible cycle has length at least 7, then G is 5-ergodic.

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2. For good 4-templates T_1 and T_2, there are 5-colorings φ_1, φ_2 s.t. each φ_i has T_i and φ_1 and φ_2 are 4-equiv. **Pf:** Easy.

Summary

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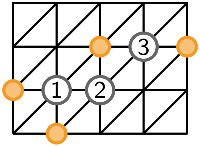

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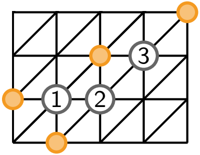

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2. For good 4-templates T_1 and T_2, there are 5-colorings φ_1, φ_2 s.t. each φ_i has T_i and φ_1 and φ_2 are 4-equiv. **Pf:** Easy.
3. Every 5-coloring of G is 5-equivalent to a 5-coloring containing a good 4-template. **Pf:** Hard.